CHAPTER - 5

STABLE NUMERICAL SOLUTIONS OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS USING LEGENDRE SCALING FUNCTIONS OPERATIONAL MATRIX

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5.1 Introduction

Instaneous change in application sometime depends on the past. That is why it is important to study fractional order differentiation. Now a day's people are adopting the fractional order differentiation in the Modelling in order to better understand the physical processes of the various problems such as fractional diffusion equations (Li (2012)) and fractional diffusion-wave equations (Daftardar-Gejji and Bhalekar (2008), Bhrawy *et al.* (2015), Mainardi (1996)) etc. Such problems are being used widely in many branches of science and engineering. For example, these equations represent propagation of mechanical waves in visco-elastic media (Mainardi (1996), Mainardi *et al.* (2001)) and many more could be found in (Caldernon and Vinagre (2006), Tavazoei and Haeri (2008), Zhu and Fan (2012), Kazem *et al.* (2013), Kumar *et al.* (2014) and Maleki *et al.* (2012)). These applications of the fractional partial differential equations motivate us to consider such equations and propose a numerical method to solve them. Here, we consider the fractional partial differential equation defined as,

$$\frac{\partial^{\alpha} y(x,t)}{\partial x^{\alpha}} + \frac{\partial^{\beta} y(x,t)}{\partial t^{\beta}} = g(x,t),\ 0 < \alpha, \beta \le 2, \tag{5.1.1}$$

with initial conditions $y(x,0) = a(x), D_t^{'} y(x,0) = c(x)$ and boundary conditions $y(0,t) = b(t), D_x^{'} y(0,t) = d(t),$ where $g(x,t)$ is known continuous function and $y(x,t)$ is the unknown function.

Analytical solutions of such equations are rare in many cases hence numerical methods become important to solve them. The most commonly used methods are like Variational Iteration Method (Odibat (2010)), Generalised Differential Transform Method (Odibat and Momani (2008), Momani and Odibat (2007)), Adomian Decomposition Method (El-Kalla (2008), Hosseini (2006)), Finite Difference Method (Zhang (2009)), Wavelet Method (Chen and Wu (2010)) and operational matrix method (Yi *et al.* (2013), Wu (2009), Saadatmandi and Dehghan (2010)).

In Yi *et al.* (2013)), the Eq. (5.1.1) is solved using block pulse functions as basis with constant initial and boundary conditions.

In our chapter we study the above equation for general initial and boundary data. The method we use here is the operational matrix approach taking two dimensional Legendre scaling functions as a basis which is more efficient, stable and convenient.

The present chapter is arranged as follows. In section-5.2, we describe basic preliminaries for fractional calculus and Legendre scaling functions. In section-5.3, we construct operational matrix of fractional integration using Legendre scaling functions as basis. In section-5.4, we describe algorithm for the approximate solution. In section-5.5, we give the error analysis of the proposed method. In section-5.6, we discuss the stability of our method based on maximum absolute error and root mean square error. In section-5.7, we present numerical experiment to show the effectiveness of the proposed method.

5.2 Preliminaries

There are several definitions of fractional order derivatives and integrals. These are not necessarily equivalent. In this chapter, the fractional order differentiations and integrations are Caputo and Riemann-Liouville sense respectively.

Definition 5.2.1. The Riemann-Liouville fractional order integral operator is given as Miller and Ross (1993).

$$I^{\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1} f(t)dt \qquad \alpha > 0, x > 0,$$

$$I^0 f(x) = f(x).$$

For the fractional Riemann-Liouville integration

$$I^{\alpha} x^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} x^{k+\alpha}$$

Definition 5.2.2. The Caputo fractional derivative of order β are defined as

$$D^{\beta} f(x) = I^{m-\beta} D^m f(x) = \frac{1}{\Gamma(m-\beta)}\int_0^x (x-t)^{m-\beta-1} \frac{d^m}{dt^m} f(t)dt,$$

$$m-1 < \beta < m, x > 0.$$

For the Caputo derivative, we have

$D^{\beta} A = 0$ (A is a constant), (Diethelm *et al.* (2005))

$$D^{\beta} x^{k} = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k+1-\beta)} x^{k-\beta}, & \text{for } k \in N_0 \textit{ and } k \geq \lceil \beta \rceil \textit{ or } k \in N \textit{ and } k \in \lfloor \beta \rfloor; \\ 0 \qquad\qquad\qquad , & \textit{for } k \in N_0 \textit{ and } k < \lceil \beta \rceil, \end{cases}$$

Where $\lceil \beta \rceil$ and $\lfloor \beta \rfloor$ are the ceiling and floor functions respectively, while $\mathrm{N} = \{1,2,3,......\}$ and $\mathrm{N}_0 = \{0,1,2,......\}$. The Caputo fractional differentiation is a linear operator similar to the integer order differential operator. The Legendre scaling functions $\{\phi_i(t)\}$ in one dimension are defined by

$$\phi_i(t) = \begin{cases} \sqrt{(2i+1)} P_i(2t-1), & \text{for } 0 \leq t < 1. \\ 0, & \textit{otherwise}, \end{cases}$$

where $P_i(t)$ is Legendre polynomials of order i on the interval $[-1,1]$, given explicitly by the following formula;

$$P_i(t) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{t^k}{(k!)^2} \quad . \tag{5.2.1}$$

Using one dimensional Legendre scaling functions, we construct two dimensional Legendre scaling function ϕ_{i_1,i_2},

$$\phi_{i_1,i_2}(x,t) = \phi_{i_1}(x)\phi_{i_2}(t), \;\; i_1, i_2 \in N_0 .$$

An explicit expression of two dimensional Legendre scaling functions are given as

$$\phi_{i_1,i_2}(x,t) = \begin{cases} \sqrt{(2i_1+1)}\sqrt{(2i_2+1)} P_{i_1}(2x-1) P_{i_2}(2t-1), & \text{for } 0 \leq x < 1, 0 \leq t < 1. \\ 0, & \textit{otherwise.} \end{cases} \tag{5.2.2}$$

From the above formula it is clear that two dimensional Legendre scaling functions are orthogonal;

$$\int_0^1\int_0^1 \phi_{i_1,i_2}(x,t)\phi_{j_1,j_2}(x,t)dxdt = \begin{cases} 1, & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0, & \text{otherwise.} \end{cases} \quad (5.2.3)$$

$\left(\phi_{i_1,i_2}\right)$ forms a complete orthonormal basis (proof is given in section 5.5).

So a function $f(x,t) \in L^2([0,1]\times[0,1])$, can be approximated as

$$f(x,t) \cong \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} c_{i_1,i_2}\phi_{i_1,i_2}(x,t) = C^T\phi(x,t), \quad (5.2.4)$$

where,

$$C = [c_{0,0},....,c_{0,n_2},...,c_{n_1,1},...,c_{n_1,n_2}]^T$$

and $\phi(x,t) = [\phi_{0,0}(x,t),....,\phi_{0,n_2}(x,t),...,\phi_{n_1,1}(x,t),...,\phi_{n_1,n_2}(x,t)]^T.$

The coefficients c_{i_1,i_2} in the Fourier expansions of $f(x,t)$ are given by the formula,

$$c_{i_1,i_2} = \int_0^1\int_0^1 f(x,t)\phi_{i_1,i_2}(x,t)dxdt. \quad (5.2.5)$$

Using matrix notation Eq. (5.2.4) can be written as,

$$f(x,t) \cong \phi^T(x)\ C\ \phi(t), \quad (5.2.6)$$

where,

$\phi(x) = [\phi_0(x),......,\phi_{n1}(x)]^T$, $\phi(t) = [\phi_0(t),......,\phi_{n2}(t)]^T$, and $C = \left(c_{i_1,i_2}\right)_{(n_1+1)\times(n_2+1)}$.

5.3 Operational Matrix

Theorem 5.3.1. Let $\phi(x) = [\phi_0(x), \phi_1(x), ..., \phi_n(x)]^T$, be Legendre scaling vector and consider $\alpha > 0$, then

$$I^{\alpha}\phi_i(x) = I^{(\alpha)}\phi(x), \tag{5.3.1}$$

where $I^{(\alpha)}$ is $(n+1)\times(n+1)$ operational matrix of fractional integral of order α and its entries are given as

$$\omega(i,j) = (2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(\alpha+k+l+1)\Gamma(\alpha+k+1)},$$

where $0 \le i, j \le n.$

Proof. Using the Legendre scaling function of degree i, we get

$$\begin{aligned} I^{\alpha}\phi_i(x) &= (2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{1}{(k!)^2}I^{\alpha}x^k \\ &= (2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!(k)!\Gamma(\alpha+k+1)}x^{\alpha+k}, \end{aligned}$$

using Legendre scaling function approximation for $x^{\alpha+k}$, we have

$$x^{\alpha+k} = \sum_{j=0}^{n}c_j\phi_j(x), \text{where } c_j = (2j+1)^{1/2}\sum_{l=0}^{j}(-1)^{l+j}\frac{(j+l)!}{(j-l)!}\frac{1}{(l!)^2}\frac{1}{(\alpha+k+l+1)}, \tag{5.3.2}$$

then,
$$I^{\alpha}\phi_i(x) = \sum_{j=0}^{n}\omega(i,j)\phi_j(x), \tag{5.3.3}$$

where,

$$\omega(i,j) = (2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(\alpha+k+l+1)\Gamma(\alpha+k+1)}. \tag{5.3.4}$$

Hence,

$$I^{\alpha}\phi_i(x) = [\omega(i,0), \omega(i,1), \omega(i,2), ..., \omega(i,n)]\Phi(x). \tag{5.3.5}$$

5.4 Method of Solution

In this section we take $n_1 = n_2 = n,$ for any approximation.

Using (5.2.6), let solution of (5.1.1) be approximated as,

$$y(x,t) \cong \phi^T(x)C\phi(t). \tag{5.4.1}$$

Taking fractional integral of order β and α in Eq. (5.4.1), with respect to t and x respectively, we get

$$I_t^\beta y(x,t) = I_t^\beta(\phi^T(x)C\phi(t)) = \phi^T(x)CI_t^\beta\phi(t) = \phi^T(x)CI^{(\beta)}\phi(t), \tag{5.4.2}$$

$$I_x^\alpha y(x,t) = I_x^\alpha(\phi^T(x)C\phi(t)) = I_x^\alpha(\phi^T(x))C\phi(t) = \phi^T(x)(I^{(\alpha)})^T C\phi(t). \tag{5.4.3}$$

Where $I^{(\beta)}$ and $I^{(\alpha)}$ are $(n+1)\times(n+1)$ operational matrices of fractional integration of order β and α respectively. Similarly approximating non homogeneous term,

$$g(x,t) \cong \phi^T(x)G\phi(t), \tag{5.4.4}$$

where,

$$G = (g_{ij})_{0\le i,j\le n}. \tag{5.4.5}$$

Given partial differential equation is,

$$\frac{\partial^\alpha y(x,t)}{\partial x^\alpha} + \frac{\partial^\beta y(x,t)}{\partial t^\beta} = g(x,t), \tag{5.4.6}$$

with initial conditions $y(x,0) = a(x), D_t^{'} y(x,0) = c(x)$ and boundary conditions $y(0,t) = b(t), D_x^{'} y(0,t) = d(t).$

Taking fractional integral of order α with respect to x in Eq. (5.4.6), we get

$$y(x,t)-y(0,t)-xD_x^{'}y(0,t)+I_x^{\alpha}(D_t^{\beta})y(x,t)=I_x^{\alpha}g(x,t). \tag{5.4.7}$$

Taking fractional integral of order β with respect to t in Eq. (5.4.7), we get

$$I_t^{\beta}y(x,t)-I_t^{\beta}(y(0,t)+xD_x^{'}y(0,t))+I_t^{\beta}I_x^{\alpha}(D_t^{\beta})y(x,t)=I_t^{\beta}I_x^{\alpha}g(x,t). \tag{5.4.8}$$

Eq. (5.4.8), can be written as,

$$I_t^{\beta}y(x,t)-I_t^{\beta}(y(0,t)+xD_x^{'}y(0,t))+I_x^{\alpha}y(x,t)-I_x^{\alpha}(y(x,0)+tD_t^{'}y(x,0))=I_t^{\beta}I_x^{\alpha}g(x,t). \tag{5.4.9}$$

Let

$$I_t^{\beta}(y(0,t)+xD_x^{'}y(0,t)+I_x^{\alpha}(y(x,0)+tD_t^{'}y(x,0))=f(x,t), \tag{5.4.10}$$

using (5.2.6), approximating $f(x,t)$,

$$f(x,t)=\phi^T(x)A\phi(t). \tag{5.4.11}$$

Now using Eqs. (5.4.2), (5.4.3), (5.4.4), (5.4.10) and (5.4.11) in (5.4.9) we get,

$$\phi^T(x)CI^{(\beta)}\phi(t)+\phi^T(x)(I^{(\alpha)})^T C\phi(t)=\phi^T(x)(I^{(\alpha)})^T GI^{(\beta)}\phi(t)+\phi^T(x)A\phi(t), \tag{5.4.12}$$

where matrices A and G are known. Eq. (5.4.12) can be written as

$$CI^{(\beta)}+(I^{(\alpha)})^T C=(I^{(\alpha)})^T GI^{(\beta)}+A. \tag{5.4.13}$$

Eq. (5.4.13) gives a system of linear algebraic equations which can be solved easily using Sylvester's approach.

5.5 Error Analysis

Lemma 5.5.1. $\left(\phi_{i_1,i_2}\right)$ form a complete orthonormal basis for the Hilbert space $L^2([0,1]\times[0,1])$.

Proof.

Space of continuous functions on $C([0,1]\times[0,1])$ is a dense subspace of $L^2([0,1]\times[0,1])$. Now consider the following sub algebra A of $C([0,1]\times[0,1])$.

$$A=\left\{\sum_{i_1=1}^{n}\sum_{i_2=1}^{m}c_{i_1i_2}\phi_{i_1i_2}\,\middle|\,c_{i_1i_2}\in R;\quad n,m\in N_0\right\},\tag{5.5.1}$$

A contains constants and separates points. So by Weirstrass theorem A is a dense in $C([0,1]\times[0,1])$.

Using orthogonality of one dimensional Legendre scaling functions, it is easy to see (ϕ_{i_1,i_2}) forms an orthogonal set. Now to show this is complete, let $f(x,t)$ be any function in $L^2([0,1]\times[0,1])$

$$\int_0^1\int_0^1 f(x,t)\phi_{ij}(x,t)dxdt=0 \qquad \forall\, i,j,\tag{5.5.2}$$

by linearity,

$$\int_0^1\int_0^1 f(x,t)\phi(x,t)dxdt=0, \qquad \forall\,\phi\in A,\tag{5.5.3}$$

since A is dense in $C([0,1]\times[0,1])$, we get

$$\int_0^1\int_0^1 f(x,t)\psi(x,t)dxdt=0, \qquad \forall\,\psi\in C[0,1]\times[0,1],\tag{5.5.4}$$

again since $C[0,1]\times[0,1]$ is dense in $L^2([0,1]\times[0,1])$

$$\int_0^1\int_0^1 f(x,t)\psi(x,t)dxdt=0, \qquad \forall\,\psi\in L^2[0,1]\times[0,1],\tag{5.5.5}$$

chose $\psi=f(x,t)$, so

$$\int_0^1\int_0^1 f^2(x,t)dxdt = 0 \Leftrightarrow f(x,t) = 0 \ a.e. \tag{5.5.6}$$

So (ϕ_{i_1,i_2}) form a complete orthonormal basis for the Hilbert space $L^2([0,1]\times[0,1])$.

Theorem 5.5.1. Let $f(x,t) \in L^2([0,1]\times[0,1])$, and $f_n(x,t)$ be its approximation obtained by using $(n+1)^2$, 2-dimensional Legendre scaling vectors. Assuming $\left|\frac{\partial^4 f(x,t)}{\partial x^2 \partial t^2}\right| \le K$, we have the fallowing upper bound for error

$$\|f(x,t) - f_n(x,t)\|_{L^2} < \left(\frac{K}{256}\right)\left(F_3(-\frac{1}{2}+n)\right) \tag{5.5.7}$$

where, $\|f(x,t)\|_{L^2} = \left(\int_0^1\int_0^1 |f(x,t)|^2 \, dx\, dt\right)^{\frac{1}{2}}$.

Where $F_n(z)$ is the Polygamma function defined by,

$$F_n(z) = (-1)^{n+1}\underline{\lfloor n}\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}. \tag{5.5.8}$$

Proof.

$$\text{Let } f(x,t) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} c_{i_1,i_2}\phi_{i_1,i_2}(x,t).$$

Truncating the above equation upto the level n, we get

$$f_n(x,t) = \sum_{i_1=0}^{n}\sum_{i_2=0}^{n} c_{i_1,i_2}\phi_{i_1,i_2}(x,t),$$

thus,

$$f(x,t) - f_n(x,t) = \sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} c_{i_1,i_2} \phi_{i_1,i_2}(x,t), \tag{5.5.9}$$

$$\|f(x,t) - f_n(x,t)\|^2_{L^2} = \int_0^1 \int_0^1 \left(f(x,t) - f_n(x,t) \right)^2 dx\,dt$$

$$= \int_0^1 \int_0^1 \left(\sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} c_{i_1,i_2} \phi_{i_1,i_2}(x,t) \right)^2 dx\,dt$$

$$= \sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} c^2_{i_1,i_2}, \tag{5.5.10}$$

where, $c_{i_1,i_2} = \int_0^1 \int_0^1 f(x,t)\phi_{i_1,i_2}(x,t)dx\,dt = \int_0^1 \int_0^1 f(x,t)\phi_{i_1}(x)\phi_{i_2}(t)dx\,dt$,

now using similar process as in Heydari et al. (2013), we get,

$$\left| c_{i_1,i_2} \right|^2 < \frac{9K^2}{64(2i_1 - 3)^4 (2i_2 - 3)^4}. \tag{5.5.11}$$

Using (5.5.11), we can write,

$$\sum_{i_1=n}^{\infty} \sum_{i_2=n}^{\infty} c^2_{i_1,i_2} < \sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} \frac{9K^2}{64(2i_1 - 3)^4 (2i_2 - 3)^4}.$$

$$= \frac{9K^2}{64} \left(\frac{1}{9216} \right) \left(F_3(-\frac{1}{2} + n) \right)^2$$

$$= \left(\frac{K^2}{65536} \right) \left(F_3(-\frac{1}{2} + n) \right)^2,$$

hence, from (5.5.10),

$$\|f(x,t) - f_n(x,t)\|_{L^2} < \left(\frac{K}{256} \right) \left(F_3(-\frac{1}{2} + n) \right). \tag{5.5.12}$$

5.6 Stability Analysis

The accuracy of proposed method is demonstrated by calculating absolute error, average deviation σ also known as root mean square error (RMS). They are calculated using the following equations

$$\Delta y(x_i,t_j)=\left|y_e(x_i,t_j)-y_a(x_i,t_j)\right|, \tag{5.6.1}$$

and

$$\sigma_{(N+1)^2}=\left\{\frac{1}{(N+1)^2}\sum_{i=0}^{N}\sum_{j=0}^{N}[y_e(x_i,t_j)-y_a(x_i,t_j)]^2\right\}^{1/2}, \tag{5.6.2}$$

where $y_e(x_i,t_j)$ is the exact value of output function at point (x_i,t_j) and $y_a(x_i,t_j)$ is the approximate value of output function at the same point.

In all the examples, the input function with and without noise are denoted by $g^{\delta}(x,t)$ and $g(x,t)$ respectively, where $g^{\delta}(x,t)$ is obtained by adding a noise δ to $g(x,t)$ such that $g^{\delta}(x_i,t_j)=g(x_i,t_j)+\delta\theta_{ij}$, where $x_i=ih, i=1,2,.....,N,\ Nh=1;$ $t_j=jk,\ \ j=1,2,.....,N,\ Nk=1$ and θ_{ij} is the uniform random variable with values in [-1, 1] such that

$$\max_{\substack{1\le i\le N\\ 1\le j\le N}}\left|g^{\delta}(x_i,t_j)-g(x_i,t_j)\right|\le\delta. \tag{5.6.3}$$

In section 5.7, five examples are solved with and without noise to illustrate the stability of the proposed method. In all the five examples, we add the noise $\delta=\sigma_{(N+1)^2}$, for two different values of $N=10,20.$ for different values of N we calculate maximum absolute error and root mean square errors denoted by E_1 and E_2 respectively for input functions without noise term. Similarly, these respective errors are denoted by E_1^* and E_2^* for input function with noise respectively. In Table 5.3., we have listed the different values of E_1, E_2, E_1^* and E_2^* for $N = 10, 20.$ from the table it is clear that the there is a very small change in errors when we add noise term in input function showing the stability of our method.

5.7 Numerical Results

In this section we discuss the implementation of our proposed numerical methods based on fractional operational matrix of integration for Legendre scaling function. We show the accuracy of our method by the graph of absolute errors in the various examples. In Tables 5.1. and 5.2. we have compared numerical solutions obtained by our method and method given in (Yi *et al* (2013)), it is observed our method gives better approximation. It is also observed that desired accuracy is obtained even for $n = 4$.

Example 5.7.1. Consider the following partial differential equation,

$$\frac{\partial^{1/4} y(x,t)}{\partial x^{1/4}} + \frac{\partial^{1/5} y(x,t)}{\partial t^{1/5}} = g(x,t), \qquad 0 \le x, t \le 1, \tag{5.7.1}$$

with initial-boundary conditions as $y(x,0) = x^2$, $y(0,t) = t$ respectively,

where $g(x,t) = \dfrac{\Gamma 3}{\Gamma \frac{11}{4}} x^{7/4} + \dfrac{\Gamma 2}{\Gamma \frac{9}{5}} t^{4/5}$ and the exact solution is $y(x,t) = x^2 + t$.

Figures 5.1.1, 5.1.2 and 5.1.3 represent absolute error of Eq. (5.7.1) for n=4, 7 and 10 respectively.

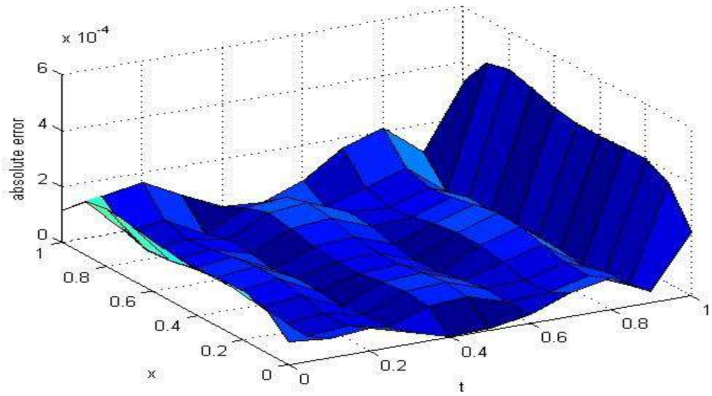

Figure 5.1.1 Absolute error for n=4.

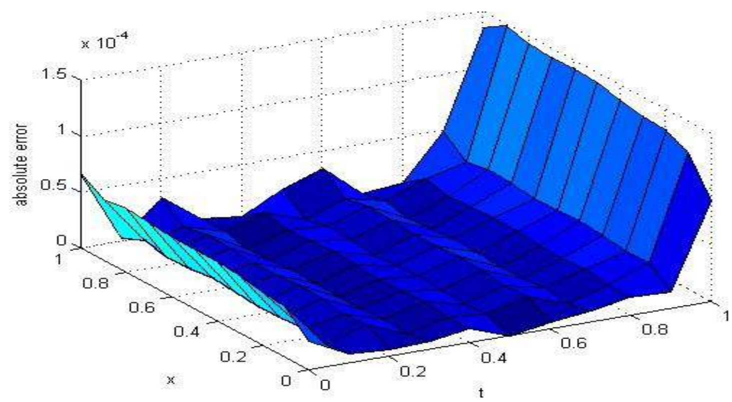

Figure 5.1.2 Absolute error for n=7.

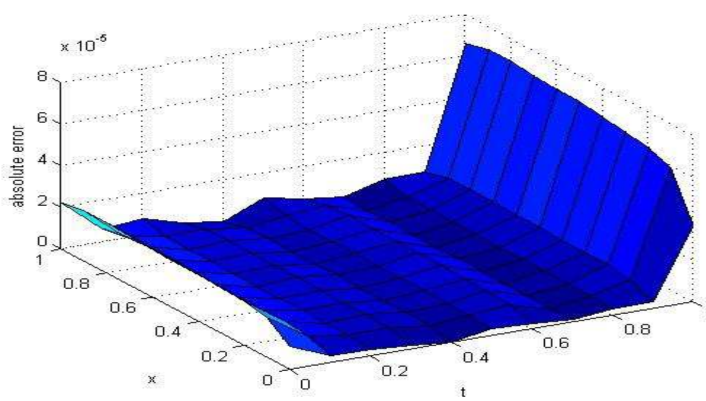

Figure 5.1.3 Absolute error for n=10.

Example 5.7.2. Consider the following partial differential equation,

$$\frac{\partial^{1/8} y(x,t)}{\partial x^{1/8}} + \frac{\partial^{1/3} y(x,t)}{\partial t^{1/3}} = g(x,t), \qquad 0 \le x,t \le 1, \qquad (5.7.2)$$

with initial-boundary conditions as $y(x,0) = x,\ y(0,t) = 2t$ respectively,

where $g(x,t) = \frac{\Gamma 2}{\Gamma \frac{15}{8}} x^{7/8} + \frac{2\Gamma 2}{\Gamma \frac{5}{3}} t^{2/3}$ and the exact solution $y(x,t) = x + 2t$.

Figures 5.2.1, 5.2.2 and 5.2.3 represent absolute error of Eq. (5.7.2) for n=4, 7 and 10 respectively.

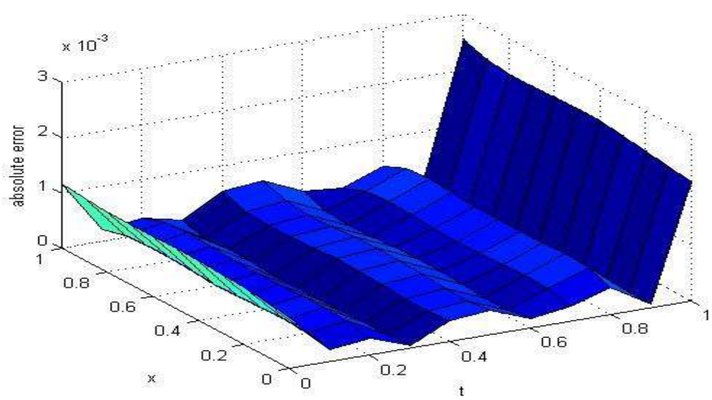

Figure 5.2.1 Absolute error for n=4.

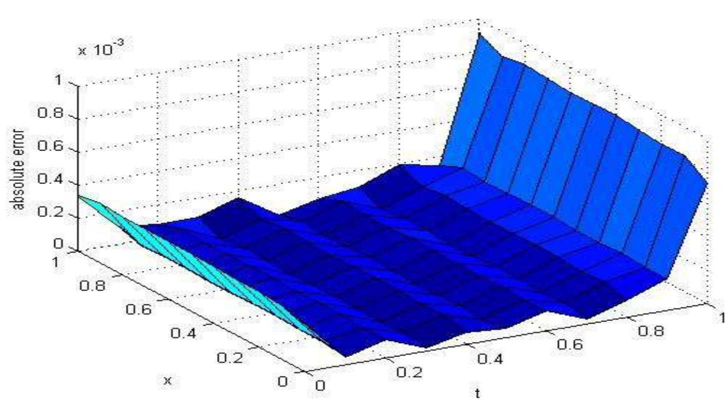

Figure 5.2.2 Absolute error for n=7.

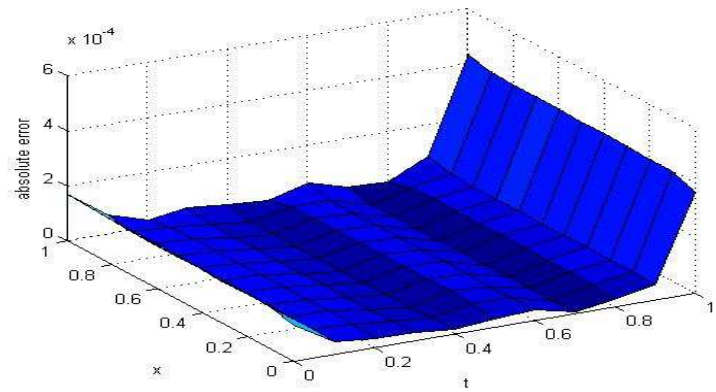

Figure 5.2.3 Absolute error for n=10.

Example 5.7.3. Consider the following non homogeneous partial differential equation (Yi *et al.* (2013)),

$$\frac{\partial y(x,t)}{\partial x}+\frac{\partial y(x,t)}{\partial t}=\sin(x+t),\ 0\le x,t\le 1, \tag{5.7.3}$$

With initial-boundary conditions as $y(x,0)=y(0,t)=0$ respectively, and the exact solution $y(x,t)=\sin x\sin t$. Figures 5.3.1, 5.3.2 and 5.3.3 represent absolute error of Eq. (5.7.3) for n=4, 7 and 10 respectively.

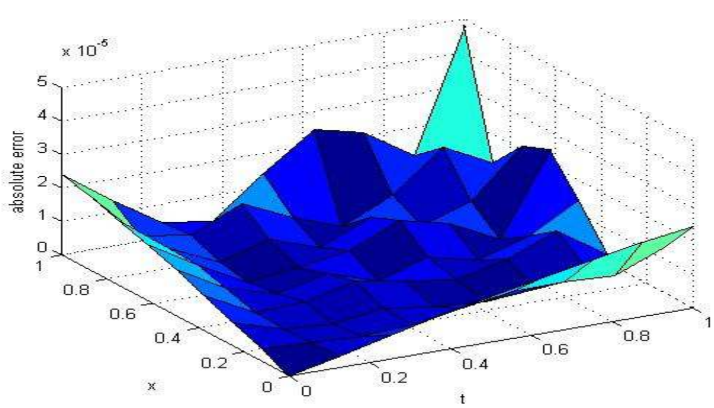

Figure 5.3.1 Absolute error for n=4.

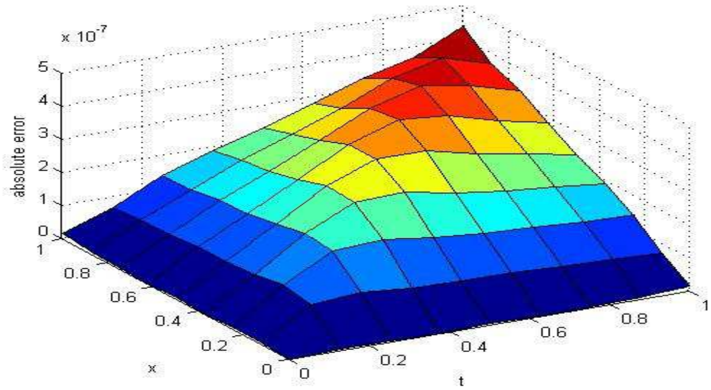

Figure 5.3.2 Absolute error for n=7.

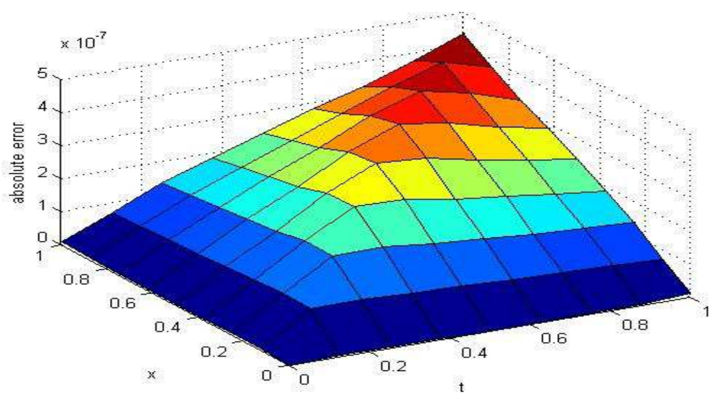

Figure 5.3.3 Absolute error for n=10.

In table 5.1., given below, we compare our result with the method of (Yi *et al.* (2013)).

Table 5.1.

Comparison of exact solution with numerical values obtained from our method and method in (Yi *et al.* (2013))

(x,t)	Exact solution	Present Method		Method in Yi *et al.* (2013)
		n=4	n=8	n=8
(0,0)	0	0	0	0
(1/8, 1/8)	0.0155	0.0155	0.0155	0.0155
(2/8, 2/8)	0.0612	0.0612	0.0612	0.0611
(3/8, 3/8)	0.1342	0.1342	0.1342	0.1340
(4/8, 4/8)	0.2298	0.2298	0.2298	0.2295
(5/8, 5/8)	0.3423	0.3423	0.3423	0.3419
(6/8, 6/8)	0.4646	0.4646	0.4646	0.4640
(7/8, 7/8)	0.5891	0.5891	0.5891	0.5884

Example 5.7.4. Consider the following non homogeneous fractional partial differential equation (Yi *et al.* (2013)),

$$\frac{\partial^{1/4} y(x,t)}{\partial x^{1/4}} + \frac{\partial^{1/4} y(x,t)}{\partial t^{1/4}} = g(x,t), \qquad 0 \le x,t \le 1, \tag{5.7.4}$$

with initial-boundary conditions as

$$y(x,0) = y(0,t) = 0.$$

Where, $g(x,t) = \dfrac{4(x^{3/4}t + xt^{3/4})}{3\left|\frac{3}{4}\right.}$ and the exact solution $y(x,t) = xt$.

Figures 5.4.1, 5.4.2 and 5.4.3 represent absolute error of Eq. (5.7.4) for n=4, 7 and 10 respectively.

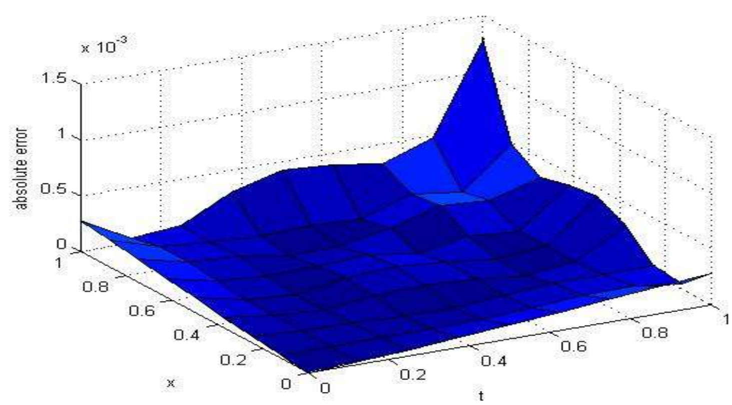

Figure 5.4.1 Absolute error for n=4.

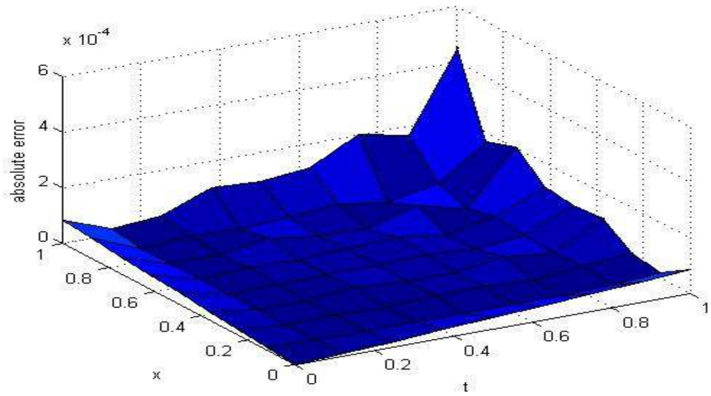

Figure 5.4.2 Absolute error for n=7.

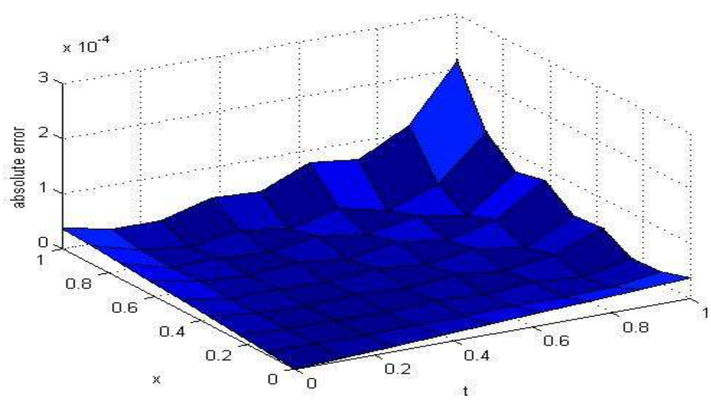

Figure 5.4.3 Absolute error for n=10.

In table 5.2., given below, we compare our result with that of (Yi *et al.* (2013)).

Table 5.2.

Comparison of exact solution with numerical values obtained from our method and method in Yi *et al.* (2013).

(x,t)	Exact solution	Present Method		Method in Yi *et al.* (2013)
		n=4	n=8	n=8
(0,0)	0	0	0	0
(1/8, 1/8)	0.0156	0.0156	0.0156	0.0150
(2/8, 2/8)	0.0625	0.0625	0.0625	0.0620
(3/8, 3/8)	0.1406	0.1407	0.1406	0.1401
(4/8, 4/8)	0.2500	0.2501	0.2500	0.2494
(5/8, 5/8)	0.3906	0.3906	0.3906	0.3900
(6/8, 6/8)	0.5625	0.5623	0.5625	0.5619
(7/8, 7/8)	0.7656	0.7655	0.7656	0.7650

Example 5.7.5. Consider the following non homogeneous fractional partial differential equation,

$$\frac{\partial^{1.5} y(x,t)}{\partial x^{1.5}} + \frac{\partial^{1.2} y(x,t)}{\partial t^{1.2}} = g(x,t), \qquad 0 \le x,t \le 1, \qquad (5.7.5)$$

with initial-boundary conditions as

$y(x,0) = x^2$, $D_t' y(x,0) = 0$ and $y(0,t) = t^2$, $D_x' y(0,t) = 0$ respectively.

Where, $g(x,t) = \frac{\Gamma 3}{\Gamma 1.5} x^{1/2} + \frac{\Gamma 3}{\Gamma 1.8} t^{4/5}$ and the exact solution $y(x,t) = x^2 + t^2$.

Figures 5.5.1, 5.5.2 and 5.5.3 represent absolute error of Eq. (5.7.5) for n=4, 7 and 10 respectively.

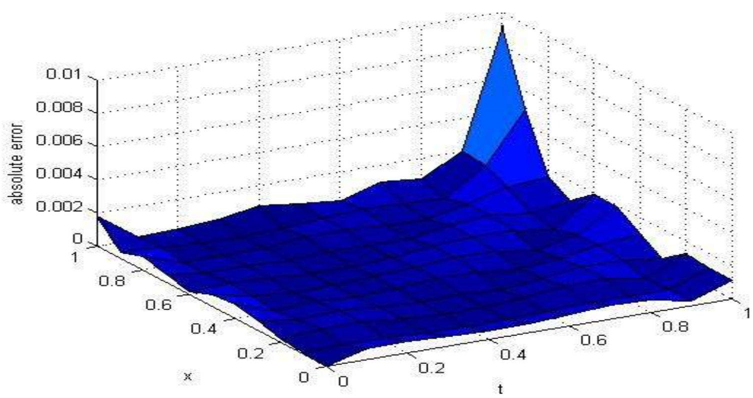

Figure 5.5.1 Absolute error for n=4.

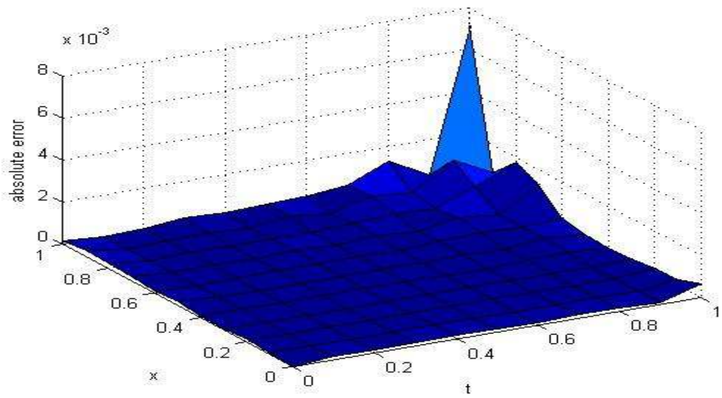

Figure 5.5.2 Absolute error for n=7.

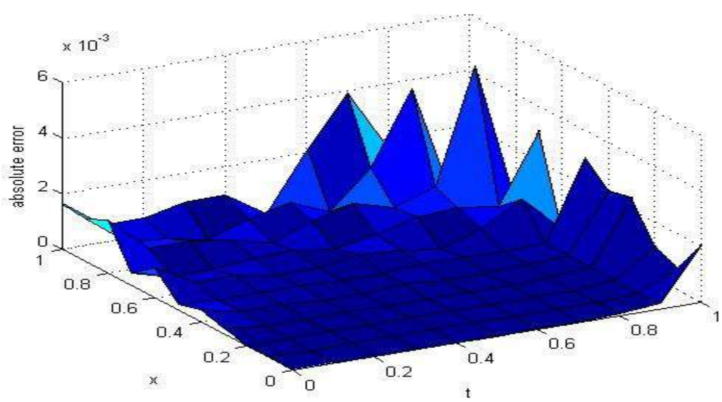

Figure 5.5.3 Absolute error for n=10.

In table 5.3., given below, we list the errors E_1, E_2, E_1^* and E_2^* to show the stability of our method.

Table 5.3.

Example No.	N	Errors without noise		Errors with noise $\delta = \sigma_{(n+1)^2}$	
		E_1	E_2	E_1^*	E_2^*
1	10	9.9693×10^{-6}	2.0190×10^{-7}	1.0058×10^{-5}	2.0284×10^{-7}
	20	1.4386×10^{-5}	7.3441×10^{-8}	1.4352×10^{-5}	7.3495×10^{-8}
2	10	6.3873×10^{-5}	1.2155×10^{-6}	6.4411×10^{-5}	1.2174×10^{-6}
	20	1.1114×10^{-4}	5.0195×10^{-7}	1.1091×10^{-4}	5.0176×10^{-7}
3	10	4.2034×10^{-7}	9.4835×10^{-9}	4.2710×10^{-7}	9.6074×10^{-9}
	20	4.2181×10^{-7}	3.4345×10^{-9}	4.2441×10^{-7}	3.4510×10^{-9}
4	10	2.6613×10^{-5}	3.4135×10^{-7}	4.9749×10^{-5}	1.7266×10^{-7}
	20	4.9905×10^{-5}	1.7277×10^{-7}	4.9826×10^{-5}	1.7271×10^{-7}
5	10	7.1002×10^{-3}	1.6594×10^{-5}	7.1735×10^{-3}	1.7150×10^{-5}
	20	7.1002×10^{-3}	8.9995×10^{-6}	7.1735×10^{-3}	9.1354×10^{-6}

The computational order for the numerical results is calculated in the paper (Dehghan *et al.* (2015)). The computational order for the numerical results are given as

Order = $\log_2\left[\frac{E_n}{E_{2n}}\right]$ where E_n is maximum absolute error $(\max_{1\le i,j\le N} E(x_i,t_j))$ for approximation having n number of basis elements. In table 5.4., we list the computational order for the numerical results.

Table 5.4.

Computational order obtained for all examples.

Example No.	n	E_n	Order
1	2	1.5789×10^{-3}	-
	4	4.6130×10^{-4}	1.7751
	8	1.1369×10^{-4}	2.0206
2	2	7.2824×10^{-3}	-
	4	2.5544×10^{-3}	1.5114
	8	6.9316×10^{-4}	1.8817
3	2	1.2215×10^{-2}	-
	4	4.7945×10^{-5}	7.9931
	8	4.3849×10^{-7}	6.7727
4	2	3.8175×10^{-3}	-
	4	1.2933×10^{-3}	1.5616
	8	3.4485×10^{-4}	1.9070
5	2	9.4023×10^{-3}	-
	4	9.2421×10^{-3}	0.0248
	8	2.1573×10^{-3}	2.0989

5.8 Conclusions

From figures 4.1.1-4.5.3, it is observed that absolute error decreases with the increasing n. Similarly as we increase the dimension of basis function, we obtain more accurate numerical solution. Operational matrix of Haar wavelet in (Wu *et al*. (2009)) is used to solve fractional partial differential equation of the same type but the procedure of construction of such operational matrix is very complicated. Our method based on two dimensional Legendre scaling functions is easy in comparison to that of (Yi *et al*. (2013)). Further our method is more accurate than that in (Yi *et al*. (2013)). The stability with respect to the data is restored and accuracy is good even for high noise levels in the data. An error analysis and stability analysis is also given.