Chapter 6

The Continuous Bessel Wavelet Transform of Distributions in β'_μ-Space

6.1 Introduction

The conventional Hankel transformation to distributions belonging to $H'_\mu-$space was extended by Zemanian [63] and discussed many properties of $H'_\mu-$space. Taking the Zemanian theory of conventional Hankel transform, various properties were found by Dube [18] on different functional spaces and their duals. With help of concepts of Hankel convolution and Hankel transform, many problems were solved by Betancor et al. [3, 4, 6–9, 11] and Arteaga and Marrero [1] on functional spaces and their duals. In the $n-$dimensional Hankel transformation case for $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, important observations were made by Molina [34, 35] on $n-$dimensional Zemanian space H_μ and its dual H'_μ-space. Using the results of [34], recently Liouville theorem for some generalized operators was obtained by Galli et al. [20] by exploiting the theory of $n-$dimensional Hankel transform.

The theory of Hankel transform and its important results were given by Haimo [21], Hirschman [24] and Cholewinski [15]. Exploiting the results of [15, 21, 24], the theory of the Bessel wavelet transform was discussed by Pathak and Dixit [42] and found the Parseval formula, other properties and applications. Later on, by considering the Zemanian theory of Hankel transform, the continuous Bessel wavelet transform and its various properties were found by Upadhyay et al. [56]. Many results of the continuous Bessel wavelet transform are obtained by Upadhyay and Singh [51–54] by using the Zemanian Hankel transform tool.

The characterizations of the inversion formula of the continuous Bessel wavelet transform in $H'_\mu(\mathbb{R}^+)$ were discussed by author in the previous chapter and found applications in Heat equation and Calderón reproducing formula in $H'_\mu(\mathbb{R}^+)$. Our main objective in the present chapter is to find the inversion formula of the continuous Bessel wavelet transform on β'_μ-space because the inversion formula and its properties in β'_μ are more general than the characterizations of the inversion formula of the continuous Bessel wavelet transform of distributions in $H'_\mu(\mathbb{R}^+)$.

Contents of the manuscript are organized in the following way:
Section 6.1 is introductory. It gives motivation and a brief history of the continuous Bessel wavelet transform and its inversion formula. In Section 6.2, various definitions and useful results which are related to other sections are provided. The continuous Bessel wavelet transform and its properties in β'_μ-space are discussed. In Section 6.3, useful lemmas and theorems related to the inversion formula of the continuous Bessel wavelet transform are given, and the main theorem of inversion formula of the continuous Bessel wavelet transform of distributions in $\beta'_{\mu,F}$-space is investigated.

6.2 The continuous Bessel wavelet transform and its properties

In this section, the various properties of the continuous Bessel wavelet transform on the Zemanian spaces β_μ and β'_μ are obtained by exploiting the theory of Hankel transform.

Lemma 6.2.1. *Let $\psi \in \beta_\mu$ with $\psi(x) = 0$ for $x \geq c$, $c > 0$ and $\mu > -\frac{1}{2}$. Then, ψ is a basic Bessel wavelet if and only if $\int_0^\infty x^{\mu+\frac{1}{2}}\psi(x)dx = 0$.*

Proof. Let $\psi \in \beta_\mu$ be the basic Bessel wavelet which satisfies the admissibility condition (1.5.1). Since $\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega)$ continuous on $\mathbb{R}^+$, therefore

$$\begin{aligned}\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega) &= \omega^{-\mu-\frac{1}{2}}\int_0^\infty (x\omega)^{\frac{1}{2}}J_\mu(x\omega)\psi(x)dx\\ &= \int_0^\infty (x\omega)^{-\mu}J_\mu(x\omega)x^{\mu+\frac{1}{2}}\psi(x)dx.\end{aligned}$$

If we take $\omega \to 0$ then $(x\omega)^{-\mu}J_\mu(x\omega) \to \frac{1}{2^\mu\Gamma(\mu+1)}$ and from (1.5.1) it is clear that $\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega)$ must go to zero. Then, we have

$$\frac{1}{2^\mu\Gamma(\mu+1)}\int_0^\infty x^{\mu+\frac{1}{2}}\psi(x)dx = 0.$$

Hence

$$\int_0^\infty x^{\mu+\frac{1}{2}}\psi(x)dx = 0. \tag{6.2.1}$$

Conversely, for $\psi \in \beta_{\mu,c} \subset L^2(\mathbb{R}^+)$, we can write

$$\int_0^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega = \int_0^1 \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega + \int_1^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega. \tag{6.2.2}$$

The second integral of the R.H.S. of (6.2.2) is estimated as

$$\begin{aligned}\int_1^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega &\leq \int_1^\infty |(h_\mu\psi)(\omega)|^2 d\omega\\ &\leq \int_0^\infty |(h_\mu\psi)(\omega)|^2 d\omega.\end{aligned}$$

Using the Parseval formula of the Hankel transform, we get

$$\int_1^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega \leq \int_0^\infty |\psi(x)|^2 dx.$$

Therefore,

$$\int_1^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega \leq \|\psi(x)\|_2^2. \tag{6.2.3}$$

Next, we find the bound of the first integral of the R.H.S. of (6.2.2). Since $\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega) \to 0$ as $\omega \to 0$, then

$$\begin{aligned}\left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega)\right| &= \left|\lim_{\epsilon\to 0}\int_\epsilon^\omega \frac{\partial}{\partial y}y^{-\mu-\frac{1}{2}}(h_\mu\psi)(y)dy\right|\\ &\leq \lim_{\epsilon\to 0}\int_\epsilon^\omega \left|y^{-\mu-\frac{1}{2}}y^{\mu+\frac{1}{2}}\frac{\partial}{\partial y}y^{-\mu-\frac{1}{2}}(h_\mu\psi)(y)\right|dy\\ &\leq \lim_{\epsilon\to 0}\int_\epsilon^\omega \left|yy^{-\mu-1-\frac{1}{2}}N_\mu(h_\mu\psi)(y)\right|dy\\ &\leq \sup_{y\in\mathbb{R}^+}\left|y^{-\mu-1-\frac{1}{2}}N_\mu(h_\mu\psi)(y)\right|\lim_{\epsilon\to 0}\int_\epsilon^\omega ydy\\ &\leq \frac{\omega^2}{2}\sup_{\eta\in\mathbb{C}}\left|e^{-c|\mathrm{Im}\eta|}\eta^{-\mu-1-\frac{1}{2}}N_\mu(h_\mu\psi)(\eta)\right|\\ &\leq \frac{\omega^2}{2}\alpha_{c,0}^{\mu+1}(N_\mu(h_\mu\psi)).\end{aligned}$$

So that

$$\left|\omega^{-\mu-1}(h_\mu\psi)(\omega)\right| \leq \frac{\omega^{\frac{3}{2}}}{2}\alpha_{c,0}^{\mu+1}(N_\mu(h_\mu\psi)).$$

Therefore,

$$\int_0^1 \left|\omega^{-\mu-1}(h_\mu\psi)(\omega)\right|^2 d\omega \le \frac{1}{4}\left\{\alpha_{c,0}^{\mu+1}(N_\mu(h_\mu\psi))\right\}^2 \int_0^1 \omega^3 \, d\omega.$$

This implies that

$$\int_0^1 \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}} d\omega \le \frac{1}{16}\left\{\alpha_{c,0}^{\mu+1}(N_\mu(h_\mu\psi))\right\}^2, \tag{6.2.4}$$

for $\psi \in \beta_{\mu,c}$ and $N_\mu(h_\mu\psi) \in \mathscr{Y}_{\mu+1,c}$.

From (6.2.3) and (6.2.4), the bound of (6.2.2) is

$$\int_0^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}} d\omega < \infty.$$

□

Definition 6.2.2. Let $\psi \in \beta_\mu$ be the Bessel wavelet and $f \in \beta'_\mu$, then the Bessel wavelet transform of f is defined by

$$(B_\psi f)(b,a) = (f\#\psi_a)(b) = \left\langle f(t), a^{\mu-\frac{1}{2}}\psi\left(\frac{t}{a},\frac{b}{a}\right)\right\rangle, \tag{6.2.5}$$

where $a > 0, b > 0$ and $\mu > -\frac{1}{2}$.

Remark 6.2.3. From [61, p. 681], β_μ is a subspace of H_μ and the topology of β_μ is stronger than the topology induced on it by H_μ. Therefore, the restrictions of the elements of H'_μ to β_μ comprise a subspace of β'_μ. So we observed that $x^{\mu+\frac{1}{2}}$ belongs to H'_μ, then this implies that $x^{\mu+\frac{1}{2}} \in \beta'_\mu$.

Theorem 6.2.4. *Let $\psi \in \beta_\mu$ be the Bessel wavelet, then the continuous Bessel wavelet transform of distributions $Cx^{\mu+\frac{1}{2}}$ is zero for $C > 0$ which is a constant.*

Proof. From Definition 6.2.2, the continuous Bessel wavelet transform of the distribution $C.x^{\mu+\frac{1}{2}}$ is

$$\begin{aligned}\Big(B_\psi Cx^{\mu+\frac{1}{2}}\Big)(b,a) =&\Big\langle Cx^{\mu+\frac{1}{2}}, a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\rangle\\ =&\int_0^\infty Cx^{\mu+\frac{1}{2}}a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)dx\\ =&\int_0^\infty Cx^{\mu+\frac{1}{2}}a^{\mu-\frac{1}{2}}\int_0^\infty \psi(z)D_\mu\Big(\frac{x}{a},\frac{b}{a},z\Big)dzdx.\end{aligned}$$

By Fubuni's theorem, we have

$$\Big(B_\psi Cx^{\mu+\frac{1}{2}}\Big)(b,a) = Ca^{\mu-\frac{1}{2}}\int_0^\infty \psi(z)\int_0^\infty x^{\mu+\frac{1}{2}}D_\mu\Big(\frac{x}{a},\frac{b}{a},z\Big)dxdz.$$

Taking $\frac{x}{a}=u$, then and using (1.4.8), we find

$$\begin{aligned}\Big(B_\psi Cx^{\mu+\frac{1}{2}}\Big)(b,a) =&Ca^{\mu-\frac{1}{2}}\int_0^\infty \psi(z)\int_0^\infty (au)^{\mu+\frac{1}{2}}D_\mu\Big(u,\frac{b}{a},z\Big)adudz\\ =&\int_0^\infty Ca^{2\mu+1}\psi(z)\frac{\Big(\frac{bz}{a}\Big)^{\mu+\frac{1}{2}}}{2^\mu\Gamma(\mu+1)}dz\\ =&C\frac{(ab)^{\mu+\frac{1}{2}}}{2^\mu\Gamma(\mu+1)}\int_0^\infty z^{\mu+\frac{1}{2}}\psi(z)dz\\ =&0.\end{aligned}$$

That is

$$\Big(B_\psi Cx^{\mu+\frac{1}{2}}\Big)(b,a) = 0.$$

□

Lemma 6.2.5. *Let $\mu > -\frac{1}{2}$, $a>0$, $b>0$ and $c>0$. Let $\psi\in\beta_{\mu,\frac{c}{a}}$, then $\psi(\frac{x}{a},\frac{b}{a})$ belongs to $\beta_{\mu,c+b}$.*

Proof. From Refs. [9, 56], for $\psi \in \beta_\mu \subset L^2(\mathbb{R}^+)$ and $a > 0,\ b > 0$, the following relations hold

$$h_\mu\{\psi(\frac{x}{a}, \frac{b}{a})\}(t) = a^{-\mu+\frac{1}{2}} t^{-\mu-\frac{1}{2}} (bt)^{\frac{1}{2}} J_\mu(bt)(h_\mu\psi)(at), \quad t \in \mathbb{R}^+. \tag{6.2.6}$$

$$h_\mu\{\psi(\frac{x}{a}, \frac{b}{a})\}(\eta) = a^{-\mu+\frac{1}{2}} \eta^{-\mu-\frac{1}{2}} (b\eta)^{\frac{1}{2}} J_\mu(b\eta)(h_\mu\psi)(a\eta), \quad \eta \in \mathbb{C}. \tag{6.2.7}$$

Since $z^{-\mu}J_\mu(z)$ is an even entire function therefore there exists $M(\mu) > 0$ such that

$$|e^{-b|\mathrm{Im}\eta|}(b\eta)^{-\mu} J_\mu(b\eta)| \leq M(\mu), \quad \eta \in \mathbb{C}. \tag{6.2.8}$$

For proving Lemma 6.2.5, we take $h_\mu\psi \in \mathscr{Y}_{\mu,\frac{c}{a}}$, using from [9, p. 285] $\eta^{-\mu-\frac{1}{2}} h_\mu\{\psi(\frac{x}{a}, \frac{b}{a})\}(\eta)$ is even entire function, then we have

$$\alpha^\mu_{c+b,k}(h_\mu\{\psi(\frac{x}{a}, \frac{b}{a})\}(\eta)) = \sup_\eta |e^{-(c+b)|\mathrm{Im}\eta|} \eta^{2k-\mu-\frac{1}{2}} h_\mu\{\psi(\frac{x}{a}, \frac{b}{a})\}(\eta)|.$$

In view of (6.2.7), we get

$$\begin{aligned}
&\alpha^\mu_{c+b,k}(h_\mu\{\psi(\frac{x}{a}, \frac{b}{a})\}(\eta)) \\
&\quad = \sup_\eta |e^{-(c+b)|\mathrm{Im}\eta|} \eta^{2k-\mu-\frac{1}{2}} a^{-\mu+\frac{1}{2}} \eta^{-\mu-\frac{1}{2}} (b\eta)^{\frac{1}{2}} J_\mu(b\eta)(h_\mu\psi)(a\eta)| \\
&\quad = \sup_\eta |e^{-(c+b)|\mathrm{Im}\eta|} \eta^{2k-\mu-\frac{1}{2}} a^{-\mu+\frac{1}{2}} b^{\mu+\frac{1}{2}} (b\eta)^{-\mu} J_\mu(b\eta)(h_\mu\psi)(a\eta)| \\
&\quad = b^{\mu+\frac{1}{2}} \sup_\eta |e^{-b|\mathrm{Im}\eta|} (b\eta)^{-\mu} J_\mu(b\eta) e^{-c|\mathrm{Im}\eta|} \eta^{2k-\mu-\frac{1}{2}} a^{-\mu+\frac{1}{2}} (h_\mu\psi)(a\eta)|.
\end{aligned}$$

Using (6.2.8), the last expression yields

$$\alpha^\mu_{c+b,k}(h_\mu\{\psi(\frac{x}{a}, \frac{b}{a})\}(\eta)) \leq b^{\mu+\frac{1}{2}} M(\mu) \sup_\eta |e^{-c|\mathrm{Im}\eta|} \eta^{2k-\mu-\frac{1}{2}} a^{-\mu+\frac{1}{2}} (h_\mu\psi)(a\eta)|.$$

Now, substituting $a\eta = z$, we get

$$\begin{aligned}\alpha^{\mu}_{c+b,k}(h_\mu\{\psi(\frac{x}{a},\frac{b}{a})\}(\eta)) &\leq b^{\mu+\frac{1}{2}}M(\mu)\sup_z|e^{-\frac{c}{a}|\mathrm{Im}z|}(\frac{z}{a})^{2k-\mu-\frac{1}{2}}a^{-\mu+\frac{1}{2}}(h_\mu\psi)(z)|\\ &\leq b^{\mu+\frac{1}{2}}a^{-2k+1}M(\mu)\sup_z|e^{-\frac{c}{a}|\mathrm{Im}z|}z^{2k-\mu-\frac{1}{2}}(h_\mu\psi)(z)|\\ &\leq b^{\mu+\frac{1}{2}}a^{-2k+1}M(\mu)\alpha^{\mu}_{\frac{c}{a},k}(h_\mu\psi).\end{aligned}$$

Thus $h_\mu\{\psi(\frac{x}{a},\frac{b}{a})\} \in \mathscr{Y}_{\mu,c+b}$. Then by [61, Theorem 1, p.683], it follows that $\psi(\frac{x}{a},\frac{b}{a}) \in \beta_{\mu,c+b}$. □

Theorem 6.2.6. *Let $\mu > -\frac{1}{2}$, $a > 0$ and $c > 0$. Let $\psi \in \beta_{\mu,\frac{c}{a}}$ be a Bessel wavelet and $f \in \beta_{\mu,d}$, then the function*

$$(B_\psi f)(b,a) = (f\#\psi_a)(b), \quad b > 0 \tag{6.2.9}$$

belongs to $\beta_{\mu,d+c}$.

Proof. By Betancor et al. [9, Proposition 3.4, p. 286],

$$\begin{aligned}h_\mu\{(B_\psi f)(b,a)\}(\eta) = h_\mu(f\#\psi_a)(\eta) &= \eta^{-\mu-\frac{1}{2}}(h_\mu f)(\eta)(h_\mu\psi_a)(\eta)\\ &= \eta^{-\mu-\frac{1}{2}}(h_\mu f)(\eta)(h_\mu\psi)(a\eta). \qquad (6.2.10)\end{aligned}$$

Now, for $k \in \mathbb{N}_0$

$$\alpha^{\mu}_{d+c,k}\{h_\mu((B_\psi f)(b,a))(\eta)\} = \sup_\eta|e^{-(d+c)|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}h_\mu((B_\psi f)(b,a))(\eta)|.$$

Using (6.2.10), we find

$$\alpha^{\mu}_{d+c,k}\{h_\mu((B_\psi f)(b,a))(\eta)\} = \sup_\eta|e^{-(d+c)|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}\eta^{-\mu-\frac{1}{2}}(h_\mu f)(\eta)(h_\mu\psi)(a\eta)|$$

$$\leq \sup_{\eta} |e^{-d|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}(h_\mu f)(\eta)| \sup_{\eta} |e^{-c|\mathrm{Im}\eta|}\eta^{-\mu-\frac{1}{2}}(h_\mu \psi)(a\eta)|$$
$$\leq \alpha^{\mu}_{d,k}(h_\mu f) \sup_{\eta} |e^{-c|\mathrm{Im}\eta|}\eta^{-\mu-\frac{1}{2}}(h_\mu \psi)(a\eta)|.$$

Taking $a\eta = z$, we get

$$\begin{aligned}\alpha^{\mu}_{d+c,k}\{h_\mu((B_\psi f)(b,a))(\eta)\} &\leq \alpha^{\mu}_{d,k}(h_\mu f) \sup_{z} |e^{-\frac{c}{a}|\mathrm{Im}z|}\left(\frac{z}{a}\right)^{-\mu-\frac{1}{2}}(h_\mu \psi)(z)| \\ &\leq a^{\mu+\frac{1}{2}}\alpha^{\mu}_{d,k}(h_\mu f) \sup_{z} |e^{-\frac{c}{a}|\mathrm{Im}z|}z^{-\mu-\frac{1}{2}}(h_\mu \psi)(z)| \\ &\leq a^{\mu+\frac{1}{2}}\alpha^{\mu}_{d,k}(h_\mu f)\alpha^{\mu}_{\frac{c}{a},0}(h_\mu \psi),\end{aligned}$$

for every $k \in \mathbb{N}_0$.

Since $f \in \beta_{\mu,d}$ and $\psi \in \beta_{\mu,\frac{c}{a}}$, then by Zemanian [61, Theorem1, p. 683], $(h_\mu f) \in \mathscr{Y}_{\mu,d}$ and $(h_\mu \psi) \in \mathscr{Y}_{\mu,\frac{c}{a}}$, then we obtain $h_\mu((B_\psi f)(b,a)) \in \mathscr{Y}_{\mu,d+c}$. Thus by Zemanian [61, Theorem1, p. 683] $(B_\psi f)(b,a) = (f\#\psi_a)(b) \in \beta_{\mu,d+c}$. □

Theorem 6.2.7. *Let $f \in \mathscr{E}'_\mu(\mathbb{R}^+)$, $\mu > -\frac{1}{2}$ and $a > 0$, $c > 0$. Let $\psi \in \beta_{\mu,\frac{c}{a}}$ be a Bessel wavelet, then*

$$(B_\psi f)(b,a) = (f\#\psi_a)(b), \quad b > 0$$

belongs to $\beta_{\mu,n+c}$ for suitable $n \in \mathbb{N}_0$.

Proof. From Betancor et al. [9, Proposition 4.7, p. 293],

$$\begin{aligned}h_\mu((B_\psi f)(b,a))(\eta) = h_\mu(f\#\psi_a)(\eta) &= \eta^{-\mu-\frac{1}{2}}(h'_\mu f)(\eta)(h_\mu \psi_a)(\eta) \\ &= \eta^{-\mu-\frac{1}{2}}(h'_\mu f)(\eta)(h_\mu \psi)(a\eta).\end{aligned}$$

The above expression becomes

$$
\begin{aligned}
&|e^{-(n+c)|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}h_\mu((B_\psi f)(b,a))(\eta)| \\
&\qquad = |e^{-(n+c)|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}\eta^{-\mu-\frac{1}{2}}(h'_\mu f)(\eta)(h_\mu\psi)(a\eta)| \\
&\qquad \leq |e^{-n|\mathrm{Im}\eta|}\eta^{-\mu-\frac{1}{2}}(h'_\mu f)(\eta)||e^{-c|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}(h_\mu\psi)(a\eta)|.
\end{aligned}
$$

From Betancor et al. [9, Proposition 4.5], for some $n, r \in \mathbb{N}_0$ and all $\eta \in \mathbb{C}$, using the result $|\eta^{-\mu-\frac{1}{2}}h'_\mu(f)(\eta)| \leq B(1+|\eta|^{2r})e^{n|\mathrm{Im}\eta|}$ where B is constant then, we obtain

$$
\begin{aligned}
&|e^{-(n+c)|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}h_\mu((B_\psi f)(b,a))(\eta)| \\
&\qquad \leq B|e^{-n|\mathrm{Im}\eta|}(1+|\eta|^{2r})e^{n|\mathrm{Im}\eta|}||e^{-c|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}(h_\mu\psi)(a\eta)| \\
&\qquad \leq B\{|e^{-c|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}(h_\mu\psi)(a\eta)| + |e^{-c|\mathrm{Im}\eta|}\eta^{2(k+r)-\mu-\frac{1}{2}}(h_\mu\psi)(a\eta)|\}.
\end{aligned}
$$

So that

$$
\begin{aligned}
&\sup_{\eta\in\mathbb{C}}|e^{-(n+c)|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}h_\mu((B_\psi f)(b,a))(\eta)| \\
&\qquad \leq B\{\sup_{\eta\in\mathbb{C}}|e^{-c|\mathrm{Im}\eta|}\eta^{2k-\mu-\frac{1}{2}}(h_\mu\psi)(a\eta)| \\
&\qquad\qquad + \sup_{\eta\in\mathbb{C}}|e^{-c|\mathrm{Im}\eta|}\eta^{2(k+r)-\mu-\frac{1}{2}}(h_\mu\psi)(a\eta)|\}.
\end{aligned}
$$

Next step will be obtained by substituting $a\eta = z$, then

$$
\begin{aligned}
&\alpha^\mu_{n+c,k}\{h_\mu((B_\psi f)(b,a))(\eta)\} \\
&\qquad \leq B\{\sup_{z}|e^{-\frac{c}{a}|\mathrm{Im}z|}\left(\frac{z}{a}\right)^{2k-\mu-\frac{1}{2}}(h_\mu\psi)(z)| \\
&\qquad\qquad + \sup_{z}|e^{-\frac{c}{a}|\mathrm{Im}z|}\left(\frac{z}{a}\right)^{2(k+r)-\mu-\frac{1}{2}}(h_\mu\psi)(z)|\}
\end{aligned}
$$

$$\leq Ba^{-2k+\mu+\frac{1}{2}}\{\sup_z |e^{-\frac{c}{a}|\mathrm{Im}z|}z^{2k-\mu-\frac{1}{2}}(h_\mu\psi)(z)| \\ + a^{-2r}\sup_z |e^{-\frac{c}{a}|\mathrm{Im}z|}z^{2(k+r)-\mu-\frac{1}{2}}(h_\mu\psi)(z)|\}$$

$$\leq Ba^{-2k+\mu+\frac{1}{2}}\{\alpha^{\mu}_{\frac{c}{a},k}(h_\mu\psi)| + a^{-2r}\alpha^{\mu}_{\frac{c}{a},k+r}(h_\mu\psi)\}.$$

for every $k \in \mathbb{N}_0$. Since $\psi \in \beta_{\mu,\frac{c}{a}}$ by Zemanian [61, Theorem1, p. 683], $(h_\mu\psi) \in \mathscr{Y}_{\mu,\frac{c}{a}}$, therefore $h_\mu((B_\psi f)(b,a)) \in \mathscr{Y}_{\mu,n+c}$. Thus by Zemanian [61, Theorem1, p. 683], $(B_\psi f)(b,a) = (f\#\psi_a)(b) \in \beta_{\mu,n+c}$. □

6.3 Inversion formula of the continuous Bessel wavelet transform of the distributions

In this section, the inversion formula of the Bessel wavelet transform of distributions in β'_μ and associated results are studied by exploiting the theory of Hankel transform.

From Theorem 6.2.4, we find that the continuous Bessel wavelet transform of the distribution of type $Cx^{\mu+\frac{1}{2}}$ will be zero. Therefore, the inversion formula of the Bessel wavelet transform in β'_μ can not be defined. This implies that the inversion formula of the continuous Bessel wavelet transform will be valid for modulo a $Cx^{\mu+\frac{1}{2}}$ distributions.

For investigating the inversion formula of the continuous Bessel wavelet transform of distributions, our result will be based on the following considerations:

(I) Inversion formula of the continuous Bessel wavelet transform of distributions will be found on $\beta'_{\mu,F}$, which can be obtained by filtering of β'_μ. This filtering can be done by the following way:

(i) removing all the non-zero $Cx^{\mu+\frac{1}{2}}$ distributions from β'_μ.

(ii) removing all non-zero $Cx^{\mu+\frac{1}{2}}$ that appear with a distribution as a union. For example in the distribution $\frac{x^{\mu+\frac{5}{2}}}{1+x^2} = x^{\mu+\frac{1}{2}} - \frac{x^{\mu+\frac{1}{2}}}{1+x^2}$, we would omit $x^{\mu+\frac{1}{2}}$ and retain only $-\frac{x^{\mu+\frac{1}{2}}}{1+x^2}$.

(II) The Bessel wavelet kernels under consideration for determining the Bessel wavelet transform are those Bessel wavelets whose all the moments weighted by $x^{\mu+\frac{1}{2}}$ are non-zero. Let β_μ^{2m} be a collection of such Bessel wavelet kernel.

Lemma 6.3.1. *Let $\mu > -\frac{1}{2}$ and $\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)$ be Hankel translation and dilation of $\psi \in \beta_\mu(\mathbb{R}^+)$, then*

$$S_{\mu,x}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big) = S_{\mu,b}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big).$$

Proof. From (1.4.4), we have

$$S_{\mu,x}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big) = S_{\mu,x}\int_0^\infty \psi(z) D_\mu\Big(\frac{x}{a}, \frac{b}{a}, z\Big)dz.$$

Using (1.4.5), we get

$$\begin{aligned}
&S_{\mu,x}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\\
&= S_{\mu,x}\int_0^\infty \psi(z)\Big(\int_0^\infty t^{-\mu-\frac{1}{2}}\Big(\frac{xt}{a}\Big)^{\frac{1}{2}} J_\mu\Big(\frac{xt}{a}\Big)\Big(\frac{bt}{a}\Big)^{\frac{1}{2}} J_\mu\Big(\frac{bt}{a}\Big)(zt)^{\frac{1}{2}} J_\mu(zt)dt\Big)dz\\
&= \int_0^\infty \psi(z)\Big(\int_0^\infty t^{-\mu-\frac{1}{2}} S_{\mu,x}\Big\{\Big(\frac{xt}{a}\Big)^{\frac{1}{2}} J_\mu\Big(\frac{xt}{a}\Big)\Big\}\Big(\frac{bt}{a}\Big)^{\frac{1}{2}} J_\mu\Big(\frac{bt}{a}\Big)(zt)^{\frac{1}{2}} J_\mu(zt)dt\Big)dz.
\end{aligned}$$

From (1.7.6) and (1.7.7), we find that

$$\begin{aligned}
S_{\mu,x}x^{\frac{1}{2}} J_\mu\Big(\frac{xt}{a}\Big) &= x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1} D_x x^{-\mu} J_\mu\Big(\frac{xt}{a}\Big)\\
&= x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1}\Big(-\frac{t}{a}\Big)x^{-\mu} J_{\mu+1}\Big(\frac{xt}{a}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(-\frac{t}{a}\Big)x^{-\mu-\frac{1}{2}}D_x x^{\mu+1}J_{\mu+1}\Big(\frac{xt}{a}\Big)\\
&= -\Big(\frac{t}{a}\Big)^2 x^{-\mu-\frac{1}{2}}x^{\mu+1}J_\mu\Big(\frac{xt}{a}\Big)\\
&= -\Big(\frac{t}{a}\Big)^2 x^{\frac{1}{2}}J_\mu\Big(\frac{xt}{a}\Big).
\end{aligned}
$$

Similarily,

$$
S_{\mu,b}b^{\frac{1}{2}}J_\mu\Big(\frac{bt}{a}\Big) = -\Big(\frac{t}{a}\Big)^2 b^{\frac{1}{2}}J_\mu\Big(\frac{bt}{a}\Big). \tag{6.3.1}
$$

Using (6.3.1), we find

$$
\begin{aligned}
&S_{\mu,x}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\\
&\quad= \int_0^\infty \psi(z)\Big(\int_0^\infty t^{-\mu-\frac{1}{2}}(-1)\Big(\frac{t}{a}\Big)^2\Big(\frac{xt}{a}\Big)^{\frac{1}{2}}J_\mu\Big(\frac{xt}{a}\Big)\Big(\frac{bt}{a}\Big)^{\frac{1}{2}}J_\mu\Big(\frac{bt}{a}\Big)(zt)^{\frac{1}{2}}J_\mu(zt)dt\Big)dz\\
&\quad= \int_0^\infty \psi(z)\Big(\int_0^\infty t^{-\mu-\frac{1}{2}}\Big(\frac{xt}{a}\Big)^{\frac{1}{2}}J_\mu\Big(\frac{xt}{a}\Big)S_{\mu,b}\Big\{\Big(\frac{bt}{a}\Big)^{\frac{1}{2}}J_\mu\Big(\frac{bt}{a}\Big)\Big\}(zt)^{\frac{1}{2}}J_\mu(zt)dt\Big)dz\\
&\quad= \int_0^\infty \psi(z)S_{\mu,b}\Big(\int_0^\infty t^{-\mu-\frac{1}{2}}\Big(\frac{xt}{a}\Big)^{\frac{1}{2}}J_\mu\Big(\frac{xt}{a}\Big)\Big(\frac{bt}{a}\Big)^{\frac{1}{2}}J_\mu\Big(\frac{bt}{a}\Big)(zt)^{\frac{1}{2}}J_\mu(zt)dt\Big)dz\\
&\quad= S_{\mu,b}\int_0^\infty \psi(z)D_\mu\Big(\frac{x}{a},\frac{b}{a},z\Big)dz\\
&\quad= S_{\mu,b}\psi\Big(\frac{x}{a},\frac{b}{a}\Big).
\end{aligned}
$$

Therefore

$$
S_{\mu,x}\psi\Big(\frac{x}{a},\frac{b}{a}\Big) = S_{\mu,b}\psi\Big(\frac{x}{a},\frac{b}{a}\Big).
$$

□

Theorem 6.3.2. *Let $\psi \in \beta_\mu$ be a Bessel wavelet with $\psi(x) = 0,\ x \geq c,\ c > 0$. Then for $f \in \beta'_\mu$, the continuous Bessel wavelet transform $(B_\psi f)(b,a)$ is a continuous function of a, b in a compact neighbourhood of the point $(b,a), a > 0, b > 0$.*

Proof. For $\psi \in \beta_\mu$ is a Bessel wavelet and $f \in \beta'_\mu$, then

$$(B_\psi f)(b,a) = \Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle, \quad a>0,\ b>0.$$

To check the continuity of $(B_\psi f)(b,a)$, we have

$$\begin{aligned}
&(B_\psi f)(b+\Delta b, a+\Delta a) - (B_\psi f)(b,a)\\
&\qquad = \Big\langle f(t), (a+\Delta a)^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a+\Delta a},\frac{b+\Delta b}{a+\Delta a}\Big)\Big\rangle - \Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle\\
&\qquad = \Big\langle f(t), (a+\Delta a)^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a+\Delta a},\frac{b+\Delta b}{a+\Delta a}\Big) - a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle.
\end{aligned}$$

Therefore, we have to show the expression

$$(a+\Delta a)^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a+\Delta a},\frac{b+\Delta b}{a+\Delta a}\Big) - a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big) \to 0,$$

in the topology of β_μ as $\Delta a, \Delta b \to 0$.

Using (1.4.4) and [56, p. 318], we obtain

$$\begin{aligned}
&(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}}\Big\{(a+\Delta a)^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a+\Delta a},\frac{b+\Delta b}{a+\Delta a}\Big) - a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\}\\
&\quad = (t^{-1}D_t)^k t^{-\mu-\frac{1}{2}}\Big[(a+\Delta a)^{\mu-\frac{1}{2}}.(a+\Delta a)^{-\mu+\frac{1}{2}}\\
&\qquad\qquad \times h_\mu\Big\{y^{-\mu-\frac{1}{2}}(ty)^{\frac{1}{2}}J_\mu(ty)(h_\mu\psi)((a+\delta a)y)\Big\}(b+\delta b)\\
&\qquad\qquad - a^{\mu-\frac{1}{2}}.a^{-\mu+\frac{1}{2}}h_\mu\Big\{y^{-\mu-\frac{1}{2}}(ty)^{\frac{1}{2}}J_\mu(ty)(h_\mu\psi)(ay)\Big\}(b)\Big]\\
&\quad = h_\mu\Big\{y^{-\mu}(t^{-1}D_t)^k t^{-\mu}J_\mu(ty)(h_\mu\psi)((a+\delta a)y)\Big\}(b+\delta b)\\
&\qquad\qquad - h_\mu\Big\{y^{-\mu}(t^{-1}D_t)^k t^{-\mu}J_\mu(ty)(h_\mu\psi)(ay)\Big\}(b)\\
&\quad = h_\mu\Big\{y^{-\mu}(-y)^k t^{-\mu-k}J_{\mu+k}(ty)(h_\mu\psi)((a+\delta a)y)\Big\}(b+\delta b)\\
&\qquad\qquad - h_\mu\Big\{y^{-\mu}(-y)^k t^{-\mu-k}J_{\mu+k}(ty)(h_\mu\psi)(ay)\Big\}(b)
\end{aligned}$$

$$
\begin{aligned}
&= h_\mu\Big\{(-y^2)^k (yt)^{-\mu-k} J_{\mu+k}(ty)(h_\mu\psi)((a+\delta a)y)\Big\}(b+\delta b)\\
&\qquad - h_\mu\Big\{(-y^2)^k (yt)^{-\mu-k} J_{\mu+k}(ty)(h_\mu\psi)(ay)\Big\}(b)\\
&= h_\mu\Big\{(yt)^{-\mu-k} J_{\mu+k}(ty)(h_\mu(S_\mu^k\psi))((a+\delta a)y).(a+\delta a)^{2k}(-1))^k\Big\}(b+\delta b)\\
&\qquad - h_\mu\Big\{(yt)^{-\mu-k} J_{\mu+k}(ty)(h_\mu(S_\mu^k\psi))(ay).(-a^2)^k\Big\}(b).
\end{aligned}
$$

Since

$$
\begin{aligned}
\Big| h_\mu\Big\{(yt)^{-\mu-k} J_{\mu+k}(ty)(h_\mu(S_\mu^k\psi))((a+\delta a)y).(a+\delta a)^{2k}(-1))^k\Big\}(b+\delta b)&\\
- h_\mu\Big\{(yt)^{-\mu-k} J_{\mu+k}(ty)(h_\mu(S_\mu^k\psi))(ay).(-a^2)^k\Big\}(b)\Big|&
\end{aligned}
$$

converges to zero as $\Delta a, \Delta b \to 0$.

Therefore,

$$
\sup_{t\in\mathbb{R}^+}\Big|(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}}\Big\{(a+\Delta a)^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a+\Delta a},\frac{b+\Delta b}{a+\Delta a}\Big) - a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\}\Big| \to 0,
$$

as $\Delta a, \Delta b \to 0$.

Thus,

$$
(a+\Delta a)^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a+\Delta a},\frac{b+\Delta b}{a+\Delta a}\Big) - a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big) \to 0,
$$

in the topology of β_μ as $\Delta a, \Delta b \to 0$. □

Theorem 6.3.3. *Let $f \in \beta'_{\mu,F}$ and $\phi \in \beta_\mu$. For $\mu > -\frac{1}{2}$, assume that ψ is a basic Bessel wavelet belonging to β_μ, then*

$$\Big\langle \frac{1}{C_{\mu,\psi}} \int_\nu^A \int_0^B \Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle$$
$$= \Big\langle f(t), \frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx\Big\rangle. \quad (6.3.2)$$

To prove the Theorem 6.3.3, we required the following lemmas:

Lemma 6.3.4. *Let $F_{A,B,C,\nu}(t)$ be the function which is defined by*

$$F_{A,B,C,\nu}(t) = \frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\phi(x)\frac{dbdadx}{a^3}, \quad a, b, x, t > 0 \quad (6.3.3)$$

then $F_{A,B,C,\nu}(t) \neq 0$ for $t \in (0, C + 2C_1)$, where $\phi, \psi \in \beta_\mu$ with $\phi(x) = 0$, for $x \geq C$ and $\psi(x) = 0$, for $x \geq C_1$.

Proof. For showing $F_{A,B,C,\nu}(t) \neq 0$ for $t \in (0, C + 2C_1)$, we have to need the following steps:

Step 1: For showing the support of $\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)$ in terms of t, we have to prove that $\psi\Big(\frac{t}{a}, \frac{b}{a}\Big) \neq 0$.

In view of Betancor [5, p. 58], we take

$$\psi\Big(\frac{t}{a}, \frac{b}{a}\Big) = \int_0^\infty \psi(z) D_\mu\Big(\frac{t}{a}, \frac{b}{a}, z\Big)dz, \quad (6.3.4)$$

where

$$D_\mu\Big(\frac{t}{a},\frac{b}{a},z\Big) = \begin{cases} \frac{1}{2^{3\mu-1}\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\Big(\frac{tbz}{a^2}\Big)^{\frac{1}{2}-\mu}\Big[z^2-\Big(\frac{t-b}{a}\Big)^2\Big]^{\mu-\frac{1}{2}}\Big[\Big(\frac{t+b}{a}\Big)^2-z^2\Big]^{\mu-\frac{1}{2}}, & |\frac{t-b}{a}|<z<\frac{t+b}{a} \\ 0, & z<|\frac{t-b}{a}| \text{ or } z>\frac{t+b}{a}. \end{cases} \tag{6.3.5}$$

Then $\psi(z), D_\mu\Big(\frac{t}{a},\frac{b}{a},z\Big)$ can be non-zero under the following condtions:

(i) $0<z<C_1;$ as $\psi\in\beta_\mu,$ so $\psi\in\beta_{\mu,C_1}$ for $C_1>0,$

(ii) $\Big|\frac{t-b}{a}\Big|<z<\frac{t+b}{a};$ as $D_\mu\Big(\frac{t}{a},\frac{b}{a},z\Big)\neq 0$ from(6.3.5).

From (i), we get

(iii) $-C_1<-z<0.$

Now, adding (ii) and (iii), we find

$$\Big|\frac{t-b}{a}\Big|-C_1<z-z<\frac{t+b}{a},$$

so that

$$\Big|\frac{t-b}{a}\Big|<C_1 \qquad \text{and} \qquad \frac{t+b}{a}>0,$$

Hence $\psi\Big(\frac{t}{a},\frac{b}{a}\Big)$ is non-zero for $-C_1<\frac{t-b}{a}<C_1,\ t>0.$

Step 2. Like step 1, we can prove $\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\neq 0$ for $-C_1<\frac{x-b}{a}<C_1$ or $-C_1<\frac{x-b}{a}<C_1.$

Step 3. As $\phi \in \beta_\mu$, so $\phi(x) \neq 0$ for $0 < x < C,\ C > 0$.

Now, from step 1, step 2 and step 3, we conclude that $F_{A,B,C,\nu}(t)$ is non-zero for

(1) $-C_1 < \frac{t-b}{a} < C_1$.

(2) $-C_1 < \frac{b-x}{a} < C_1$.

(3) $0 < x < C$.

Adding (1) and (2), we get

$$-2C_1 < \frac{t-b}{a} + \frac{b-x}{a} < 2C_1.$$

This implies that

$$\left|\frac{t-x}{a}\right| < 2C_1.$$

For dilation parameter a, we found the following cases:

Case I: If $a \leq 1$,

$$\left|\frac{t-x}{a}\right| < 2C_1.$$

This implies that

$$|t-x| < 2C_1.$$

This shows that $F_{A,B,C,\nu}(t) \neq 0$ if $|t-x| < 2C_1$.

Case II: If $a \geq 1$, then

$$\left|\frac{t-x}{a}\right| \geq 2C_1.$$

This gives

$$|t - x| \geq 2C_1.$$

Hence, we find that $F_{A,B,C,\nu}(t) = 0$ for $|t - x| \geq 2C_1$.

From the aforesaid steps and cases, we find the support of $F_{A,B,C,\nu}(t)$

$$i.e. \qquad t = t - x + x,$$

which implies

$$t \leq |t - x| + |x| < 2C_1 + C.$$

Then, the support of $F_{A,B,C,\nu}(t)$ is contained in $(0, C + 2C_1)$. □

After showing the support of $F_{A,B,C,\nu}(t)$, with help of following Lemma we shall prove that $F_{A,B,C,\nu}(t) \in \beta_\mu$.

Lemma 6.3.5. *Let $F_{A,B,C,\nu}(t)$ be the function which is defined by*

$$F_{A,B,C,\nu}(t) = \frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\phi(x)\frac{dbdadx}{a^3}, \quad a, b, x, t > 0 \tag{6.3.6}$$

then $F_{A,B,C,\nu}(t)$ belongs to β_μ.

Proof. From Lemma 6.2.5, we have $\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)$ is an element of β_μ, so $\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)$ be a smooth function of variable t. Hence $F_{A,B,C,\nu}(t)$ be smooth.

From (6.3.6) and [56, p.318], we find

$$\begin{aligned}
&\Big|(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}} F_{A,B,C,\nu}(t)\Big| \\
&\quad = \Big|(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}} \frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\phi(x)\frac{dbdadx}{a^3}\Big| \\
&\quad = \Big|\frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B (t^{-1}D_t)^k t^{-\mu-\frac{1}{2}} h_\mu\Big(y^{-\mu-\frac{1}{2}}(ty)^{\frac{1}{2}} J_\mu(ty)(h_\mu\psi)(ay)\Big)(b) \\
&\qquad\qquad \times h_\mu\Big(\xi^{-\mu-\frac{1}{2}}(x\xi)^{\frac{1}{2}} J_\mu(x\xi)(h_\mu\psi)(a\xi)\Big)(b)\phi(x)\frac{dbdadx}{a^{2\mu+2}}\Big| \\
&\quad = \Big|\frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B h_\mu\Big(y^{-\mu-\frac{1}{2}} y^{\frac{1}{2}} (t^{-1}D_t)^k t^{-\mu} J_\mu(ty)(h_\mu\psi)(ay)\Big)(b) \\
&\qquad\qquad \times h_\mu\Big(\xi^{-\mu-\frac{1}{2}}(x\xi)^{\frac{1}{2}} J_\mu(x\xi)(h_\mu\psi)(a\xi)\Big)(b)\phi(x)\frac{dbdadx}{a^{2\mu+2}}\Big|.
\end{aligned}$$

Applying $(t^{-1}D_t)^k t^{-\mu} J_\mu(yt) = (-y)^k t^{-\mu-k} J_{\mu+k}(yt)$, we get

$$\begin{aligned}
&\Big|(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}} F_{A,B,C,\nu}(t)\Big| \\
&\quad = \Big|\frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B h_\mu\Big(y^{-\mu}(-y)^k t^{-\mu-k} J_{\mu+k}(ty)(h_\mu\psi)(ay)\Big)(b) \\
&\qquad\qquad \times h_\mu\Big(\xi^{-\mu-\frac{1}{2}}(x\xi)^{\frac{1}{2}} J_\mu(x\xi)(h_\mu\psi)(a\xi)\Big)(b)\phi(x)\frac{dbdadx}{a^{2\mu+2}}\Big| \\
&\quad = \Big|\frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B h_\mu\Big((-y^2)^k (ty)^{-\mu-k} J_{\mu+k}(ty)(h_\mu\psi)(ay)\Big)(b) \\
&\qquad\qquad \times x^{\mu+\frac{1}{2}} h_\mu\Big((x\xi)^{-\mu} J_\mu(x\xi)(h_\mu\psi)(a\xi)\Big)(b)\phi(x)\frac{dbdadx}{a^{2\mu+2}}\Big|.
\end{aligned}$$

From (1.7.8) taking formula $h_\mu(S_{\mu,x}\phi(x))(y) = -y^2(h_\mu\phi)(y)$, we have

$$\begin{aligned}
&\Big|(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}} F_{A,B,C,\nu}(t)\Big| \\
&\quad \le \frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B \Big|h_\mu\Big((ty)^{-\mu-k} J_{\mu+k}(ty)(h_\mu(S^k_{\mu,x}\psi))(ay)a^{-2k}\Big)(b) \\
&\qquad\qquad \times h_\mu\Big((x\xi)^{-\mu} J_\mu(x\xi)(h_\mu\psi)(a\xi)\Big)(b) x^{\mu+\frac{1}{2}}\phi(x)\Big|\frac{dbdadx}{a^{2\mu+2}}
\end{aligned}$$

$$\leq \frac{1}{C_{\mu,\psi}} \int_0^C \int_\nu^A \int_0^B \Big| h_\mu\Big((ty)^{-\mu-k} J_{\mu+k}(ty)(h_\mu(S^k_{\mu,x}\psi))(ay)\Big)(b)$$
$$\times h_\mu\Big((x\xi)^{-\mu} J_\mu(x\xi)(h_\mu\psi)(a\xi)\Big)(b) x^{\mu+\frac{1}{2}} \frac{\phi(x)}{a^{2k+2\mu+2}}\Big| dbdadx.$$

Since the integrand in the above integral

$$h_\mu\Big((ty)^{-\mu-k} J_{\mu+k}(ty)(h_\mu(S^k_{\mu,x}\psi))(ay)\Big)(b) h_\mu\Big((x\xi)^{-\mu} J_\mu(x\xi)(h_\mu\psi)(a\xi)\Big)(b) \frac{x^{\mu+\frac{1}{2}}\phi(x)}{a^{2k+2\mu+2}}$$

is uniformly continuous in the compact set

$$\Lambda = \{(a,b,x,t) : \nu \leq a \leq A, 0 \leq b \leq B, 0 \leq x \leq C, s_1 \leq t \leq s_2\},$$

and it is bounded over Λ.

Thus

$$\sup_{t\in\mathbb{R}^+} \Big|(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}} F_{A,B,C,\nu}(t)\Big| < \infty. \tag{6.3.7}$$

From aforesaid facts and Lemma 6.3.4, (6.3.7) it follows that $F_{A,B,C,\nu}(t)$ is smooth function with $F_{A,B,C,\nu}(t) = 0$ for $t \geq C + 2C_1$ such that

$$\gamma^\mu_k(F_{A,B,C,\nu}) = \sup_{t\in\mathbb{R}^+} \Big|(t^{-1}D_t)^k t^{-\mu-\frac{1}{2}} F_{A,B,C,\nu}(t)\Big| < \infty, \forall\ k \in \mathbb{N}_0.$$

This implies that $F_{A,B,C,\nu}(t) \in \beta_{\mu,C+2C_1} \subset \beta_\mu$. □

Lemma 6.3.6. *For $\psi \in \beta_\mu$, a Bessel wavelet and $\phi \in \beta_\mu, \mu > -\frac{1}{2}$, we take*

$$F(t) = \int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx, \qquad a,b,x,t > 0 \tag{6.3.8}$$

then

$$t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(t) = \int_0^C\int_\nu^A\int_0^B h_\mu\Big((yt)^{-\mu}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b) \times h_\mu\Big((\xi x)^{-\mu}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b)x^{\mu+\frac{1}{2}}S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^{2\mu+2}}.$$

.

Proof. From (6.3.8), we have

$$\begin{aligned}t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(t) &= t^{-\mu-\frac{1}{2}}S^k_{\mu,t}\int_0^C\int_\nu^A\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx\\ &= t^{-\mu-\frac{1}{2}}\int_0^C\int_\nu^A\int_0^B S^k_{\mu,t}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx.\end{aligned}$$

From Lemma 6.3.1, we find the relation

$$S^k_{\mu,t}\psi\Big(\frac{t}{a},\frac{b}{a}\Big) = S^k_{\mu,b}\psi\Big(\frac{t}{a},\frac{b}{a}\Big).$$

Using aforesaid relation, we get

$$t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(t) = t^{-\mu-\frac{1}{2}}\int_0^C\int_\nu^A\int_0^B \Big\{S^k_{\mu,b}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbdadx}{a^3}.$$

By integrating by parts, we get

$$t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(t) = t^{-\mu-\frac{1}{2}}\int_0^C\int_\nu^A\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\{S^k_{\mu,b}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\}\phi(x)\frac{dbdadx}{a^3}.$$

Note that in the integration by parts with respect to b the boundary terms when evaluated at 0 and B will vanish in the limiting process as $B \to \infty$. So, we omit the remainder terms when evaluated at points 0 and B.

Again we take

$$S^k_{\mu,b}\psi\Big(\frac{x}{a},\frac{b}{a}\Big) = S^k_{\mu,x}\psi\Big(\frac{x}{a},\frac{b}{a}\Big).$$

So that

$$\begin{aligned}
t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(t) &= t^{-\mu-\frac{1}{2}}\int_0^C\int_\nu^A\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\{S^k_{\mu,x}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\}\phi(x)\frac{dbdadx}{a^3}\\
&= \Big\langle S^k_{\mu,x}\int_\nu^A\int_0^B t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\frac{dbda}{a^3},\phi(x)\Big\rangle.
\end{aligned}$$

By exploiting distributional derivative, we get

$$\begin{aligned}
t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(t) &= \Big\langle \int_\nu^A\int_0^B t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\frac{dbda}{a^3}, S^k_{\mu,x}\phi(x)\Big\rangle\\
&= \int_0^C\int_\nu^A\int_0^B t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^3}. \qquad (6.3.9)
\end{aligned}$$

In view of [56, p.318], we find

$$\begin{aligned}
t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(t) &= \int_0^C\int_\nu^A\int_0^B t^{-\mu-\frac{1}{2}}h_\mu\Big(y^{-\mu-\frac{1}{2}}(yt)^{\frac{1}{2}}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b)\\
&\qquad\times h_\mu\Big(\xi^{-\mu-\frac{1}{2}}(\xi x)^{\frac{1}{2}}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b)S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^{2\mu+2}}\\
&= \int_0^C\int_\nu^A\int_0^B h_\mu\Big((yt)^{-\mu}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b)\\
&\qquad\times h_\mu\Big((\xi x)^{-\mu}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b)x^{\mu+\frac{1}{2}}S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^{2\mu+2}}.
\end{aligned}$$

$\square$

Lemma 6.3.7. *Let $F_{A,B,C,\nu}(t)$ be a function which is defined in Lemma 6.3.4, then*

$$\int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx$$
$$= \lim_{m\to\infty}\Big[\sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m}\psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\phi(x_l)\frac{1}{a_i^3}\square_{i,j,l}\Big],$$

in the topology of β_μ, where

$$\square_{i,j,l} = (a_i - a_{i-1})(b_j - b_{j-1})(x_l - x_{l-1}) = \triangle\ a \triangle\ b \triangle\ x,$$

is volume of the elementary cuboid of the partition of the region $\{(a,b,x) : \nu \le a \le A, 0 \le b \le B, 0 \le x \le C\}$.

Proof. We have from (6.3.3)

$$F_{A,B,C,\nu}(t) = \frac{1}{C_{\mu,\psi}}\int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx.$$

From the R.H.S. of the above integral, we denote

$$F(a,b,x,t) = \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{1}{a^3}.$$

From [56, p.318], we have

$$\begin{aligned}F(a,b,x,t) &= a^{-\mu+\frac{1}{2}}h_\mu\Big(y^{-\mu-\frac{1}{2}}(yt)^{\frac{1}{2}}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b)\\ &\qquad\times a^{-\mu+\frac{1}{2}}h_\mu\Big(\xi^{-\mu-\frac{1}{2}}(\xi x)^{\frac{1}{2}}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b)\phi(x)\frac{1}{a^3}\\ &= h_\mu\Big((yt)^{-\mu}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b)h_\mu\Big((\xi x)^{-\mu}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b)\\ &\qquad\times (xt)^{\mu+\frac{1}{2}}\phi(x)\frac{1}{a^{2\mu+2}}.\end{aligned}$$

Since from Lemma 6.3.5, we get $F_{A,B,C,\nu}(t) \in \beta_{\mu,C+2C_1} \subset \beta_\mu$. Then it clearly shows that $F(a,b,x,t)$ is a continuous function of a,b,x and t over a compact set in $\mathbb{R}^{+4}(= \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$ which is not intersecting the axis $a = 0$.

Let $\Lambda = \{(a,b,x,t) : \nu \leq a \leq A, 0 \leq b \leq B, 0 \leq x \leq C, s_1 \leq t \leq s_2\}$. Then $F(a,b,x,t)$ is uniformly continuous over the compact set Λ as a function of a,b,x and t. Hence for a given $\epsilon > 0$, there exists $\delta_0 > 0$ (independent of a,b,x and t) such that

$$\Big|F(a+\triangle a, b+\triangle b, x+\triangle x, t+\triangle t) - F(a,b,x,t)\Big| < \epsilon, \tag{6.3.10}$$

whenever $\|(\triangle a, \triangle b, \triangle x, \triangle t)\|_1 < \delta_0$. Using the technique of [36, Lemma 4.3, p.2018], the topology generated by $\|(a,b,x,t)\|_1 = max\{|a|,|b|,|x|,|t|\}$ and $\|(a,b,x,t)\| = \sqrt{a^2+b^2+x^2+t^2}$ is the same over $\mathbb{R}^{+4}(= \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$ as

$$\|(a,b,x,t)\|_1 \leq \|(a,b,x,t)\| \leq 2\|(a,b,x,t)\|_1.$$

To avoid the singularity at $a = 0$, we shall consider $m-$partitions on each of the axes B, X and $m+1$-partitions are taken on A axis equal in length, that is

$$a_{i+1} - a_i = a_{i+2} - a_{i+1} = \dots$$
$$b_{j+1} - b_j = b_{j+2} - b_{j+1} = \dots$$
$$x_{l+1} - x_l = x_{l+2} - x_{l+1} = \dots$$

such that

$$\nu = a_0 < a_1 < \dots < a_{m+1} = A, \quad 0 = b_0 < b_1 < \dots < b_m = B$$
$$\text{and} \quad 0 = x_0 < x_1 < \dots < x_m = C.$$

Taking m is sufficiently large partitions, the length of each of the sides of the subrectangles of the partitioning is less than δ_0.

We shall use Lemma 6.3.6, we get

$$\begin{aligned} t^{-\mu-\frac{1}{2}} S^k_{\mu,t} \int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbdadx}{a^3} \\ = \int_0^C \int_\nu^A \int_0^B t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big) S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^3}. \end{aligned} \tag{6.3.11}$$

From [56, p.318], we find

$$\begin{aligned} t^{-\mu-\frac{1}{2}} S^k_{\mu,t} \int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbdadx}{a^3} \\ = \int_0^C \int_\nu^A \int_0^B h_\mu\Big((yt)^{-\mu}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b) \\ \times h_\mu\Big((\xi x)^{-\mu}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b)x^{\mu+\frac{1}{2}} S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^{2\mu+2}}. \end{aligned} \tag{6.3.12}$$

Now, the integrand of the above integral becomes

$$\begin{aligned} & t^{-\mu-\frac{1}{2}} S^k_{\mu,t} F(a,b,x,t) \\ & = h_\mu\Big((yt)^{-\mu}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b)x^{\mu+\frac{1}{2}} h_\mu\Big((\xi x)^{-\mu}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b)\frac{S^k_{\mu,x}\phi(x)}{a^{2\mu+2}}. \end{aligned}$$

The above expression is uniformly continuous over any compact subset Λ as mentioned earliar(6.3.10). If we put $\triangle\ t = 0$ in (6.3.10), then after using (1.7.9), we get

$$\begin{aligned}
&\Big|t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(a+\triangle a, b+\triangle b, x+\triangle x, t)-t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F(a,b,x,t)\Big| \\
&\qquad = \Big|\sum_{j=0}^{k}A_j t^{2j}\Big(t^{-1}\frac{\partial}{\partial t}\Big)^{k+j}t^{-\mu-\frac{1}{2}}F(a+\triangle a, b+\triangle b, x+\triangle x, t) \\
&\qquad\qquad -\sum_{j=0}^{k}A_j t^{2j}\Big(t^{-1}\frac{\partial}{\partial t}\Big)^{k+j}t^{-\mu-\frac{1}{2}}F(a,b,x,t)\Big| < \epsilon, \qquad (6.3.13)
\end{aligned}$$

whenever $\|(\triangle a, \triangle b, \triangle x, 0)\|_1 < \delta_k,\ s_1 \le t \le s_2,\ k = 0,1,2,\ldots.$

The volume of the elementary cuboid of the partition for the Riemann sum is

$$\Box_{i,j,l} = (a_i - a_{i-1})(b_j - b_{j-1})(x_l - x_{l-1}) = \triangle a \triangle b \triangle x,$$

where $i = 0,1,\ldots,m+1,\ j = 0,1,\ldots,m$ and $l = 0,1,2,\ldots,m$. Therefore, the volume of the region $\{(a,b,x) : \nu \le a \le A, 0 \le b \le B, 0 \le x \le C\}$ can be written as $\cup_{i,j,l}\Box_{i,j,l} = (A-\nu)BC$.

Now we have to prove that

$$\begin{aligned}
&\Big|t^{-\mu-\frac{1}{2}}\int_0^C\int_\nu^A\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\{S^k_{\mu,x}\phi(x)\Big\}\frac{dbdadx}{a^3} \\
&\quad -\sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m}t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\Big\{S^k_{\mu,x}\phi(x)\Big\}\Big|_{x=x_l}\frac{1}{a_i^3}\Box_{i,j,l}\Big| \to 0,
\end{aligned}$$

as $m \to \infty$.

Using the concept of (6.3.11), (6.3.12) and [56, p.318], the last expression yields

$$\begin{aligned}
&\Big|t^{-\mu-\frac{1}{2}}\int_0^C\int_\nu^A\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\{S^k_{\mu,x}\phi(x)\Big\}\frac{dbdadx}{a^3} \\
&\quad -\sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m}t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\Big\{S^k_{\mu,x}\phi(x)\Big\}\Big|_{x=x_l}\frac{1}{a_i^3}\Box_{i,j,l}\Big|
\end{aligned}$$

$$= \Big| \int_{\cup_{i,j,l}\square_{i,j,l}} h_\mu\Big((yt)^{-\mu}J_\mu(yt)(h_\mu\psi)(ay)\Big)(b) h_\mu\Big((\xi x)^{-\mu}J_\mu(\xi x)(h_\mu\psi)(a\xi)\Big)(b) \times x^{\mu+\frac{1}{2}}S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^{2\mu+2}}$$

$$- \sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m} h_\mu\Big((yt)^{-\mu}J_\mu(yt)(h_\mu\psi)(a_i y)\Big)(b_j) \times h_\mu\Big((\xi x_l)^{-\mu}J_\mu(\xi x_l)(h_\mu\psi)(a_i\xi)\Big)(b_j)x_l^{\mu+\frac{1}{2}}\Big\{S^k_{\mu,x}\phi(x)\Big\}\Big|_{x=x_l}\frac{1}{a_i^{2\mu+2}}\square_{i,j,l}\Big|.$$

From (6.3.13), we get

$$\Big|t^{-\mu-\frac{1}{2}}\int_0^C\int_\nu^A\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\{S^k_{\mu,x}\phi(x)\Big\}\frac{dbdadx}{a^3} - \sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m} t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\Big\{S^k_{\mu,x}\phi(x)\Big\}\Big|_{x=x_l}\frac{1}{a_i^3}\square_{i,j,l}\Big| \leq \epsilon BC(A-\nu), \tag{6.3.14}$$

whenever $\|(a_i - a_{i-1}, b_j - b_{j-1}, x_l - x_{l-1}, 0)\|_1 < \delta_k,\ \forall\ i = 1,2,\ldots,m+1,\ j = 1,2,\ldots,m,\ l = 1,2,\ldots,m$ and $k = 0,1,2,\ldots$ and as $\epsilon > 0$ is an arbitrarily small number. It follows that the last expression (6.3.14) converges to zero uniformly for all $t \in \mathbb{R}^+$ as $m \to \infty$, for each $k = 0,1,2,\ldots$.

Then we get

$$\int_0^C\int_\nu^A\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx = \lim_{m\to\infty}\Big[\sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m}\psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\phi(x_l)\frac{1}{a_i^3}\square_{i,j,l}\Big],$$

in the topology of β_μ. $\square$

Proof of the main Theorem 6.3.3:

Proof. By definition of the distributions, we again write

$$\Big\langle \frac{1}{C_{\mu,\psi}} \int_\nu^A \int_0^B \Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle$$
$$= \int_0^C \int_\nu^A \int_0^B \frac{1}{C_{\mu,\psi}}\Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}\phi(x)dx, \quad (6.3.15)$$

where $\phi \in \beta_{\mu,C} \subset \beta_\mu$ for $C > 0$.

Taking the partition of the region $\{(a,b,x) : \nu \le a \le A, 0 \le b \le B, 0 \le x \le C\}$, we denote as

$$\square_{i,j,l} = (a_i - a_{i-1})(b_j - b_{j-1})(x_l - x_{l-1}) = \triangle\, a \,\triangle\, b \,\triangle\, x,$$

where $i = 0, 1, \ldots, m+1$, $j = 0, l, \ldots, m$ and $l = 0, 1, 2, \ldots, m$.

Therefore, using the concept of Riemann integral as a sum we find

$$\int_0^C \int_\nu^A \int_0^B \frac{1}{C_{\mu,\psi}}\Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}\phi(x)dx$$
$$= \frac{1}{C_{\mu,\psi}} \lim_{m\to\infty} \Big[\sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m} \Big\langle f(t), a_i^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\Big\rangle a_i^{\mu-\frac{1}{2}}\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\frac{\phi(x_l)}{a_i^{2\mu+2}}\square_{i,j,l}\Big]$$
$$= \lim_{m\to\infty} \Big\langle f(t), \frac{1}{C_{\mu,\psi}}\sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m} \psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\frac{\phi(x_l)}{a_i^3}\square_{i,j,l}\Big\rangle$$
$$= \Big\langle f(t), \frac{1}{C_{\mu,\psi}}\lim_{m\to\infty}\sum_{i=1}^{m+1}\sum_{j=1}^{m}\sum_{l=1}^{m} \psi\Big(\frac{t}{a_i},\frac{b_j}{a_i}\Big)\psi\Big(\frac{x_l}{a_i},\frac{b_j}{a_i}\Big)\frac{\phi(x_l)}{a_i^3}\square_{i,j,l}\Big\rangle.$$

In view of Lemma 6.3.7, we observe

$$\int_0^C \int_\nu^A \int_0^B \frac{1}{C_{\mu,\psi}}\Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}\phi(x)dx$$
$$= \Big\langle f(t), \frac{1}{C_{\mu,\psi}}\int_0^C \int_\nu^A \int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx\Big\rangle.$$

Employing the definition of distributions, we find

$$\Big\langle \frac{1}{C_{\mu,\psi}} \int_{\nu}^{A} \int_{0}^{B} \Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle$$
$$= \Big\langle f(t), \frac{1}{C_{\mu,\psi}} \int_{0}^{C}\int_{\nu}^{A}\int_{0}^{B} \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx\Big\rangle.$$

□

Lemma 6.3.8. *Let $\psi \in \beta_\mu$ be a Bessel wavelet with $\psi(x) = 0$ for $x \geq C_1,\ C_1 > 0$ and let $\phi \in \beta_\mu$ with $\phi(x) = 0,\ x \geq C, C > 0$. Then*

$$t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F_{A,B,C,\nu}(t) = \frac{1}{C_{\mu,\psi}}\int_{0}^{C}\int_{\nu}^{A}\int_{0}^{B} t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^3},$$

converges uniformly to $t^{-\mu-\frac{1}{2}}S^k_{\mu,t}\phi(t)$ for all $t \in \mathbb{R}^+$ as $A, B, C \to \infty$ and $\nu \to 0$.

Proof. Invoking Lemma 6.3.6, we have

$$t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F_{A,B,C,\nu}(t) = \frac{1}{C_{\mu,\psi}}\int_{0}^{C}\int_{\nu}^{A}\int_{0}^{B} t^{-\mu-\frac{1}{2}}\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)S^k_{\mu,x}\phi(x)\frac{dbdadx}{a^3}. \tag{6.3.16}$$

Now, we have to prove that the expression in (6.3.16) converges to $t^{-\mu-\frac{1}{2}}S^k_{\mu,t}\phi(t)$ uniformly for all $t \in \mathbb{R}^+$ as $A, B, C \to \infty$ and $\nu \to 0$ for each $k = 0, 1, 2, \ldots$.

If we take $B \to \infty$ in the following expression, then

$$\int_{0}^{B} \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)db \to \int_{0}^{\infty} \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)db$$
$$= \Big\langle \psi\Big(\frac{t}{a},\frac{b}{a}\Big), \psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\rangle.$$

Using the Parseval relation of the Hankel transform (1.4.12), we get

$$\begin{aligned}\int_0^B \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)db \to \Big\langle h_\mu^b\psi\Big(\frac{t}{a},\frac{b}{a}\Big), h_\mu^b\psi\Big(\frac{x}{a},\frac{b}{a}\Big)\Big\rangle\\ = \int_0^\infty a^{-\mu+\frac{1}{2}}y^{-\mu-\frac{1}{2}}(ty)^{\frac{1}{2}}J_\mu(ty)(h_\mu\psi)(ay)\\ \times a^{-\mu+\frac{1}{2}}y^{-\mu-\frac{1}{2}}(xy)^{\frac{1}{2}}J_\mu(xy)(h_\mu\psi)(ay)dy\\ = a\int_0^\infty aa^{-2\mu-1}y^{-2\mu-1}(ty)^{\frac{1}{2}}J_\mu(ty)(xy)^{\frac{1}{2}}J_\mu(xy)\\ \times |(h_\mu\psi)(ay)|^2dy.\end{aligned} \tag{6.3.17}$$

Using (6.3.17), then we have

$$\begin{aligned}&\Big|\int_0^\infty aa^{-2\mu-1}y^{-2\mu-1}(ty)^{\frac{1}{2}}J_\mu(ty)(xy)^{\frac{1}{2}}J_\mu(xy)|(h_\mu\psi)(ay)|^2dy\Big|\\ &\le \int_0^\infty aa^{-2\mu-1}y^{-2\mu-1}\Big|(ty)^{\frac{1}{2}}J_\mu(ty)\Big|\Big|(xy)^{\frac{1}{2}}J_\mu(xy)\Big||(h_\mu\psi)(ay)|^2dy\\ &\le A_\mu B_\mu\int_0^\infty aa^{-2\mu-1}y^{-2\mu-1}|(h_\mu\psi)(ay)|^2dy.\end{aligned}$$

Putting $ay = u$, we find

$$\begin{aligned}&\Big|\int_0^\infty aa^{-2\mu-1}y^{-2\mu-1}(ty)^{\frac{1}{2}}J_\mu(ty)(xy)^{\frac{1}{2}}J_\mu(xy)|(h_\mu\psi)(ay)|^2dy\Big|\\ &\le A_\mu B_\mu\int_0^\infty \Big|u^{-\mu-\frac{1}{2}}(h_\mu\psi)(u)\Big|^2du\\ &\le A_\mu B_\mu\int_0^\infty \Big|(1+u^2)u^{-\mu-\frac{1}{2}}(h_\mu\psi)(u)\Big|^2\frac{1}{(1+u^2)^2}du\\ &\le A_\mu B_\mu\Big\{\sup_{u\in\mathbb{R}^+}\Big|(1+u^2)u^{-\mu-\frac{1}{2}}(h_\mu\psi)(u)\Big|\Big\}^2\int_0^\infty\frac{1}{(1+u^2)^2}du.\end{aligned}$$

From [61, p.684], the last expression yields

$$\begin{aligned}&\Big|\int_0^\infty aa^{-2\mu-1}y^{-2\mu-1}(ty)^{\frac{1}{2}}J_\mu(ty)(xy)^{\frac{1}{2}}J_\mu(xy)|(h_\mu\psi)(ay)|^2dy\Big|\\ &\le A_\mu B_\mu\Big\{\sup_{z\in\mathbb{C}}\Big|e^{-C_1|\mathrm{Im}z|}(1+z^2)z^{-\mu-\frac{1}{2}}(h_\mu\psi)(z)\Big|\Big\}^2\frac{\pi}{4}\end{aligned}$$

$$\leq \frac{\pi}{4}A_\mu B_\mu\Big\{\sup_{z\in\mathbb{C}}\Big|e^{-C_1|\mathrm{Im}z|}z^{-\mu-\frac{1}{2}}(h_\mu\psi)(z)\Big| \\ +\sup_{z\in\mathbb{C}}\Big|e^{-C_1|\mathrm{Im}z|}z^2z^{-\mu-\frac{1}{2}}(h_\mu\psi)(z)\Big|\Big\}^2 \\ \leq \frac{\pi}{4}A_\mu B_\mu\Big\{\alpha^\mu_{C_1,0}(\psi)+\alpha^\mu_{C_1,1}(\psi)\Big\}^2<\infty.$$

Hence the right hand side of (6.3.17)

$$\int_0^\infty a^{-2\mu+1}y^{-2\mu-1}(ty)^{\frac{1}{2}}J_\mu(ty)\Big(h_\mu\psi\Big)(ay)(xy)^{\frac{1}{2}}J_\mu(xy)\Big(h_\mu\psi\Big)(ay)dy<\infty.$$

Therefore, we conclude that (6.3.17) i.e.

$$\int_0^B\psi\Big(\frac{t}{a},\frac{b}{a}\Big)\psi\Big(\frac{x}{a},\frac{b}{a}\Big)db\to\int_0^\infty\Big\{aa^{-2\mu-1}y^{-2\mu-1}(ty)^{\frac{1}{2}}J_\mu(ty) \\ \times(xy)^{\frac{1}{2}}J_\mu(xy)\Big|\Big(h_\mu\psi\Big)(ay)\Big|^2\Big\}dy$$

uniformly for all $x,t\in\mathbb{R}^+$ when $B\to\infty$.

In view of (6.3.16) and the above expression, we find that

$$\lim_{\substack{A,B,C\to\infty\\ \nu\to0+}}t^{-\mu-\frac{1}{2}}S^k_{\mu,t}F_{A,B,C,\nu}(t) \\ =\frac{1}{C_{\mu,\psi}}\lim_{\substack{A,C\to\infty\\ \nu\to0+}}\int_0^C\int_\nu^A t^{-\mu-\frac{1}{2}}\Big(\int_0^\infty a^{-2\mu+1}y^{-2\mu-1}(ty)^{\frac{1}{2}}J_\mu(ty)(xy)^{\frac{1}{2}}J_\mu(xy) \\ \times|(h_\mu\psi)(ay)|^2dy\Big)S^k_{\mu,x}\phi(x)\frac{dadx}{a^3}.$$

Using the Fubini's theorem, in the right hand side of the aforesaid integral we can make

$$\lim_{\substack{A,B,C\to\infty\\ \nu\to 0+}} t^{-\mu-\frac{1}{2}} S^k_{\mu,t} F_{A,B,C,\nu}(t) = \frac{1}{C_{\mu,\psi}} \lim_{\substack{A,C\to\infty\\ \nu\to 0+}} \int_0^C \int_0^\infty t^{-\mu-\frac{1}{2}} (ty)^{\frac{1}{2}} J_\mu(ty)(xy)^{\frac{1}{2}} J_\mu(xy)$$
$$\times \left(\int_\nu^A \frac{|(h_\mu\psi)(ay)|^2}{a^{2\mu+2} y^{2\mu+1}} da \right) S^k_{\mu,x}\phi(x) dy dx.$$

Using the concept of convergence, the last expression becomes

$$\lim_{\substack{A,B,C\to\infty\\ \nu\to 0+}} t^{-\mu-\frac{1}{2}} S^k_{\mu,t} F_{A,B,C,\nu}(t) = \frac{1}{C_{\mu,\psi}} \lim_{C\to\infty} \int_0^C \int_0^\infty t^{-\mu-\frac{1}{2}} (ty)^{\frac{1}{2}} J_\mu(ty)(xy)^{\frac{1}{2}} J_\mu(xy)$$
$$\times \left(\lim_{\substack{A\to\infty\\ \nu\to 0+}} \int_\nu^A \frac{\left|\left(h_\mu\psi\right)(ay)\right|^2}{a^{2\mu+2} y^{2\mu+1}} da \right) S^k_{\mu,x}\phi(x) dy dx.$$
$$= \frac{1}{C_{\mu,\psi}} \lim_{C\to\infty} \int_0^C \int_0^\infty t^{-\mu-\frac{1}{2}} (ty)^{\frac{1}{2}} J_\mu(ty)(xy)^{\frac{1}{2}} J_\mu(xy)$$
$$\times \left(\int_0^\infty \frac{\left|\left(h_\mu\psi\right)(ay)\right|^2}{a^{2\mu+2} y^{2\mu+1}} da \right) S^k_{\mu,x}\phi(x) dy dx.$$

Using the admissibility condition (1.5.1), $\int_0^\infty \frac{|(h_\mu\psi)(ay)|^2}{a^{2\mu+2}y^{2\mu+1}} da = \int_0^\infty \frac{|(h_\mu\psi)(u)|^2}{u^{2\mu+2}} du = C_{\mu,\psi}$, we get

$$\lim_{\substack{A,B,C\to\infty\\ \nu\to 0+}} t^{-\mu-\frac{1}{2}} S^k_{\mu,t} F_{A,B,C,\nu}(t)$$
$$= \frac{t^{-\mu-\frac{1}{2}}}{C_{\mu,\psi}} C_{\mu,\psi} \lim_{C\to\infty} \int_0^\infty \int_0^\infty (ty)^{\frac{1}{2}} J_\mu(ty)(xy)^{\frac{1}{2}} J_\mu(xy) S^k_{\mu,x}\phi(x) dx dy$$
$$= t^{-\mu-\frac{1}{2}} \int_0^\infty \int_0^\infty (ty)^{\frac{1}{2}} J_\mu(ty)(xy)^{\frac{1}{2}} J_\mu(xy) S^k_{\mu,x}\phi(x) dx dy$$
$$= t^{-\mu-\frac{1}{2}} S^k_{\mu,t}\phi(t).$$

By Weierstrass M-test, the above implies that $t^{-\mu-\frac{1}{2}} S^k_{\mu,t} F_{A,B,C,\nu}(t)$ converges to $t^{-\mu-\frac{1}{2}} S^k_{\mu,t}\phi(t)$ uniformly for all $t \in \mathbb{R}^+$ as $A, B, C \to \infty$ and $\nu \to 0+$ for each $k = 0, 1, 2, \ldots$. □

Lemma 6.3.9. *Let $\phi \in \beta_\mu$ with $\phi(x) = 0,\ x \geq C,\ C > 0$ and $\psi \in \beta_\mu$ be a Bessel wavelet with $\psi(x) = 0$ for $x \geq C_1,\ C_1 > 0$. Then the sequence $\left\{\phi_n(t)\right\}_{n=1}^{\infty}$ defined by*

$$\phi_n(t) = \frac{1}{C_{\mu,\psi}} \int_0^n \int_{1/n}^n \int_0^n \phi(x)\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbdadx}{a^3}, \tag{6.3.18}$$

is in β_μ and $\phi_n(t) \to \phi(t)$ as $n \to \infty$ in the topology of β_μ.

Proof. If we use Lemma 6.3.5 then, we get $\phi_n(t) \in \beta_\mu$ for each $n = 1, 2, 3, \ldots$. It follows that the sequence $\left\{\phi_n(t)\right\}_{n=1}^{\infty}$ is in β_μ.

In (6.3.18), applying (6.3.9) we obtain

$$t^{-\mu-\frac{1}{2}} S^k_{\mu,t}\phi_n(t) = \frac{1}{C_{\mu,\psi}} \int_0^n \int_{1/n}^n \int_0^n t^{-\mu-\frac{1}{2}}\{S^k_{\mu,x}\phi(x)\}\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbdadx}{a^3}.$$

With help of Lemma 6.3.8, we observe that $t^{-\mu-\frac{1}{2}} S^k_{\mu,t}\phi_n(t)$ converges uniformly to $t^{-\mu-\frac{1}{2}} S^k_{\mu,t}\phi(t)$ for all $t \in \mathbb{R}^+$ and for each $k = 0, 1, 2, \ldots$ as $n \to \infty$.

Therefore, the sequence $\phi_n(t) \to \phi(t)$ in the topology of β_μ as $n \to \infty$. □

Remark 6.3.10. In view of Theorem 6.3.2, the Bessel wavelet transform $B_\psi f(b, a)$ is a continuous function of $a, b > 0$. Therefore the integral

$$\int_{1/n}^n \int_0^n (B_\psi f)(b, a)a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}$$

is meaningful. Our main aim is to prove that

$$\frac{1}{C_{\mu,\psi}} \int_{1/n}^n \int_0^n (B_\psi f)(b, a)a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}} \to f \text{ as } n \to \infty$$

in the weak topology of β'_μ.

Or in other words

$$\Big\langle \frac{1}{C_{\mu,\psi}} \int_{1/n}^{n} \int_{0}^{n} (B_\psi f)(b,a) a^{\mu-\frac{1}{2}} \psi\Big(\frac{x}{a},\frac{b}{a}\Big) \frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle \to \langle f,\phi\rangle \text{ as } n\to\infty.$$

Main result of the inversion formula of the continuous Bessel wavelet transform of distributions is:

Theorem 6.3.11. *Let $f \in \beta'_{\mu,F}$ and ψ be a Bessel wavelet in $\beta_\mu^{2m} \subset \beta_\mu$. Then the following inversion formula of the Bessel wavelet transform holds*

$$\Big\langle \frac{1}{C_{\mu,\psi}} \int_{1/n}^{n} \int_{0}^{n} \Big\langle f(t), a^{\mu-\frac{1}{2}} \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}} \psi\Big(\frac{x}{a},\frac{b}{a}\Big) \frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle \to \langle f,\phi\rangle, \quad (6.3.19)$$

as $n\to\infty$, for all $\phi\in\beta_\mu$.

Proof. Taking n sufficiently large, the support of $\phi(x)$ is contained in $[0,n]$.

Therefore, from (6.3.19)

$$\begin{aligned}
&\Big\langle \frac{1}{C_{\mu,\psi}} \int_{1/n}^{n} \int_{0}^{n} \Big\langle f(t), a^{\mu-\frac{1}{2}} \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}} \psi\Big(\frac{x}{a},\frac{b}{a}\Big) \frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle \\
&\qquad = \frac{1}{C_{\mu,\psi}} \int_0^n \int_{1/n}^{n} \int_{0}^{n} \Big\langle f(t), a^{\mu-\frac{1}{2}} \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}} \psi\Big(\frac{x}{a},\frac{b}{a}\Big) \phi(x) \frac{dbdadx}{a^{2\mu+2}} \\
&\qquad = \frac{1}{C_{\mu,\psi}} \int_0^n \int_{1/n}^{n} \int_{0}^{n} \Big\langle f(t), \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle \psi\Big(\frac{x}{a},\frac{b}{a}\Big) \phi(x) \frac{dbdadx}{a^{3}}. \qquad (6.3.20)
\end{aligned}$$

By Theorem 6.3.3, the above yields

$$\begin{aligned}
&\frac{1}{C_{\mu,\psi}} \int_0^n \int_{1/n}^{n} \int_{0}^{n} \Big\langle f(t), \psi\Big(\frac{t}{a},\frac{b}{a}\Big)\Big\rangle \psi\Big(\frac{x}{a},\frac{b}{a}\Big) \phi(x) \frac{dbdadx}{a^{3}} \\
&\qquad = \Big\langle f(t), \frac{1}{C_{\mu,\psi}} \int_0^n \int_{1/n}^{n} \int_{0}^{n} \psi\Big(\frac{t}{a},\frac{b}{a}\Big) \psi\Big(\frac{x}{a},\frac{b}{a}\Big) \phi(x) \frac{dbda}{a^{3}} dx\Big\rangle \\
&\qquad = \langle f(t), \phi_n(t)\rangle, \qquad (6.3.21)
\end{aligned}$$

where

$$\phi_n(t) = \frac{1}{C_{\mu,\psi}} \int_0^n \int_{1/n}^n \int_0^n \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\phi(x)\frac{dbda}{a^3}dx.$$

Exploiting Lemma 6.3.5, the sequence $\Big\{\phi_n(t)\Big\}_{n=1}^{\infty}$ which belongs in β_μ and its each term being zero outside the interval $(0, C + 2C_1)$. Hence

$$\phi_n(t) \to \phi(t) \text{ in } \beta_\mu \text{ as } n \to \infty.$$

With help of (6.3.20) and (6.3.21), we find that

$$\lim_{n\to\infty} \Big\langle \frac{1}{C_{\mu,\psi}} \int_{1/n}^n \int_0^n \Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle$$
$$= \lim_{n\to\infty} \langle f(t), \phi_n(t)\rangle = \langle f(t), \phi(t)\rangle.$$

This implies that

$$\Big\langle \frac{1}{C_{\mu,\psi}} \int_{1/n}^n \int_0^n \Big\langle f(t), a^{\mu-\frac{1}{2}}\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\Big\rangle a^{\mu-\frac{1}{2}}\psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbda}{a^{2\mu+2}}, \phi(x)\Big\rangle \to \langle f, \phi\rangle,$$

as $n \to \infty$. □

Theorem 6.3.12. *Let $f \in \beta'_\mu$ and ψ be a Bessel wavelet belonging to $\beta_\mu^{2m} \subset \beta_\mu$. Then*

$$\Big\langle \frac{1}{C_{\mu,\psi}} \int_{1/n}^n \int_0^n \Big\langle f(t), \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\Big\rangle \psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbda}{a^3}, \phi(x)\Big\rangle \to \langle f - Kx^{\mu+\frac{1}{2}}, \phi\rangle, \quad (6.3.22)$$

as $n \to \infty$, where K is non-zero constant.

Proof. If we take $f \in \beta'_\mu$ then

$$f(x) = g(x) + K.x^{\mu+\frac{1}{2}}, \quad \text{where} \quad g \in \beta'_{\mu,F}. \tag{6.3.23}$$

Using Theorem 6.2.4, we have found that

$$\left\langle K.t^{\mu+\frac{1}{2}}, \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle = \int_0^\infty K.t^{\mu+\frac{1}{2}}\psi\Big(\frac{t}{a}, \frac{b}{a}\Big)dt = 0. \tag{6.3.24}$$

Then with the help of (6.3.23), we get

$$\begin{aligned}
\left\langle f(t), \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle &= \left\langle g(t) + K.t^{\mu+\frac{1}{2}}, \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle \\
&= \left\langle g(t), \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle + \left\langle K.t^{\mu+\frac{1}{2}}, \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle \\
&= \left\langle g(t), \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle. \qquad \left[\text{from}(6.3.24),\ \left\langle K.t^{\mu+\frac{1}{2}}, \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle = 0\right]
\end{aligned}$$

Therefore, from Theorem 6.3.11, we finally find that

$$\begin{aligned}
&\left\langle \frac{1}{C_{\mu,\psi}}\int_{1/n}^{n}\int_0^n \left\langle f(t), \psi\Big(\frac{t}{a}, \frac{b}{a}\Big)\right\rangle \psi\Big(\frac{x}{a}, \frac{b}{a}\Big)\frac{dbda}{a^3}, \phi(x)\right\rangle \\
&\qquad\qquad = \langle g(x), \phi\rangle = \langle f - Kx^{\mu+\frac{1}{2}}, \phi\rangle, \ \text{as } n \to \infty.
\end{aligned}$$

From above arguement, the inversion formula is true for modulo $K.x^{\mu+\frac{1}{2}}$ distributions. But if $f \in \beta'_{\mu,F}$ and $\psi \in \beta_\mu^{2m} \subset \beta_\mu$, the uniqueness theorem holds. □

Example 6.1. *For validation of the inversion formula of continuous Bessel wavelet transform in β'_μ-space, consider*

$$\phi(x) = \begin{cases} e^{-\frac{1}{x(1-x)}}, & \textit{for} \quad 0 < x < 1 \\ 0, & \textit{for} \quad x \leq 0,\ x \geq 1 \end{cases}$$

$\phi \in D(\mathbb{R}^+) \subset \beta_\mu$ for every μ. We take $\psi(x) = M_\mu \phi(x)$, represents a Bessel wavelet in β_μ and containing non-vanishing moments of all even orders weighted by $x^{\mu+\frac{1}{2}}$.

Proof. For $\psi(x) = M_\mu \phi(x)$,

$$\begin{aligned}
\int_0^\infty x^{\mu+\frac{1}{2}} \psi(x) dx &= \int_0^1 x^{\mu+\frac{1}{2}} M_\mu \phi(x) dx \\
&= \int_0^1 x^{\mu+\frac{1}{2}} x^{-\mu-\frac{1}{2}} D_x x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} dx \\
&= \int_0^1 D_x x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} dx = \left\{ x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} \right\} \Big|_{x=0}^{1} = 0.
\end{aligned}$$

By Lemma 6.2.1, we conclude that $\psi(x) \in \beta_\mu$ is a Bessel wavelet.

Now, the moment of $\psi(x)$ of even order weighted by $x^{\mu+\frac{1}{2}}$ be

$$\begin{aligned}
\int_0^\infty x^{2m} x^{\mu+\frac{1}{2}} \psi(x) dx &= \int_0^1 x^{2m} x^{\mu+\frac{1}{2}} x^{-\mu-\frac{1}{2}} D_x x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} dx \\
&= \int_0^1 x^{2m} D_x x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} dx \\
&= \left\{ x^{2m} x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} \right\} \Big|_{x=0}^{1} - \int_0^1 2m x^{2m-1} x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} dx \\
&= -2m \int_0^1 x^{2m-1} x^{\mu+\frac{1}{2}} e^{-\frac{1}{x(1-x)}} dx \neq 0.
\end{aligned}$$

This implies that moments of even orders weighted by $x^{\mu+\frac{1}{2}}$ of $\psi(x)$ are non-zero.

□

6.4 Conclusions

Due to wide acceptability and applicability of continuous wavelet transform and its inversion formula in the classical sense, Holschneider [25], Pathak [46] and others studied the characterization of wavelet transform of Schwartz tempered distributions. Later on, Pandey et al. [36, 38] discussed the inversion formula of Schwartz tempered distributions in $S'(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$. For the Bessel wavelet transform of Zemanian distributions concern, author investigated Bessel wavelet transform of distributions in $H'_\mu(\mathbb{R}^+)$ and found its inversion formula in previous chapters. Keeping the views of these researches and seeing the vision of the development of future research works, in the present chapter, author developed the inversion formula of the continuous Bessel wavelet transform on $\beta'_{\mu,F}$ distributions and discussed their auxiliary results by exploiting the theory of Hankel transform and Hankel convolution where $\beta'_{\mu,F}$ is the dual of β_μ, which is filtered by $C.x^{\mu+\frac{1}{2}}$ type distribution. Hankel transform is frequently applied in the problems of image processing, signal processing. It can be seen from papers of [23, 26, 27, 48]. These applications of Hankel transform in image processing and signal processing can be extended in the problems of the Bessel wavelet transform and the Bessel wavelet transform of distributions.

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