Chapter 5

Numerical Wavelets Scheme for Complex Partial Differential Equation Arising from Morlet Continuous Wavelet Transform

5.1 Introduction

Numerical methods were first put into use as an effective tool for solving partial differential equations (PDEs) by John Von Neumann in the mid-1940s. In a 1949 letter von Neumann wrote " the entire computing machine is merely one component of a greater whole, namely, of the unity formed by the computing machine, the mathematical problems that go with it, and the type of planning which is called by both". The "greater whole" is viewed today as scientific computation: over the past sixty five years, scientific computation has emerged as the most versatile tool to complement theory and experiments, and numerical methods for solving partial differential equations (PDEs) are at the heart of many of today's advanced scientific computations. Here we provide a bird's eye view on the development of these numerical methods, with a particular emphasis on linear complex partial differential equation (CPDE) arising from continuous Morlet wavelet transform.

Among the kinds of Integral transform, the continuous wavelet transform is remarkable as a powerful modern tool for analyzing time series, signal and images ([199]-[200]). A continuous Morlet wavelet transform analyzed the images obtained during the second half of 2004 by Cassini interplanetary spacecraft [201]. The technique of investigation is based on the original algorithm of a continuous Morlet wavelet transform that reduces the integral transform to solving CPDE (Cauchy problem [202]-[203]). A relationship was recognized between PDEs approach and the real continuous wavelet transform by [204] where continuous wavelet transform as follows:

$$w(a,b) = \int_{-\infty}^{\infty} f(t)\psi^* \left(\frac{t-b}{a}\right) \frac{dt}{a}. \tag{5.1}$$

In [205], the equation

$$\begin{gathered} \frac{\partial w}{\partial a} = a\frac{\partial^2 w}{\partial b^2} - i\omega_0 \frac{\partial w}{\partial b}, \\ \text{with initial condition}, w(0,b) = w_0(b) \end{gathered} \tag{5.2}$$

was given, which is satisfied by complex continuous wavelet transform $Eq.(5.1)$ based on Morlet wavelet. The main purpose of this chapter is to construct an algorithm for complex partial differential equation (CPDE) $Eq.(5.2)$ and demonstrate its applicability via two dimensional Legendre and Bernoulli wavelets based on operational and almost operational matrices.

A most efficient method for solving the problem of differentiating the instant period in signal is the complex continuous wavelet transform. Therefore, in this chapter, we propose a problem based on the observation that the transformation obtained by the convolution with the complex continuous Morlet wavelet transform satisfied the proposed problem $Eq.(5.2)$ can be rewritten as the system

$$\frac{\partial u}{\partial a} = a\frac{\partial^2 u}{\partial b^2} + \omega_0 \frac{\partial v}{\partial b}, \tag{5.3}$$

$$\frac{\partial v}{\partial a} = a\frac{\partial^2 v}{\partial b^2} - \omega_0 \frac{\partial u}{\partial b}, \tag{5.4}$$

where, $w(a,b) = u(a,b) + iv(a,b)$ is wavelet transform, u ans v are the real and imaginary parts of the complex function respectively. Such an approach has been proposed for a first time in [202]. Later, it was generalized on the cases of the exact(admissible) Morlet wavelet [203] and two-dimensional CWT [206]. However, although PDE-based method provides a better accuracy in comparison with the conventional methods of CWT calculations (say, FFT-based), the standard numerical techniques for PDE (e.g. via finite differences) are, as a rule, sufficiently slow.

Thus, there emerges the idea to solve $Eqs.(5.3) - (5.4)$ via alternative methods, e.g. by the operational matrix technique ([207],[208]). This system of PDE can be twice integrated with respect to a that will result in a system of PIDEs. In addition, this representation allows for exploring of the influence of boundary conditions on the proposed problem. Finally, the integro-differential approach provides an opportunity to take into account various kinds of sophisticated boundary conditions up to nonlocal as follows:

$$u(1,b) = \int_0^1 K_1(a,b)u(a,b)da, \tag{5.5}$$

$$v(1,b) = \int_0^1 K_2(a,b)v(a,b)da. \tag{5.6}$$

Nowadays, Wavelets are powerful tool which have been used in numerical techniques. Wavelets theory are mostly used in the areas of applied engineering and sciences. Also, this allowed the accurate representation of a variety of functions and operators. Recently, two dimensional wavelets have been found their location in many applications (For example: Legendre wavelet[209], Bernoulli wavelet [210], Chebyshev wavelet [209],[37]). Wavelet transform in two-dimension with Morlet wavelet introduced almost 34 year ago as an effective tool for image processing [211] and particularly as a local spectral decomposition [212] has found applications in a various field of research and classical methods of the wavelet transform numerically deals based on Fast Fourier transform as an intermediate step and application of filter banks [213].

5.1.1 Function approximation

Suppose that $f(a,b)$ is an arbitrary function in complete space $L^2(\Omega)$, then it can be approximated as follows:

$$F(a,b) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} \sum_{n'=1}^{2^{k'}-1} \sum_{m'=0}^{\infty} F_{nmn'm'} \Phi_{nmn'm'}(a,b) = F^T \Phi(a,b), \tag{5.7}$$

and its truncation series as follows

$$F(a,b) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{k'}-1} \sum_{m'=0}^{M'-1} F_{nmn'm'} \Phi_{nmn'm'}(a,b) = F^T \Phi(a,b). \tag{5.8}$$

Where, $Fnmn'm' =< F(a,b), \Phi_n mn'm' >$ and F, Φ are $2^{k-1}2^{k'-1}MM' \times 1$, vector given as follows:

$$\begin{aligned}
F = [f_{1,0,1,0}, \cdots, f_{1,0,1,M'-1}, f_{1,0,2,0} \cdots, f_{1,0,2,M'-1}, \cdots, f_{1,0,2^{k'-1},0}, \cdots, \\
f_{1,0,2^{k'-1},M'-1}, \cdots, f_{1,M-1,1,0}, \cdots, f_{1,M-1,1,M'-1}, f_{1,M-1,2,0}, \cdots, f_{1,M-1,2,M'-1}, \\
f_{1,M-1,2^{k'-1},0} \cdots, f_{1,M-1,2^{k'-1},M'-1}, f_{2,0,1,0}, \cdots, f_{2,0,1,M'-1}, f_{2,0,2,0}, \cdots, \\
f_{2,0,2,M'-1}, \cdots, f_{2,0,2^{k'-1},0}, \cdots, f_{2,0,2^{k'-1},M'-1}, \cdots, f_{2,M-1,1,0}, \cdots, f_{2,M-1,1,M'-1}, \\
f_{2,M-1,2,0}, \cdots, f_{2,M-1,2,M'-1}, f_{2,M-1,2^{k'-1},0} \cdots, f_{2,M-1,2^{k'-1},M'-1}, \cdots, \\
f_{2^{k-1},0,1,0}, \cdots, f_{2^{k-1},0,1,M'-1}, f_{2^{k-1},0,2,0}, \cdots, f_{2^{k-1},M-1,2^{k'-1},M'-1}]^T,
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\Phi = [&\Phi_{1,0,1,0}, \cdots, \Phi_{1,0,1,M'-1}, \Phi_{1,0,2,0} \cdots, \Phi_{1,0,2,M'-1}, \cdots, \Phi_{1,0,2^{k'-1},0}, \cdots,\\ &\Phi_{1,0,2^{k'-1},M'-1}, \cdots, \Phi_{1,M-1,1,0}, \cdots, \Phi_{1,M-1,1,M'-1}, \Phi_{1,M-1,2,0}, \cdots, \Phi_{1,M-1,2,M'-1}\\ &\Phi_{1,M-1,2^{k'-1},0} \cdots, \Phi_{1,M-1,2^{k'-1},M'-1}, \Phi_{2,0,1,0}, \cdots, \Phi_{2,0,1,M'-1}, \Phi_{2,0,2,0} \cdots\\ &\Phi_{2,0,2,M'-1}, \cdots, \Phi_{2,0,2^{k'-1},0}, \cdots, \Phi_{2,0,2^{k'-1},M'-1}, \cdots, \Phi_{2,M-1,1,0}, \cdots, \Phi_{2,M-1,1,M'-1},\\ &\Phi_{2,M-1,2,0}, \cdots, \Phi_{2,M-1,2,M'-1} \Phi_{2,M-1,2^{k'-1},0} \cdots, \Phi_{2,M-1,2^{k'-1},M'-1}, \cdots, \Phi_{2^{k-1},0,1,0}, \cdots,\\ &\Phi_{2^{k-1},0,1,M'-1}, \cdots, \Phi_{2^{k-1},0,2^{k'-1},0}, \cdots, \Phi_{2^{k-1},0,2^{k'-1},0}, \cdots, \Phi_{2^{k-1},M-1,2^{k'-1},M'-1}]^T.\end{aligned} \tag{5.10}$$

5.2 Operational matrices

5.2.1 Bernoulli wavelet operational matrix of differentiation with respect to variable a

Let $\Phi^B(a,b)$ be two-dimensional Bernoulli wavelet defined in $Eq.$(1.16) then derivative matrix as follows:

$$\begin{aligned}\frac{\partial}{\partial a}\Phi^B(a,b) &= \frac{\partial}{\partial a}(\Phi^B(a) \otimes \Phi^B(b))\\ &= (D^B\Phi^B(a)) \otimes (I\Phi^B(b))\\ &= (D^B \otimes I)(\Phi^B(a) \otimes \Phi^B(b))\\ &= D_a^B\Phi^B(a,b),\end{aligned}$$

so,

$$\frac{\partial}{\partial a}\Phi^B(a,b) = D_a^B\Phi^B(a,b), \tag{5.11}$$

where, $D_a^B = D^B \otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$ (see [38]) and also I is the identity matrix.

5.2.2 Bernoulli wavelet operational matrix of differentiation with respect to variable b

Let $\Phi^B(a,b)$ be two-dimensional Bernoulli wavelet defined in $Eq.$(1.16) then derivative matrix as follows:

$$\frac{\partial}{\partial b}\Phi^B(a,b) = \frac{\partial}{\partial b}(\Phi^B(a)\otimes\Phi^B(b))$$

$$= \Phi^B(a)\otimes\frac{\partial}{\partial b}\Phi^B(b)$$

$$= (I\Phi^B(a))\otimes(D^B\Phi^B(b))$$

$$= (I\otimes D^B)(\Phi^B(a)\otimes\Phi^B(b))$$

$$= D_b^B\Phi^B(a,b),$$

so,

$$\frac{\partial}{\partial b}\Phi^B(a,b) = D_b^B\Phi^B(a,b), \tag{5.12}$$

where, $D_b^B = I\otimes D^B$ is a $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ matrix (see [38]) and also I is the identity matrix.

5.2.3 Legendre wavelet operational matrix of differentiation with respect to variable a

Let $\Phi^L(a,b)$ be two-dimensional Legendre wavelet defined in $Eq.$(1.8) then derivative matrix as follows:

$$\frac{\partial}{\partial a}\Phi^L(a,b) = \frac{\partial}{\partial a}(\Phi^L(t)\otimes\Phi^L(b))$$

$$= (D^L\Phi^L(a))\otimes(I\Phi^L(b))$$

$$= (D^L \otimes I)(\Phi^L(a) \otimes \Phi^L(b))$$

$$= D_a^L \Phi^L(a,b),$$

so,

$$\frac{\partial}{\partial a}\Phi^L(a,b) = D_a^L \Phi^L(a,b), \tag{5.13}$$

where, $D_a^L = D^L \otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$ (see [112]) and also I is the identity matrix.

5.2.4 Legendre wavelet operational matrix of differentiation with respect to variable b

Let $\Phi^L(a,b)$ be two-dimensional Legendre wavelet defined in $Eq.$(1.8) then dreivative matrix as follows:

$$\frac{\partial}{\partial b}\Phi^L(a,b) = \frac{\partial}{\partial b}(\Phi^L(a) \otimes \Phi^L(b))$$

$$= \Phi^L(a) \otimes \frac{\partial}{\partial b}\Phi^L(b)$$

$$= (I\Phi^L(a)) \otimes (D^L\Phi^L(b))$$

$$= (I \otimes D^L)(\Phi^L(a) \otimes \Phi^L(b))$$

$$= D_b^L \Phi^L(a,b),$$

so,

$$\frac{\partial}{\partial b}\Phi^L(a,b) = D_b^L \Phi^L(a,b), \tag{5.14}$$

where, $D_b^L = I \otimes D^L$ is a $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ matrix (see [112]) and also I is the identity matrix.

5.3 Numerical solution of the problem

Let us consider $Eq.$(5.3), as follows

$$\left(\frac{\partial}{\partial a} - a\frac{\partial^2}{\partial b^2}\right) u - w_0 \left(\frac{\partial}{\partial b}\right) v = 0, \tag{5.15}$$

and $Eq.$(5.4), as follows

$$\left(\frac{\partial}{\partial a} - a\frac{\partial^2}{\partial b^2}\right) v + w_0 \left(\frac{\partial}{\partial b}\right) u = 0. \tag{5.16}$$

Multiplying $Eq.$(5.15) by $\left(\frac{\partial}{\partial a} - a\frac{\partial^2}{\partial b^2}\right)$ and $Eq.$(5.16) by $\omega_0(\frac{\partial}{\partial b})$ and then adding, we get

$$\left(\frac{\partial}{\partial a} - a\frac{\partial^2}{\partial b^2}\right)\left(\frac{\partial}{\partial a} - a\frac{\partial^2}{\partial b^2}\right) u + \omega_0^2 \left(\frac{\partial^2}{\partial b^2}\right) u = 0,$$

this implies,

$$\frac{\partial^2 u}{\partial a^2} + (\omega_0^2 - 1)\frac{\partial^2 u}{\partial b^2} - a\frac{\partial^3 u}{\partial a \partial b^2} - a\frac{\partial^2 u}{\partial b^2 \partial a} + a^2\frac{\partial^4 u}{\partial b^4} = 0. \tag{5.17}$$

Since u is continuous then $\frac{\partial^3 u}{\partial a \partial b^2} = \frac{\partial^3 u}{\partial b^2 \partial a}$ and then $Eq.$(5.17) become as:

$$\frac{\partial^2 u}{\partial a^2} + (\omega_0^2 - 1)\frac{\partial^2 u}{\partial b^2} - 2a\frac{\partial^3 u}{\partial a \partial b^2} + a^2\frac{\partial^4 u}{\partial b^4} = 0. \tag{5.18}$$

Now, we multiply $Eq.$(5.15) by $\omega_0(\frac{\partial}{\partial b})$ and $Eq.$(5.16) by $\left(\frac{\partial}{\partial a} - a\frac{\partial^2}{\partial b^2}\right)$ and then after similar process as finding $Eq.$(5.18) we get

$$\frac{\partial^2 v}{\partial a^2} + (\omega_0^2 - 1)\frac{\partial^2 v}{\partial b^2} - 2a\frac{\partial^3 v}{\partial a \partial b^2} + a^2\frac{\partial^4 v}{\partial b^4} = 0. \tag{5.19}$$

Integrating $Eq.$(5.18) with respect to a, we get

$$u_a(a,b) - u_a(0,b) = 2\int_0^a r\frac{\partial^3 u}{\partial r \partial b^2} dr - (\omega_0^2 - 1)\int_0^a \frac{\partial^2 u}{\partial b^2} dr - \int_0^a r^2\frac{\partial^4 u}{\partial b^4} dr. \tag{5.20}$$

Again, integrating $Eq.(5.20)$ with respect to a, we get

$$\begin{aligned} u(a,b)-u(0,b)-au_a(0,b) = 2\int_0^a\left(\int_0^s r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)ds-(\omega_0^2-1) \\ \times\int_0^a\left(\int_0^s \frac{\partial^2 u}{\partial b^2}dr\right)ds-\int_0^a\left(\int_0^s r^2\frac{\partial^4 u}{\partial b^4}dr\right)ds, \end{aligned} \tag{5.21}$$

this implies,

$$\begin{aligned} u(a,b) = u(0,b)+au_a(0,b)+2\int_0^a\left(\int_0^s r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)ds-(\omega_0^2-1) \\ \times\int_0^a\left(\int_0^s \frac{\partial^2 u}{\partial b^2}dr\right)ds-\int_0^a\left(\int_0^s r^2\frac{\partial^4 u}{\partial b^4}dr\right)ds. \end{aligned} \tag{5.22}$$

Now, putting $a=1$ in $Eq.(5.22)$ we get

$$\begin{aligned} u(1,b) = u(0,b)+u_a(0,b)+2\int_0^1\left(\int_0^a r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)da-(\omega_0^2-1) \\ \times\int_0^1\left(\int_0^a \frac{\partial^2 u}{\partial b^2}dr\right)da-\int_0^1\left(\int_0^a r^2\frac{\partial^4 u}{\partial b^4}dr\right)da, \end{aligned} \tag{5.23}$$

this implies,

$$\begin{aligned} u_a(0,b) = u(1,b)-u(0,b)-2\int_0^1\left(\int_0^a r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)da+(\omega_0^2-1) \\ \times\int_0^1\left(\int_0^a \frac{\partial^2 u}{\partial b^2}dr\right)da+\int_0^1\left(\int_0^a r^2\frac{\partial^4 u}{\partial b^4}dr\right)da. \end{aligned} \tag{5.24}$$

Putting the value of $u_a(0,b)$ in $Eq.(5.22)$, we get

$$\begin{aligned} u(a,b) = u(0,b)+au(1,b)-au(0,b)-2a\int_0^1\left(\int_0^a r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)da \\ +(\omega_0^2-1)a\int_0^1\left(\int_0^a \frac{\partial^2 u}{\partial b^2}dr\right)da+a\int_0^1\left(\int_0^a r^2\frac{\partial^4 u}{\partial b^4}dr\right)da \\ +2\int_0^a\left(\int_0^s r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)ds-(\omega_0^2-1)\int_0^a\left(\int_0^s \frac{\partial^2 u}{\partial b^2}dr\right)ds \\ -\int_0^a\left(\int_0^s r^2\frac{\partial^4 u}{\partial b^4}dr\right)ds \end{aligned} \tag{5.25}$$

or,

$$\begin{aligned} u(a,b) = g_1(a,b) + h_1(a,b) - 2a\int_0^1\left(\int_0^a r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)da \\ +(\omega_0^2-1)a\int_0^1\left(\int_0^a \frac{\partial^2 u}{\partial b^2}dr\right)da + a\int_0^1\left(\int_0^a r^2\frac{\partial^4 u}{\partial b^4}dr\right)da \\ +2\int_0^a\left(\int_0^s r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)ds - (\omega_0^2-1)\int_0^a\left(\int_0^s \frac{\partial^2 u}{\partial b^2}dr\right)ds \\ -\int_0^a\left(\int_0^s r^2\frac{\partial^4 u}{\partial b^4}dr\right)ds, \end{aligned} \tag{5.26}$$

where, $g_1(a,b) = (1-a)u(0,b)$ and $h_1(a,b) = au(1,b)$.

Now, we approximate the functions that satisfy $Eq.(5.26)$ using the Bernoulli and Legendre wavelets given by $Eq.(5.8)$ and $Eq.(5.14)$ as follows:

$$\begin{aligned} u(a,b) &= C_1^T\Phi(a,b) = C_1^T\Phi(a)\Phi(b), \\ g_1(a,b) &= A_1^T\Phi(a,b) = A_1^T\Phi(a)\Phi(b), \\ h_1(a,b) = a\int_0^1 K_1(a,b)u(a,b)da &= C_1^T U_1\Phi(a,b) = C_1^T U_1\Phi(a)\Phi(b), \\ a\int_0^1\left(\int_0^a r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)da &= C_1^T\left(D_b\right)^2 D_a L_{r1}\Phi(a)\Phi(b), \\ a\int_0^1\left(\int_0^a \frac{\partial^2 u}{\partial b^2}dr\right)da &= C_1^T\left(D_b\right)^2 L_1\Phi(a)\Phi(b), \\ a\int_0^1\left(\int_0^a r^2\frac{\partial^4 u}{\partial b^4}dr\right)da &= C_1^T\left(D_b\right)^4 L_{r^21}\Phi(a)\Phi(b), \\ \int_0^a\left(\int_0^s r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)ds &= C_1^T\left(D_b\right)^2 D_a^T Q_{ra}\Phi(a)\Phi(b), \\ \int_0^a\left(\int_0^s \frac{\partial^2 u}{\partial b^2}dr\right)ds &= C_1^T\left(D_b\right)^2 Q_a\Phi(a)\Phi(b), \\ \int_0^a\left(\int_0^s r^2\frac{\partial^4 u}{\partial b^4}dr\right)ds &= C^T\left(D_b\right)^4 Q_{r^2a}\Phi(a)\Phi(b), \end{aligned} \tag{5.27}$$

where, C_1^T,A_1^T and B_1^T are $1\times 2^{k-1}2^{k'-1}MM'$ matrix. Also, L_{r1}, L_1, L_{r^21}, Q_{ra}, Q_a and Q_{r^2a} (see in Appendix subsections 6.3.1-6.3.6) are matrices of order $2^{k-1}2^{k'-1}MM'$. Here, C_1^T is the only unknown matrix and the rest of the matrices are known.

Representing the $Eq.(5.26)$ via expansion

Table 5.1: List of denotation

General symbol	**Uses of LWA**	**Use of BWA**
$\Phi(t,x)$	$\Phi^L(t,x)$	$\Phi^B(t,x)$
D	D^L	D^B
D_a	D_a^L	D_a^B
D_b	D_b^L	D_b^B
L_1	L_1^L	L_1^B
L_{r1}	L_{r1}^L	L_{r1}^B
L_{r^21}	$L_{r^21}^L$	$L_{r^21}^B$
Q_a	Q_a^L	Q_a^B
Q_{ra}	Q_{ra}^L	Q_{ra}^B
Q_{r^2a}	$Q_{r^2a}^L$	$Q_{r^2a}^B$

So, using approximation $Eq.(5.27)$ in $Eq.(5.26)$, we have:

$$\begin{aligned} C_1^T\Phi(a)\Phi(b) = [A_1^T + C_1^T U_1 - 2C_1^T\left(D_b\right)^2 D_a L_{r1} + (\omega_0^2+1)C_1^T\left(D_b\right)^2 L_1 \\ +C_1^T\left(D_b\right)^4 L_{r^21} + 2C_1^T\left(D_b\right)^2 D_a Q_{ra} - (\omega_0^2+1)C_1^T\left(D_b\right)^2 Q_a \\ -C_1^T\left(D_b\right)^4 Q_{r^2a}]\Phi(a)\Phi(b), \end{aligned} \quad (5.28)$$

and so by $Eq.(5.28)$ we obtained the set of algebraic equations as follows:

$$\begin{aligned} C_1^T = A_1^T[-U_1 + 2\left(D_b\right)^2 D_a L_{r1} - (\omega_0^2+1)\left(D_b\right)^2 L_1 - \left(D_b\right)^4 L_{r^21} \\ -2\left(D_b\right)^2 D_a Q_{ra} + (\omega_0^2+1)\left(D_b\right)^2 Q_a + \left(D_b\right)^4 Q_{r^2a}]^{-1}, \end{aligned} \quad (5.29)$$

and hence from $Eqs.(5.27)$ and (5.29),

$$\begin{aligned} u(a,b) = A_1^T[-U_1 + 2\left(D_b\right)^2 D_a L_{r1} - (\omega_0^2+1)\left(D_b\right)^2 L_1 - \left(D_b\right)^4 L_{r^21} \\ -2\left(D_b\right)^2 D_a Q_{ra} + (\omega_0^2+1)\left(D_b\right)^2 Q_a + \left(D_b\right)^4 Q_{r^2a}]^{-1}\Phi(a,b). \end{aligned} \quad (5.30)$$

Now, we use the collocation method for solving $Eq.(5.30)$. For this, we suppose $t = \{t_j\}_{j=1}^N = \frac{j}{N}$ and $x = \{x_h\}_{h=1}^N = \frac{h}{N}$ are the set of (N) nodes. We substitute these nodes in $Eq.(5.30)$ and then we find the numerical solution of $Eq.(5.30)$ and hence real part of w solved numerically.

Now, we are computing imaginary parts of w. For this we are integrate $Eq.(5.19)$ twice with respect to a and using similar process as we have done for $u(a,b)$, we get

$$\begin{aligned} v(a,b) = v(0,b) + av_a(0,b) + 2\int_0^a \left(\int_0^s r\frac{\partial^3 v}{\partial r\partial b^2}dr\right)ds - (\omega_0^2-1) \\ \times \int_0^a \left(\int_0^s \frac{\partial^2 v}{\partial b^2}dr\right)ds - \int_0^a \left(\int_0^s r^2\frac{\partial^4 v}{\partial b^4}dr\right)ds. \end{aligned} \quad (5.31)$$

Putting $a = 1$ in $Eq.(5.32)$,

$$\begin{aligned} v(1,b) = v(0,b) + v_a(0,b) + 2\int_0^1 \left(\int_0^a r\frac{\partial^3 u}{\partial r\partial b^2}dr\right)da - (\omega_0^2-1) \\ \times \int_0^1 \left(\int_0^a \frac{\partial^2 u}{\partial b^2}dr\right)da - \int_0^1 \left(\int_0^a r^2\frac{\partial^4 u}{\partial b^4}dr\right)da, \end{aligned}$$

this implies,

$$v_a(0,b) = v(1,b) - v(0,b) - 2\int_0^1 \left(\int_0^a r\frac{\partial^3 v}{\partial r\partial b^2}dr\right) da + (\omega_0^2 - 1) \times \int_0^1 \left(\int_0^a \frac{\partial^2 v}{\partial b^2}dr\right) da + \int_0^1 \left(\int_0^a r^2\frac{\partial^4 v}{\partial b^4}dr\right) da. \tag{5.32}$$

Grouping $Eqs.$(5.31) and (5.32) as follows:

$$v(a,b) = g_2(a,b) + h_2(a,b) - 2a\int_0^1 \left(\int_0^a r\frac{\partial^3 v}{\partial r\partial b^2}dr\right) da + (\omega_0^2 - 1) \times a\int_0^1 \left(\int_0^a \frac{\partial^2 v}{\partial b^2}dr\right) da + a\int_0^1 \left(\int_0^a r^2\frac{\partial^4 v}{\partial b^4}dr\right) da + 2\int_0^a \left(\int_0^s r\frac{\partial^3 v}{\partial r\partial b^2}dr\right) ds - (\omega_0^2 - 1)\int_0^a \left(\int_0^s \frac{\partial^2 v}{\partial b^2}dr\right) ds - \int_0^a \left(\int_0^s r^2\frac{\partial^4 v}{\partial b^4}dr\right) ds, \tag{5.33}$$

where, $g_2(a,b) = v(0,b)(1-a)$ and $h_2(a,b) = av(1,b)$.

So, for the solution of $Eq.$(5.33) we starts approximation using operational matrices as follows:

$$\begin{aligned} v(a,b) &= C_2^T\Phi(a,b) = C_2^T\Phi(a)\Phi(b), \\ g_2(a,b) &= A_2^T\Phi(a,b) = A_2^T\Phi(a)\Phi(b), \\ h_2(a,b) = a\int_0^1 K_1(a,b)v(a,b)da &= C_2^T U_2\Phi(a,b) = C_2^T U_2\Phi(a)\Phi(b), \end{aligned} \tag{5.34}$$

where, C_2^T,A_2^T and B_2^T are $1\times 2^{k-1}2^{k'-1}MM'$ matrix. Here, C_2^T is the only unknown matrix and the rest of the matrices are known. So, using approximation $Eq.$(5.34), we have:

$$\begin{aligned} C_2^T\Phi(a)\Phi(b) = A_2^T\Phi(a)\Phi(b) + C_2^T U_2\Phi(a)\Phi(b) - 2C_2^T (D_b)^2 D_a L_{r1}\Phi(a)\Phi(b) \\ +(\omega_0^2 - 1)C_2^T (D_b)^2 L_1\Phi(a)\Phi(b) + C_2^T (D_b)^4 L_{r^21}\Phi(a)\Phi(b) \\ +2C_2^T (D_b)^2 D_a Q_{ra}\Phi(a)\Phi(b) - (\omega_0^2 - 1)C_2^T (D_b)^2 Q_a\Phi(a)\Phi(b) \\ -C_2^T (D_b)^4 Q_{r^2a}\Phi(a)\Phi(b), \end{aligned} \tag{5.35}$$

this implies,

$$\begin{aligned} C_2^T\Phi(a)\Phi(b) = [A_2^T + C_2^T U_2 - 2C_2^T\left(D_b\right)^2 D_a L_{r1} + (\omega_0^2-1)C_2^T\left(D_b\right)^2 L_1 \\ +C_2^T\left(D_b\right)^4 L_{r^21} + 2C_2^T\left(D_b\right)^2 D_a Q_{ra} - (\omega_0^2-1)C_2^T\left(D_b\right)^2 Q_a \\ -C_2^T\left(D_b\right)^4 Q_{r^2a}]\Phi(a)\Phi(b), \end{aligned} \tag{5.36}$$

and so by $Eq.(5.37)$ we obtained the set of algebraic equations as follows:

$$\begin{aligned} C_2^T = A_2^T[-U_2 + 2\left(D_b\right)^2 D_a L_{r1} - (\omega_0^2-1)\left(D_b\right)^2 L_1 - \left(D_b\right)^4 L_{r^21} \\ +2C_2^T\left(D_b^T\right)^2 D_a Q_{ra} + (\omega_0^2-1)\left(D_b\right)^2 Q_a + \left(D_b\right)^4 Q_{r^2a}]^{-1}. \end{aligned} \tag{5.37}$$

Hence, using $Eq.(5.34)$ and $Eq.(5.37)$as follows:

$$\begin{aligned} v(a,b) = A_2^T[-U_2 + 2\left(D_b\right)^2 D_a L_{r1} - (\omega_0^2-1)\left(D_b\right)^2 L_1 - \left(D_b\right)^4 L_{r^21} \\ +2C_2^T\left(D_b^T\right)^2 D_a Q_{ra} + (\omega_0^2-1)\left(D_b\right)^2 Q_a + \left(D_b\right)^4 Q_{r^2a}]^{-1}\Phi(a,b) \end{aligned} \tag{5.38}$$

Now, we use the collocation method for solving $Eq.(5.38)$. For this, we suppose $t = \{t_i\}_{i=0}^N = \frac{i}{N}$ and $x = \{x_j\}_{j=0}^N = \frac{j}{N}$ are the set of $(N+1)$ nodes. We substitute these nodes in $Eq.(5.38)$ and then we find the numerical solution of $Eq.(5.38)$ and hence imaginary part of w solved numerically.

Since,

$$w(a,b) = u(a,b) + iv(a,b) \tag{5.39}$$

so, the modulus of the wavelet transform required for our analysis can be easily calculated as follows:

$$|w(a,b)| = \sqrt{u^2(a,b) + v^2(a,b)}. \tag{5.40}$$

5.4 Convergence analysis

5.4.1 For Legendre wavelet

Theorem 5.4.1 *For $\tau > 0$, let $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)_N$ be the LWA of $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)$ and assume that the mixed second derivative of $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)$ is bounded by a constant T_1.i.e. $\mid \frac{\partial^{\tau+4} u(t,x)}{\partial t^2 \partial x^{\tau+2}} \mid < T_1$, then we have the following upper bound of error:*

$$\parallel \left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right) - \left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)_N \parallel_{L^2}^2 < \frac{T_1^2 \wp^2}{2^{18}},$$

where, $\wp$ is a polygamma function ,$\wp = F_3(-3/2 + N)$.

Proof See [209].

Lemma 5.4.2 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $(u_{xx})_N(t,x)$ be the LWA of $u_{xx}(t,x)$. Assume that the mixed second derivative of $u(t,x)$ is bounded by a constant G_1.*

i.e.

$$|(\frac{\partial u^6(t,x)}{\partial t^2 \partial x^4})| < G_1,$$

then ,we have the following upper bound of error :

$$\parallel u_{xx} - (u_{xx})_N \parallel < \frac{G_1^2 \wp^2}{2^{18}},$$

where, $\wp = F_3(\frac{-3}{2} + N)$.

Proof See [209].

Lemma 5.4.3 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $(u_{tt})_N(t,x)$ be the LWA of $u_{tt}(t,x)$. Assume that the mixed second derivative of $u(t,x)$ is bounded by a constant G_2. i.e.*

$$|(\frac{\partial u^6(t,x)}{\partial t^4 \partial x^2})| < G_2,$$

then ,we have the following upper bound of error :

$$\| u_{tt} - (u_{tt})_N \|_{L^2}^2 < \frac{G_2^2 \wp^2}{2^{18}},$$

where, $\wp = F_3(\frac{-3}{2} + N)$.

Proof See [209].

5.4.2 For Bernoulli wavelet

Theorem 5.4.4 *Suppose that the function* $f : [0,1] \times [0,1] \to \boldsymbol{R}$ *are is* N *times continuously differentiable and* $C^T\psi(a,b)$ *be Bernoulli wavelet approximation of* $f(a,b)$. *Then mean error bound as follows:*

$$\left\| f(a,b) - C^T\psi(a,b) \right\| \leq [\frac{1}{(N!)^2 2^{2N(K-1)}} S]^2.$$

Proof Firstly, we divided the interval $[0,1]$ into subinterval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$,$[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}]$ and suppose $C^T\psi(a,b)$ is a Bernoulli wavelet of degree N ($N = max\{m, m'\}$) that approximate $f(a,b)$ with minimum mean error. Then, we used the maximum error estimation for the polynomial which interpolates $f(a,b)$ of order N on $[\frac{n-1}{2^{K-1}}, \frac{n}{2^{K-1}}]$ and $[\frac{n'-1}{2^{K-1}}, \frac{n'}{2^{K-1}}]$, where, $K = min\{k, k'\}$, We have,

$$\left\|f(a,b) - C^T\psi(a,b)\right\| = \int_0^1\int_0^1 \left|f(a,b) - C^T\psi(a,b)\right|^2 dadb$$
$$= \sum_{n=0}^{2^{K-1}} \int_{\frac{n-1}{2^{K-1}}}^{\frac{n}{2^{K-1}}} \sum_{n'=0}^{2^{K-1}} \int_{\frac{n'-1}{2^{K-1}}}^{\frac{n'}{2^{K-1}}} \left|f(a,b) - C^T\psi(a,b)\right|^2 dadb,$$
$$= \sum_{n=0}^{2^{K-1}} \int_{\frac{n-1}{2^{K-1}}}^{\frac{n}{2^{K-1}}} \sum_{n'=0}^{2^{K-1}} \int_{\frac{n'-1}{2^{K-1}}}^{\frac{n'}{2^{K-1}}} \left|f(a,b) - f^*(a,b)\right|^2 dadb,$$
$$= \sum_{n=0}^{2^{K-1}} \int_{\frac{n-1}{2^{K-1}}}^{\frac{n}{2^{K-1}}} \sum_{n'=0}^{2^{K-1}} \int_{\frac{n'-1}{2^{K-1}}}^{\frac{n'}{2^{K-1}}} [\frac{1}{(N!)^2 2^{2N(K-1)}}$$
$$\sup_{a\in[0,1],b\in[0,1]}((a-x_0)\frac{\partial}{\partial a} + (b-y_0)\frac{\partial}{\partial b})^N$$
$$f(2^{K-1}(x_0 + (a-x_0)\theta) - n + 1)$$
$$f(2^{K-1}(y_0 + (b-y_0)\theta) - n' + 1)))]^2 dadb$$
$$= \sum_{n=0}^{2^{K-1}} \int_{\frac{n-1}{2^{K-1}}}^{\frac{n}{2^{K-1}}} \sum_{n'=0}^{2^{K-1}} \int_{\frac{n'-1}{2^{K-1}}}^{\frac{n'}{2^{K-1}}} [\frac{1}{(N!)^2 2^{2N(K-1)}} S]^2 dadb$$
$$\leq [\frac{1}{(N!)^2 2^{2N(K-1)}} S]^2.$$

i.e.

$$\left\|f(a,b) - C^T\psi(a,b)\right\| \leq [\frac{1}{(N!)^2 2^{2N(K-1)}} S]^2.$$

Here, $f^*(a,b)$ denotes the interpolating polynomial (see [214]) of $f(a,b)$ and we have used the well known maximum error bound for interpolation and $S = \sup_{a\in[0,1],b\in[0,1]}((a-x_0)\frac{\partial}{\partial a} + (b-y_0)\frac{\partial}{\partial b})^N f(2^{K-1}(x_0 + (a-x_0)\theta) - n + 1) f(2^{K-1}(y_0 + (b-y_0)\theta) - n' + 1)))$, where, $0 < \theta < 1$.

5.5 Error analysis

5.5.1 Error for real part

Theorem 5.5.1 *Let $u_N(a,b)$ be n^{th} approximation of $u(a,b)$ and $S_2 = \|u_{bb} - (u_{bb})_N\|$, $S_3 = \|u_{abb} - (u_{abb})_N\|$, $S_4 = \|u_{bbb} - (u_{bbb})_N\|$ be bounded then,*

$$\|u(a,b) - (u(a,b))_N\| \leq A_1 + B_1 + (1+a^2)S_2 + \frac{S_3}{2}(1 + (\omega_0^2 - 1)a^2) + \frac{S_4}{2}(1+a^2).$$

Proof Let N^{th} approximation of $Eq.(5.26)$ as follows:

$$\begin{aligned} u_N(a,b) &= g_{1N}(a,b) + h_{1N}(a,b) - 2a\int_0^1 \left(\int_0^a r\left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N dr\right) da \\ &+(\omega_0^2 - 1)a\int_0^1 \left(\int_0^a \left(\frac{\partial^2 u}{\partial b^2}\right)_N dr\right) da + a\int_0^1 \left(\int_0^a r^2\left(\frac{\partial^4 u}{\partial b^4}\right)_N dr\right) da \\ &+2\int_0^a \left(\int_0^s r\left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N dr\right) ds - (\omega_0^2 - 1)\int_0^a \left(\int_0^s \left(\frac{\partial^2 u}{\partial b^2}\right)_N dr\right) ds \\ &- \int_0^a \left(\int_0^s r^2\left(\frac{\partial^4 u}{\partial b^4}\right)_N dr\right) ds. \end{aligned} \tag{5.41}$$

Subtracting $Eq.(5.41)$ from $Eq.(5.26)$, we get

$$\begin{aligned} u(a,b) - u_N(a,b) &= (g_1(a,b) - g_{1N}(a,b)) + (h_1(a,b) - h_{1N}(a,b)) \\ &-2a\int_0^1 \left(\int_0^a r\left(\frac{\partial^3 u}{\partial r\partial b^2} - \left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N dr\right)\right) da \\ &+(\omega_0^2 - 1)a\int_0^1 \left(\int_0^a \left(\frac{\partial^2 u}{\partial b^2} - \left(\frac{\partial^2 u}{\partial b^2}\right)_N dr\right)\right) da \\ &+a\int_0^1 \left(\int_0^a r^2\left(\frac{\partial^4 u}{\partial b^4} - \left(\frac{\partial^4 u}{\partial b^4}\right)_N dr\right)\right) da \\ &+2\int_0^a \left(\int_0^s r\left(\frac{\partial^3 u}{\partial r\partial b^2} - \left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N dr\right)\right) ds \\ &-(\omega_0^2 - 1)\int_0^a \left(\int_0^s \left(\frac{\partial^2 u}{\partial b^2} - \left(\frac{\partial^2 u}{\partial b^2}\right)_N dr\right)\right) ds \\ &- \int_0^a \left(\int_0^s r^2\left(\frac{\partial^4 u}{\partial b^4} - \left(\frac{\partial^4 u}{\partial b^4}\right)_N dr\right)\right) ds. \end{aligned} \tag{5.42}$$

Taking L_2 norm both side, we get

$$\begin{aligned}\|u(a,b)-u_N(a,b)\| \leq{}& \|g_1(a,b)-g_{1N}(a,b)\|+\|h_1(a,b)-h_{1N}(a,b)\| \\ &+2\left\|a\int_0^1\left(\int_0^a r\left(\frac{\partial^3 u}{\partial r\partial b^2}-\left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N dr\right)\right)da\right\| \\ &+(\omega_0^2-1)\left\|a\int_0^1\left(\int_0^a \left(\frac{\partial^2 u}{\partial b^2}-\left(\frac{\partial^2 u}{\partial b^2}\right)_N dr\right)\right)da\right\| \\ &+\left\|a\int_0^1\left(\int_0^a r^2\left(\frac{\partial^4 u}{\partial b^4}-\left(\frac{\partial^4 u}{\partial b^4}\right)_N dr\right)\right)da\right\| \\ &+2\left\|\int_0^a\left(\int_0^s r\left(\frac{\partial^3 u}{\partial r\partial b^2}-\left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N dr\right)\right)ds\right\| \\ &+(\omega_0^2-1)\left\|\int_0^a\left(\int_0^s \left(\frac{\partial^2 u}{\partial b^2}-\left(\frac{\partial^2 u}{\partial b^2}\right)_N dr\right)\right)ds\right\| \\ &+\left\|\int_0^a\left(\int_0^s r^2\left(\frac{\partial^4 u}{\partial b^4}-\left(\frac{\partial^4 u}{\partial b^4}\right)_N dr\right)\right)ds\right\|,\end{aligned}$$

or,

$$\begin{aligned}\|u(a,b)-u_N(a,b)\| <{}& \|g_1(a,b)-g_{1N}(a,b)\|+\|h_1(a,b)-h_{1N}(a,b)\| \\ &+2\|a\|\int_0^1\left(\int_0^a \|r\|\left\|\frac{\partial^3 u}{\partial r\partial b^2}-\left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N\right\| dr\right)da \\ &+(\omega_0^2-1)\|a\|\int_0^1\left(\int_0^a \left\|\frac{\partial^2 u}{\partial b^2}-\left(\frac{\partial^2 u}{\partial b^2}\right)_N\right\| dr\right)da \\ &+\|a\|\int_0^1\left(\int_0^a \|r^2\|\left\|\frac{\partial^4 u}{\partial b^4}-\left(\frac{\partial^4 u}{\partial b^4}\right)_N\right\| dr\right)da \\ &+2\int_0^a\left(\int_0^s \|r\|\left\|\frac{\partial^3 u}{\partial r\partial b^2}-\left(\frac{\partial^3 u}{\partial r\partial b^2}\right)_N\right\| dr\right)ds \\ &+(\omega_0^2-1)\int_0^a\left(\int_0^s \left\|\frac{\partial^2 u}{\partial b^2}-\left(\frac{\partial^2 u}{\partial b^2}\right)_N\right\| dr\right)ds \\ &+\int_0^a\left(\int_0^s \|r^2\|\left\|\frac{\partial^4 u}{\partial b^4}-\left(\frac{\partial^4 u}{\partial b^4}\right)_N\right\| dr\right)ds,\end{aligned}$$

since,

$$\begin{aligned}\|g_1(a,b)-g_{1N}(a,b)\| = \|au(0,b)-au_N(0,b)\| <& \\ &< A_1(say),\end{aligned}$$

and,

$$\begin{aligned}\|h_1(a,b)-h_{1N}(a,b)\| &= \left\|a\int_0^1 K_1(a)u(a,b)da - a\int_0^1 K_{1N}(a_N)u_N(a,b)da\right\| \\ &< B\int_0^1 \|K(a)\|\,\|(u(a,b)-u_N(a,b))\|\,da \\ &< BKM_1(say\ B_1),\end{aligned}$$

here, $\|K(a)\| < K$ and $max_{0\le a\le 1, 0\le b\le 1}\|(u(a,b)-u_N(a,b))\| = M_1$.

So,

$$\begin{aligned}\|u(a,b)-u_N(a,b)\| < A_1 + B_1 + 2S_2\int_0^1\left(\int_0^a dr\right)da + (\omega_0^2-1)S_3\int_0^1\left(\int_0^a dr\right)da \\ +S_4\int_0^1\left(\int_0^a dr\right)da + 2S_2\int_0^a\left(\int_0^s dr\right)ds \\ +(\omega_0^2-1)S_3\int_0^a\left(\int_0^s dr\right)ds + S_4\int_0^a\left(\int_0^s dr\right)ds,\end{aligned}$$

or,

$$\begin{aligned}\|u(a,b)-u_N(a,b)\| < A_1 + B_1 + S_2 + (\omega_0^2-1)S_3/2 + S_4/2 + 2S_2(a^2/2) \\ +(\omega_0^2-1)S_3(a^2/2) + S_4(a^2/2),\end{aligned}$$

or,

$$\|u(a,b)-u_N(a,b)\| < A_1 + B_1 + (1+a^2)S_2 + \frac{S_3}{2}(1+(\omega_0^2-1)a^2) + \frac{S_4}{2}(1+a^2).$$

5.5.2 Error for imaginary part

Theorem 5.5.2 *Let $v_N(a,b)$ be n^{th} approximation of $v(a,b)$ and $S_5 = \|v_{bb}-(v_{bb})_N\|$, $S_6 = \|v_{abb}-(v_{abb})_N\|$, $S_7 = \|v_{bbb}-(v_{bbb})_N\|$ be bounded then,*

$$\|v(a,b)-(v(a,b))_N\| \le A_2 + B_2 + (1+a^2)S_5 + \frac{S_6}{2}(1+(\omega_0^2-1)a^2) + \frac{S_7}{2}(1+a^2).$$

Proof

$$\begin{aligned} v_N(a,b) &= v(0,b)(1-a) + av(1,b) - 2a\int_0^1\left(\int_0^a r\left(\frac{\partial^3 v}{\partial r\partial b^2}\right)_N dr\right)da \\ &+a(\omega_0^2-1)\int_0^1\left(\int_0^a \left(\frac{\partial^2 v}{\partial b^2}\right)_N dr\right)da + a\int_0^1\left(\int_0^a r^2\left(\frac{\partial^4 v}{\partial b^4}\right)_N dr\right)da \\ &+2\int_0^a\left(\int_0^s r\left(\frac{\partial^3 v}{\partial r\partial b^2}\right)_N dr\right)ds - (\omega_0^2-1)\int_0^a\left(\int_0^s \left(\frac{\partial^2 v}{\partial b^2}\right)_N dr\right)ds \\ &- \int_0^a\left(\int_0^s r^2\left(\frac{\partial^4 v}{\partial b^4}\right)_N dr\right)ds. \end{aligned} \tag{5.43}$$

Subtracting $Eq.(5.43)$ from $Eq.(5.33)$ then taking L_2 norm and done as theorem 5.5.1, we get

$$\|v(a,b)-v_N(a,b)\| < A_2 + B_2 + (1+a^2)S_5 + \frac{S_6}{2}(1+(\omega_0^2-1)a^2) + \frac{S_7}{2}(1+a^2), \tag{5.44}$$

where,

$$\begin{aligned} \|g_1(a,b)-g_{1N}(a,b)\| = \|au(0,b)-au_N(0,b)\| &< \\ &< A_1(say), \end{aligned}$$

and,

$$\begin{aligned} \|h_2(a,b)-h_{2N}(a,b)\| &= \left\|a\int_0^1 K_2(a)v(a,b)da - a\int_0^1 K_{2N}(a_N)v_N(a,b)da\right\| \\ &< B\int_0^1 \|K'(a)\|\,\|(v(a,b)-v_N(a,b))\|\,da \\ &< BM_2K'(say\ B_2), \end{aligned}$$

here also, $\|K'(a)\| < K'$ and $max_{0\le a\le 1, 0\le b\le 1}\|(v(a,b)-v_N(a,b))\| = M_2$.

5.5.3 Absolute Error for w(a,b)

Since, we have

$$w(a,b) = u(a,b) + iv(a,b), \tag{5.45}$$

taking L_2 norm both side on $Eq.$(5.46), we have

$$\|w(a,b) - w_N(a,b)\| \leq \|u(a,b) - u_N(a,b)\| + \|v(a,b) - v_N(a,b)\|. \tag{5.46}$$

Grouping $Eqs.$(5.43), (5.45) and (5.47) we get

$$\begin{aligned} \|w(a,b) - w_N(a,b)\| &< A_1 + B_1 + A_2 + B_2 + (1+a^2)(S_2+S_5) \\ &+ \frac{(S_3+S_6)}{2}(1+(\omega_0^2-1)a^2) + \frac{(S_4+S_7)}{2}(1+a^2). \end{aligned} \tag{5.47}$$

5.6 Numerical examples

In this section, some test examples are given to demonstrate the applicability and accuracy of our method. For the accuracy of our method we consider two examples with known exact solution. The numerical results of these example compared with exact solution and then findout absolute error (see Table 5.2-5.5), l^2-norm error (see Table 5.6-5.9) and l^∞-norm error (see Table 5.6-5.9) of each examples. We observed that the solution has good accuracy at $k = 1, M = 4$ (see Figure 1-12 and Table 5.2-5.9),

where,

$$Absolute\ error = |g(x,y) - g_N(x,y)|, \quad (x,y) \in \Omega,$$

$$\|g(x,y)\|_2 = \sqrt{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|g(x_i,y_j)|^2},$$

$$\|g(x,y)\|_\infty = max\left\{|g(x,y)|, 0 \leq x \leq 1, 0 \leq y \leq 1\right\}.$$

and

$$|w(a,b)| = \sqrt{u^2(a,b) + v^2(a,b)}.$$

Example 5.1 Taking $Eqs.$(5.26) and (5.33) with the initial condition,

$$w(0,b) = (1+i)b^2 e^{-\frac{\omega 0^2}{2}},$$

Analytical solution of $Eqs.$(5.26) and (5.33) as follows,

$$w(a,b) = (1+i)b^2 e^{-\frac{(\omega a - \omega 0)^2}{2}}.$$

Numerical solution of $Eqs.$(5.26) and (5.33) at $\omega = 10pi, \omega 0 = 2pi$ respectively as follows:

$$u(a,b) = C_1^T \Phi(a,b),$$

and

$$v(a,b) = C_2^T \Phi(a,b)$$

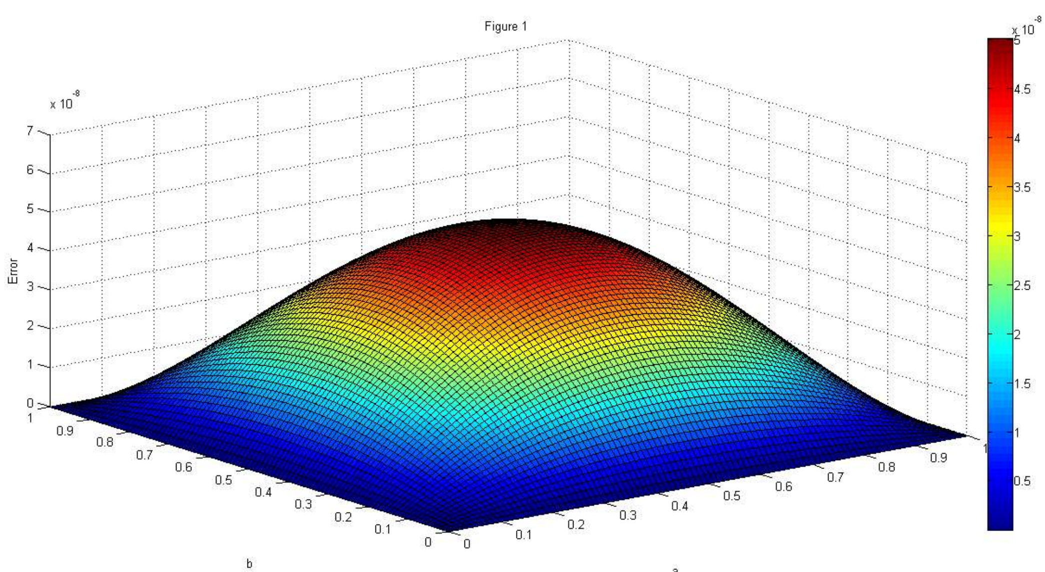

Figure 5.1: Figure 1, represent the absolute errors between the exact solution of example 5.1 for u(a,b) and its numerical solution to utilising the LWA for $k = k' = 1, M = M' = 4$.

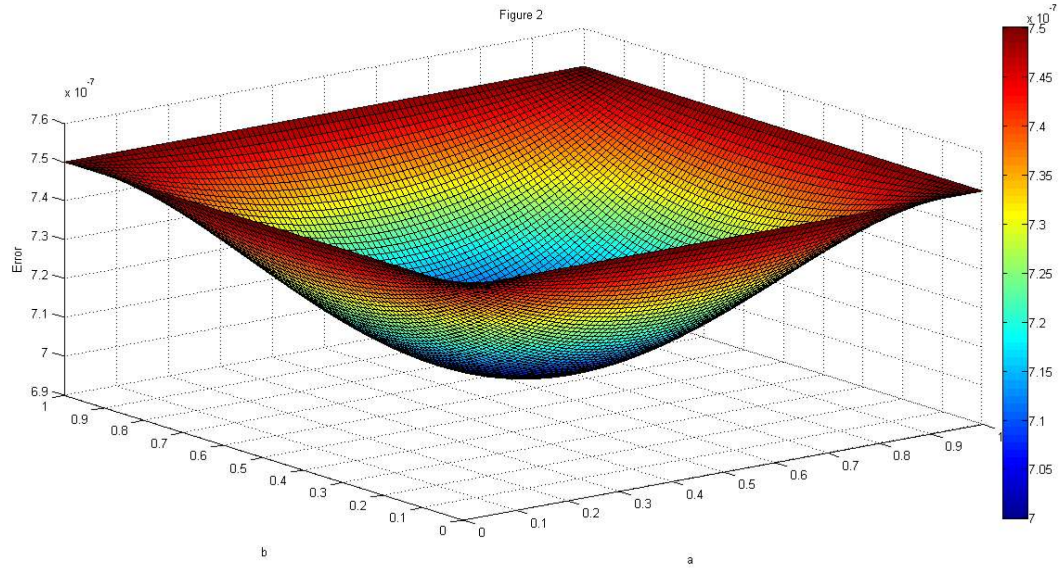

Figure 5.2: Figure 2, represent the absolute errors between the exact solution of example 5.1 for v(a,b) and its numerical solution to utilising the LWA for $k = k' = 1, M = M' = 4$.

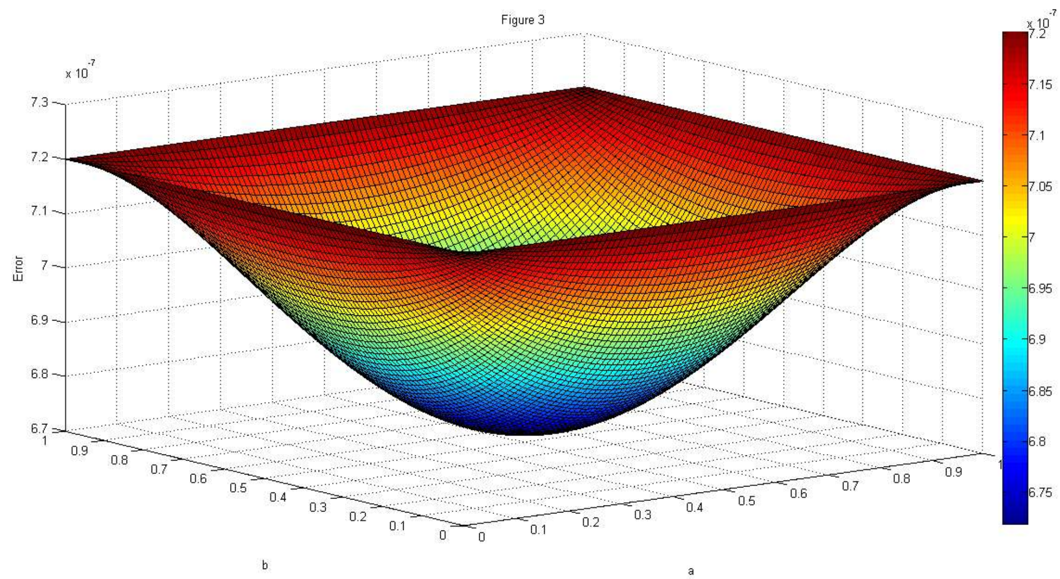

Figure 5.3: Figure 3, represent the absolute errors between the exact solution of example 5.1 for w(a,b) and its numerical solution to utilising the LWA for $k = k' = 1, M = M' = 4$.

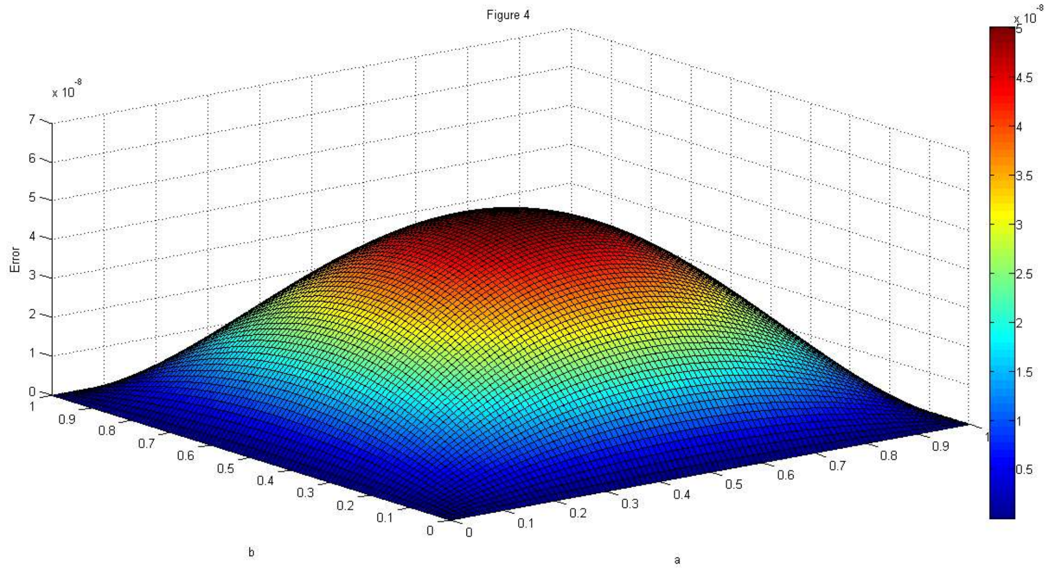

Figure 5.4: Figure 4, represent the absolute errors between the exact solution of example 5.1 for u(a,b) and its numerical solution to utilising the BWA for $k = k' = 1, M = M' = 4$.

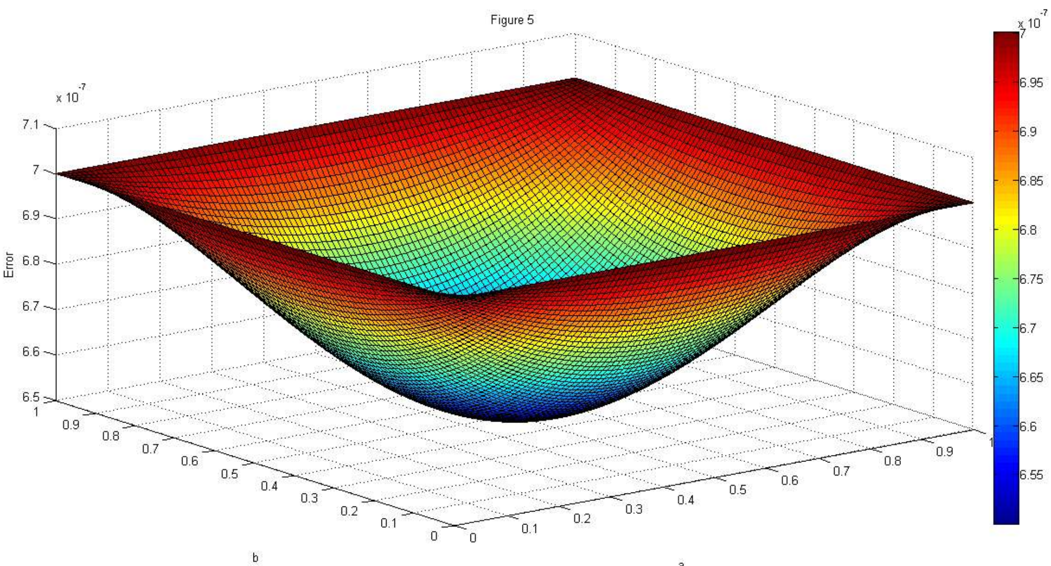

Figure 5.5: Figure 5, represent the absolute errors between the exact solution of example 5.1 for v(a,b) and its numerical solution to utilising the BWA for $k = k' = 1, M = M' = 4$.

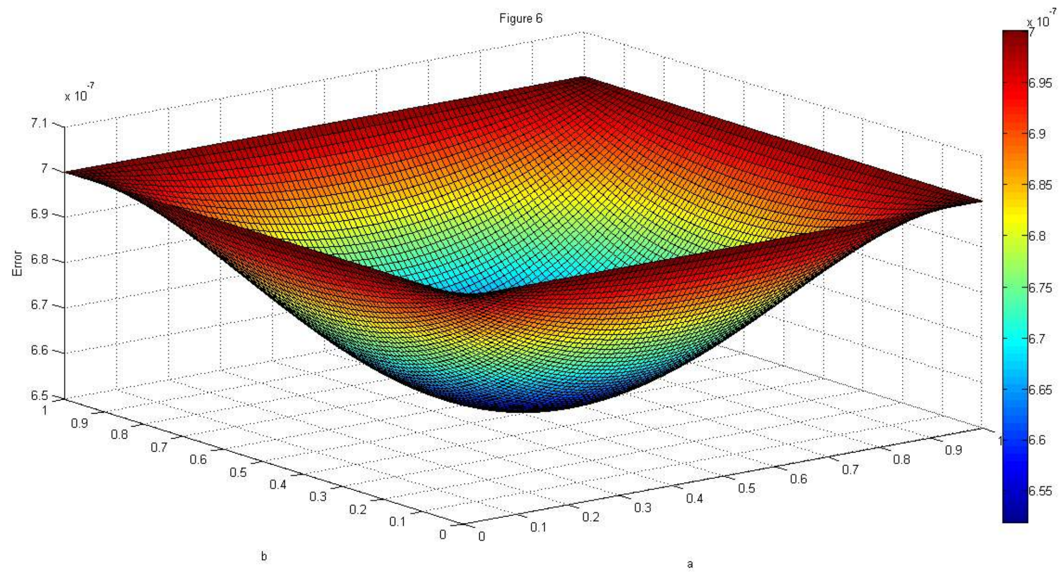

Figure 5.6: Figure 6, represent the absolute errors between the exact solution of example 5.1 for w(a,b) and its numerical solution to utilising the BWA for $k = k' = 1, M = M' = 4$.

Example 2. Taking $Eqs.$(5.26) and (5.33) with the initial condition,

$$w(0,b) = (1+i)b^3 e^{-\frac{\omega 0^2}{2}},$$

Analytical solution of $Eqs.$(5.26) and (5.33) as follows,

$$w(a,b) = (1+i)b^3 e^{-\frac{(\omega a - \omega 0)^2}{2}}.$$

Numerical solution of $Eqs.$(5.26) and (5.33) $\omega = 10pi, \omega 0 = 2pi$ respectively as follows:

$$u(a,b) = C_1^T \Phi(a,b),$$

and

$$v(a,b) = C_2^T \Phi(a,b)$$

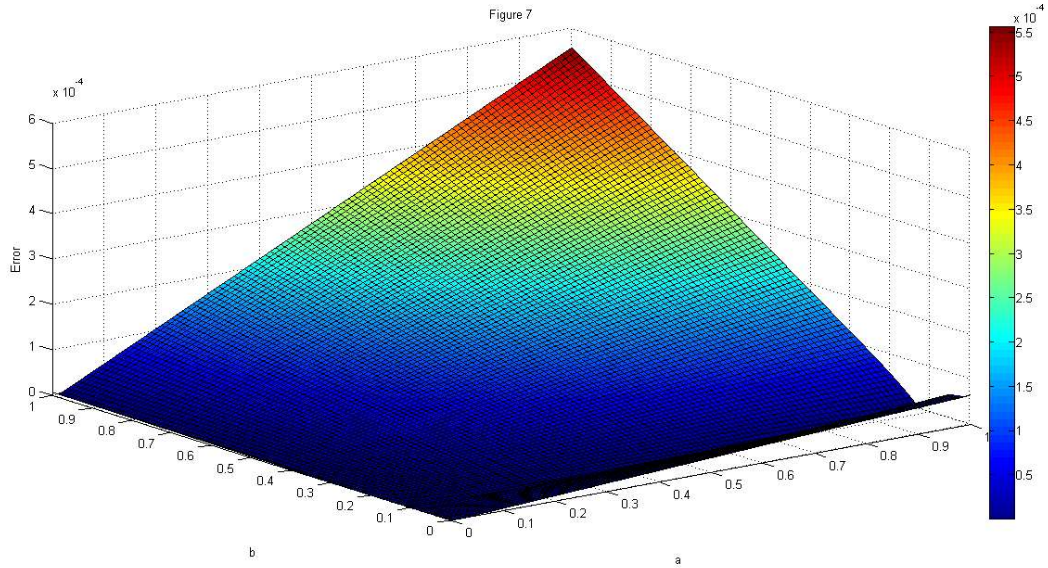

Figure 5.7: Figure 7, represent the absolute errors between the exact solution of example 5.2 for u(a,b) and its numerical solution to utilising the LWA for $k = k' = 1, M = M' = 4$.

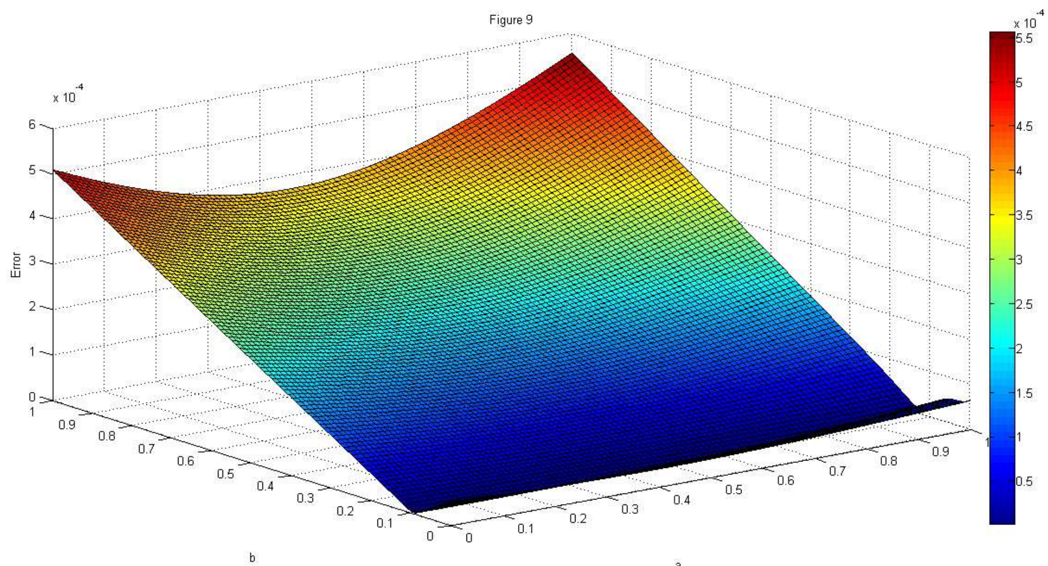

Figure 5.8: Figure 8, represent the absolute errors between the exact solution of example 5.2 for v(a,b) and its numerical solution to utilising the LWA for $k = k' = 1, M = M' = 4$.

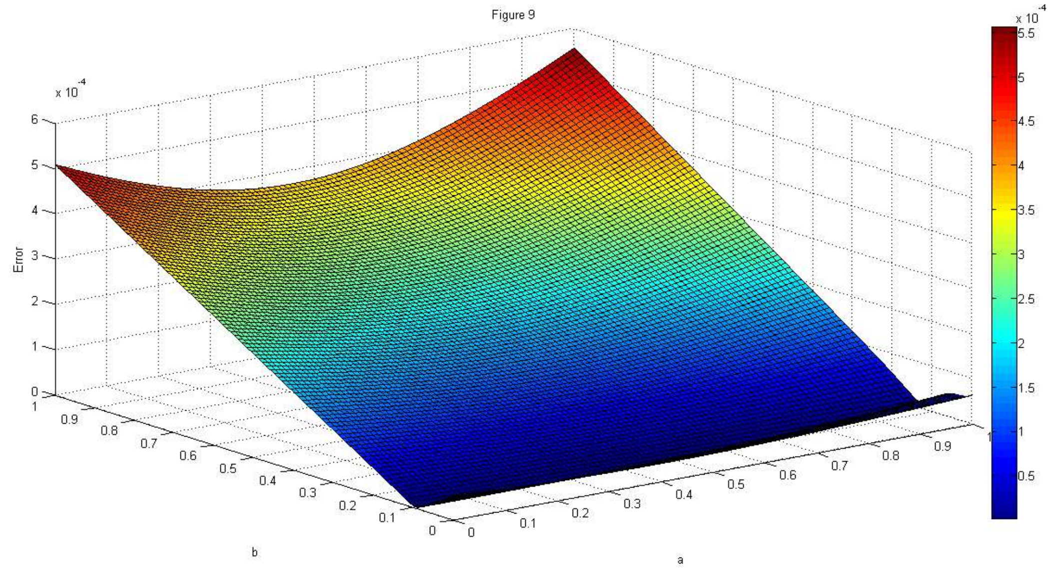

Figure 5.9: Figure 9, represent the absolute errors between the exact solution of example 5.2 for w(a,b) and its numerical solution to utilising the LWA for $k = k' = 1, M = M' = 4$.

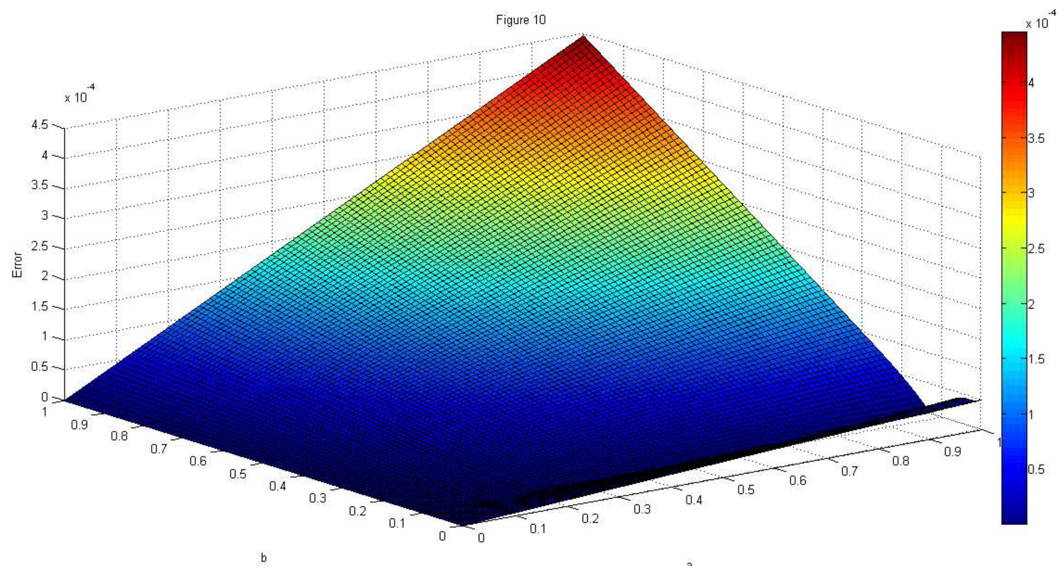

Figure 5.10: Figure 10, represent the absolute errors between the exact solution of example 5.2 for u(a,b) and its numerical solution to utilising the BWA for $k = k' = 1, M = M' = 4$.

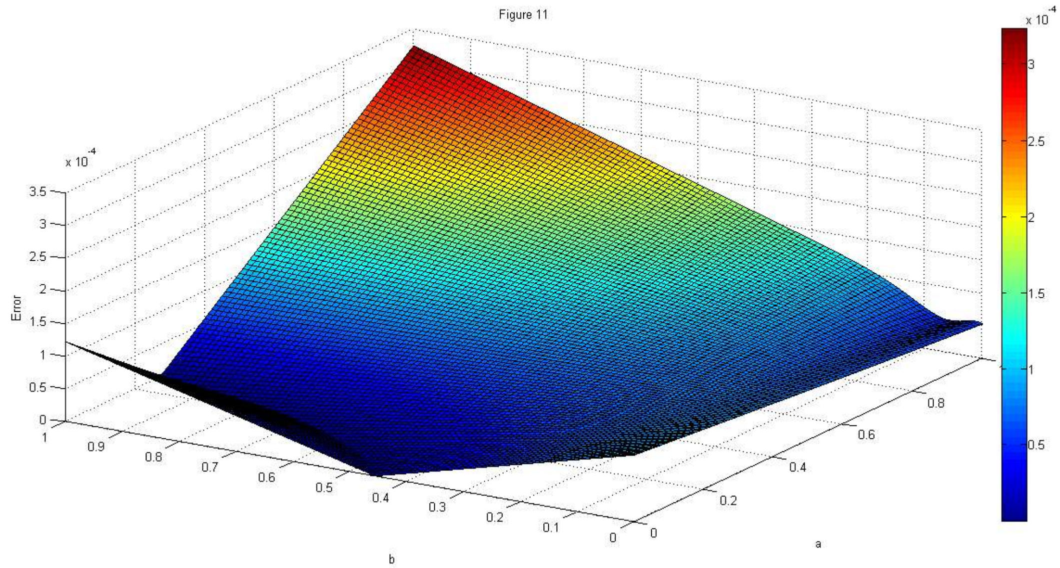

Figure 5.11: Figure 11, represent the absolute errors between the exact solution of example 5.2 for v(a,b) and its numerical solution to utilising the BWA for $k = k' = 1, M = M' = 4$.

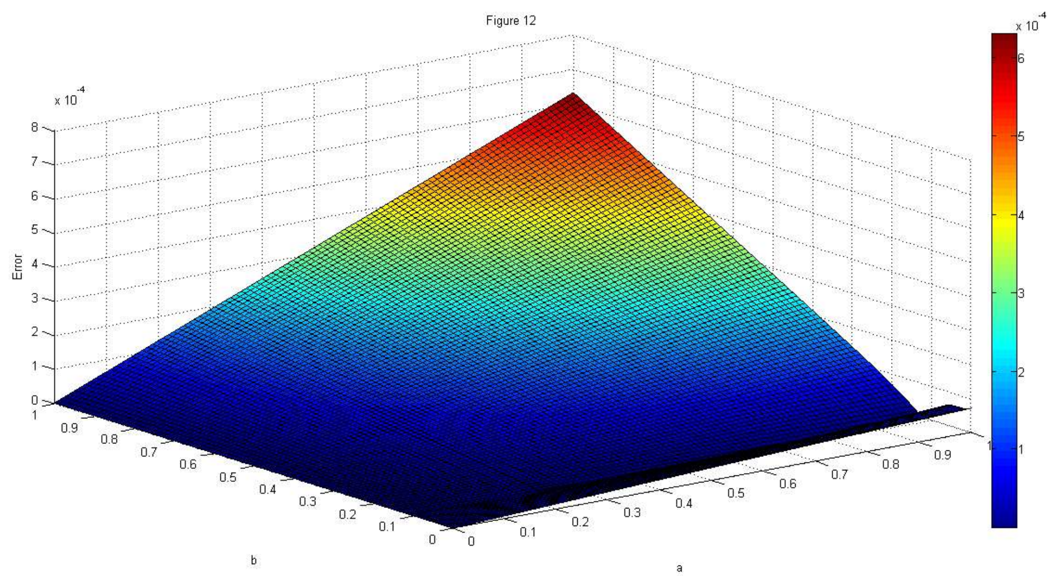

Figure 5.12: Figure 12, represent the absolute errors between the exact solution of example 5.2 for w(a,b) and its numerical solution to utilising the BWA for $k = k' = 1, M = M' = 4$.

Table 5.2: Absolute Error of Example 5.1 via Legendre wavelet at $k = k' = 1, M = M' = 4$

(a,b)	$u(a,b)$	$v(a,b)$	$w(a,b)$
$(0.1, 0.1)$	6.48×10^{-9}	7.14×10^{-8}	7.14×10^{-7}
$(0.2, 0.2)$	2.05×10^{-8}	6.99×10^{-8}	7.00×10^{-7}
$(0.3, 0.3)$	3.53×10^{-8}	6.85×10^{-8}	6.86×10^{-7}
$(0.4, 0.4)$	4.61×10^{-8}	6.74×10^{-8}	6.76×10^{-7}
$(0.5, 0.5)$	5.00×10^{-8}	6.70×10^{-8}	6.72×10^{-7}
$(0.6, 0.6)$	4.61×10^{-8}	6.74×10^{-8}	6.76×10^{-7}
$(0.7, 0.7)$	3.53×10^{-8}	6.85×10^{-8}	6.86×10^{-7}
$(0.8, 0.8)$	2.05×10^{-8}	6.99×10^{-8}	7.00×10^{-7}
$(0.9, 0.9)$	6.48×10^{-9}	7.14×10^{-8}	7.01×10^{-7}
$(1.0, 1.0)$	0.00×10^{-8}	7.20×10^{-8}	7.20×10^{-7}

Table 5.3: Absolute Error of Example 5.1 via Bernoulli wavelet at $k = k' = 1, M = M' = 4$

(a,b)	$u(a,b)$	$v(a,b)$	$w(a,b)$
$(0.1, 0.1)$	6.48×10^{-9}	6.94×10^{-8}	6.94×10^{-8}
$(0.2, 0.2)$	2.05×10^{-8}	6.79×10^{-8}	6.81×10^{-8}
$(0.3, 0.3)$	3.53×10^{-8}	6.65×10^{-8}	6.66×10^{-8}
$(0.4, 0.4)$	4.61×10^{-8}	6.54×10^{-8}	6.56×10^{-8}
$(0.5, 0.5)$	5.00×10^{-8}	6.50×10^{-8}	6.52×10^{-8}
$(0.6, 0.6)$	4.61×10^{-8}	6.54×10^{-8}	6.56×10^{-8}
$(0.7, 0.7)$	3.53×10^{-8}	6.65×10^{-8}	6.66×10^{-8}
$(0.8, 0.8)$	2.05×10^{-8}	6.79×10^{-8}	6.81×10^{-8}
$(0.9, 0.9)$	6.48×10^{-9}	6.94×10^{-8}	6.94×10^{-8}
$(1.0, 1.0)$	0.00×10^{-8}	7.00×10^{-8}	7.00×10^{-8}

Table 5.4: Absolute Error of Example 5.2 via Legendre wavelet at $k = k' = 1, M = M' = 4$

(a,b)	$u(a,b)$	$v(a,b)$	$w(a,b)$
$(0.1, 0.1)$	1.00×10^{-7}	7.00×10^{-7}	2.00×10^{-7}
$(0.2, 0.2)$	1.15×10^{-5}	1.15×10^{-5}	1.62×10^{-5}
$(0.3, 0.3)$	3.54×10^{-5}	3.37×10^{-5}	5.01×10^{-5}
$(0.4, 0.4)$	7.20×10^{-5}	6.74×10^{-5}	1.02×10^{-4}
$(0.5, 0.5)$	1.21×10^{-4}	1.12×10^{-4}	1.71×10^{-4}
$(0.6, 0.6)$	1.83×10^{-4}	1.69×10^{-4}	2.56×10^{-4}
$(0.7, 0.7)$	2.57×10^{-4}	2.37×10^{-4}	3.64×10^{-4}
$(0.8, 0.8)$	3.44×10^{-4}	3.16×10^{-4}	4.86×10^{-4}
$(0.9, 0.9)$	4.43×10^{-4}	4.07×10^{-4}	6.27×10^{-4}
$(1.0, 1.0)$	5.55×10^{-4}	5.09×10^{-4}	7.85×10^{-4}

Table 5.5: Absolute Error of Example 5.2 via Bernoulli wavelet at $k = k' = 1, M = M' = 4$

(a,b)	$u(a,b)$	$v(a,b)$	$w(a,b)$
$(0.1, 0.1)$	2.00×10^{-7}	2.00×10^{-7}	3.00×10^{-7}
$(0.2, 0.2)$	9.70×10^{-6}	9.71×10^{-6}	1.37×10^{-5}
$(0.3, 0.3)$	2.96×10^{-5}	2.97×10^{-5}	4.18×10^{-5}
$(0.4, 0.4)$	5.94×10^{-5}	5.95×10^{-5}	8.39×10^{-5}
$(0.5, 0.5)$	9.90×10^{-5}	9.90×10^{-5}	1.40×10^{-5}
$(0.6, 0.6)$	1.48×10^{-4}	1.48×10^{-4}	2.19×10^{-5}
$(0.7, 0.7)$	2.08×10^{-4}	2.07×10^{-4}	2.94×10^{-5}
$(0.8, 0.8)$	2.78×10^{-4}	2.78×10^{-4}	3.91×10^{-5}
$(0.9, 0.9)$	3.56×10^{-4}	3.56×10^{-4}	5.03×10^{-5}
$(1.0, 1.0)$	4.45×10^{-4}	4.44×10^{-4}	6.29×10^{-5}

Table 5.6: l_2 and $l_\infty - norm$ error for Example 5.1 via Legendre wavelet at $k = k' = 1, M = M' = 4$

Norm	$u(a,b)$	$v(a,b)$	$w(a,b)$
$l_2 - norm$	1.0083×10^{-7}	2.1934×10^{-7}	2.1923×10^{-6}
$l_\infty - norm$	5.0000×10^{-8}	7.2000×10^{-4}	7.2000×10^{-6}

Table 5.7: l_2 and $l_\infty - norm$ error for Example 5.1 via Bernoulli wavelet at $k = k' = 1, M = M' = 4$

Norm	$u(a,b)$	$v(a,b)$	$w(a,b)$
$l_2 - norm$	1.0083×10^{-7}	2.0187×10^{-7}	2.1339×10^{-7}
$l_\infty - norm$	5.0000×10^{-8}	6.9900×10^{-8}	7.0000×10^{-8}

Table 5.8: l_2 and $l_\infty - norm$ error for Example 5.2 via Legendre wavelet at $k = k' = 1, M = M' = 4$

Norm	$u(a,b)$	$v(a,b)$	$w(a,b)$
$l_2 - norm$	8.6218×10^{-4}	7.9217×10^{-4}	1.2000×10^{-3}
$l_\infty - norm$	5.5500×10^{-4}	5.0900×10^{-4}	7.8500×10^{-4}

Table 5.9: l_2 and $l_\infty - norm$ error for Example 5.2 via Bernoulli wavelet at $k = k' = 1, M = M' = 4$

Norm	$u(a,b)$	$v(a,b)$	$w(a,b)$
$l_2 - norm$	6.9391×10^{-4}	1.2000×10^{-4}	1.3612×10^{-4}
$l_\infty - norm$	4.4500×10^{-4}	9.9000×10^{-4}	8.3900×10^{-5}

5.7 Conclusion

In this chapter, we discussed the solution of a complex partial differential equation (CPDE) which arising from continuous wavelet transform. A most powerful scheme for solving the problem of distinguishing the instant period in signal is the complex continuous wavelet transform. We have proposed an efficient numerical technique via wavelets operational and new almost wavelets operational matrices for solving a CPDE which is arising by a continuous Morlet wavelet transform. For solving this CPDE, we transformed the CPDE into system of partial differential equations (PDEs) and then we have utilized Legendre and Bernoulli wavelets approximation with its operational matrices to convert system of PDEs into two algebraic equations and these algebraic equations solved at collocation point. Finally, from convergence analysis (see *Theorem* (5.4.1)and (5.4.4)) and *Lemma* (5.4.2)-(5.4.3))of both wavelets, error analysis (see Theorem (5.5.1)- (5.5.3)), test examples at $k = k' = 1$,

$M = M' = 4$, absolute error (see tables 5.2-5.5), $l_2 - norm$ error (see tables 5.6-5.9), $l_\infty - norm$ error (see tables 5.6-5.9), and error graphs (see $5.1 - 5.12$), we may say that our proposed new numerical technique gives solution of CPDE as accurate as possible. We may find better result if we increasing the number of basis (see error analysis which absolute error goes to zero whenever N goes to ∞) and also follow the best approximation theorem. This method can also be applied for higher dimensional continuous Morlet wavelet transform, which is a topic for future scope.

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