Chapter 3

Approximation Schemes for a Quadratic Type Generalized Isoperimetric Constraint Fractional Variational Problems

This chapter is structured as pursues. The introduction of the chapter is given in section 3.1. Section 3.2 defines the problem and briefly discusses the three different numerical schemes for GICFVPs. The error analysis of the proposed schemes is discussed in Section 3.3. In Section 3.4, the comparative analysis of all the schemes is studied, and the simulation results with convergence are discussed. Section 3.5 concludes the chapter.

3.1 Introduction

FVP with integral constraint is called an isoperimetric constraint fractional variational problem (ICFVP). In recent years, several researchers have broadly discussed ICVFPs due to its importance in control theory. Agrawal [31, 45] and Klimek [46, 53] were among the earlier ones who presented the Euler-Lagrange equations for FVPs in terms of the R-L and Caputo derivatives and explained the stationary conservation laws using linear FDEs with variable coefficients. In [4], author derived the transversality conditions for FVPs and presented the application of the formulations. Almeida et al. [57] studied the isoperimetric problem of FCV with left and right RLFD and derived the optimal conditions for FVPs and ICFVPs.

The solution to the classical variational problems can be found analytically using Euler—Lagrange equation, but the FVPs are very difficult to solve using analytical methods. Therefore, numerical methods play an essential role in solving FVPs. In [47], author presented a finite element method for FVPs. Agrawal obtained the solution in terms of power series for the ICFVP, and FDEs in [48]. In [58], author studied the fractional oscillator problem and derived the solution on finite time interval using Mellin transform and fixed point theorem. Agrawal et al. [59] discussed the numerical method for the class of parametric problem of FCV using finite element approximation and derived the fractional orthogonal functions. Pandey et al. [65] presented the Ritz method for an ICFVP with an A-operator using Bernstein's polynomial as the basis function. Later in 2015, Pandey and Agrawal discussed a comparative analysis of four numerical schemes for ICFVPs [66].

In 2012, Agrawal [8] discussed GFD, which contains a scale and a weight function. For extending the kernel in operators, we use the weight function, which is helpful to create models that are more relaxed and to change the considered domain the scale function can be used. Here, our focus is on developing numerical schemes for GICFVPs. We present three schemes, i.e. linear, quadratic and quadratic-linear approximation for GICFVPs. To examine the legitimacy of the proposed methods, we used various sets of scale and weight functions and considered the variations in α and h. The boundary conditions (BCs) of the GICFVPs are taken as zero in all the numerical schemes. Furthermore, we demonstrate that the results obtained through various numerical techniques are consistent with the solutions found by [59] and [65, 66] in particular cases.

3.2 The GICFVPs and Numerical Schemes

Here, we discuss the problem followed by three numerical schemes for solving it.

3.2.1 Problem Statement

Find a minimizer of the cost functional

$$J(y) = \int_0^1 \left[\left({}_0D^{\alpha}_{t;[z;\omega,1]} y(t)\right)\right]^2 z'(t)\, dt, \tag{3.1}$$

with respect to the integral constraint,

$$K(y) = \int_0^1 [y(t)]^2 z'(t)\, dt = 2, \tag{3.2}$$

and fixed BCs $y(0) = y(1) = 0$, where $({}_0D^{\alpha}_{t;[z;\omega,1]} y)(t)$ $\alpha \in (0, 1)$ is the generalized fractional R-L type derivative. The exact solution for the integer-order case with $z(t) = t$, $w(t) = 1$, and $\alpha = 1$ is $y(t) = \pm 2\sin(\pi t)$ [59]. The similar problem with A-operator is studied in [65, 66] and numerical solution were obtained. Here, we present three numerical scheme for the proposed problem and also establish their convergence in detail. In numerical schemes, the solution is approximated on the discretized subdomains of the domain using linear and quadratic polynomial approximation in first and second schemes, respectively. In the third scheme, the linear and quadratic functions are used together to approximate the solution. Now, we discuss the numerical schemes for GICFVPs:

3.2.2 The Linear Scheme(S1)

Here, we discretize the time interval $[0,1]$ into $\mathcal{N}$ sub-intervals with equal interval-length h $= 1/\mathcal{N}$, such that $t_k = k$h, $k = 0, 1, 2..., \mathcal{N}$. At the point t_k, the solution $y(t)$ is approximated as y_k . Now, Eq. (3.1) can be written as

$$J(y) = \int_0^1 \left[\left({}_0D^{\alpha}_{t;[z;\omega,1]}y(t)\right)\right]^2 z'(t)\ dt = \sum_{j=1}^{\mathcal{N}} \int_{t_{j-1}}^{t_j} \left[\left({}_0D^{\alpha}_{t;[z;\omega,1]}y(t)\right)\right]^2 z'(t)\ dt. \tag{3.3}$$

Let $v(t) = \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\omega(\tau)z'(\tau)y(\tau))}{(z(t)-z(\tau))^{\alpha}}\,d\tau\right]$, and at the node point t_j, $v(t)$ is given as $v(t_j) = v_j$. Approximating $v(t), z(t)$ and $\omega(t)$ on the interval $[t_{j-1}, t_j]$, Eq. (3.3) reduces to

$$J(y) = \sum_{j=1}^{\mathcal{N}} \int_{t_{j-1}}^{t_j} \left[\frac{v'(t)}{z'(t)[\omega(t)]}\right]^2 z'(t)dt \approx 4\sum_{j=1}^{\mathcal{N}} [z_j - z_{j-1}]^{-1}[v_j - v_{j-1}]^2[\omega_j + \omega_{j-1}]^{-2}. \tag{3.4}$$

Therefore, from Eqs. (3.3-3.4)

$$J(y) = V^T E V, \tag{3.5}$$

where,

$$V = [v_0, v_1, v_2, v_3, ..., v_{\mathcal{N}}]^T. \tag{3.6}$$

And

$$E = \begin{bmatrix} a_1b_1 & -a_1b_1 & 0 & 0 & \dots & 0 & 0 \\ -a_1b_1 & a_1b_1+a_2b_2 & -a_2b_2 & 0 & \dots & 0 & 0 \\ 0 & -a_2b_2 & a_2b_2+a_3b_3 & -a_3b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{\mathcal{N}}b_{\mathcal{N}}+a_{\mathcal{N}+1}b_{\mathcal{N}+1} & -a_{\mathcal{N}+1}b_{\mathcal{N}+1} \\ 0 & 0 & 0 & \dots & 0 & -a_{\mathcal{N}+1}b_{\mathcal{N}+1} & a_{\mathcal{N}+1}b_{\mathcal{N}+1} \end{bmatrix}_{(\mathcal{N}+1)\times(\mathcal{N}+1)}, \tag{3.7}$$

where, $a_ib_j = 4[z_j - z_{j-1}]^{-1}[\omega_j + \omega_{j-1}]^{-2}$.

On interval $[t_{j-1}, t_j]$, approximation of $y(t)$ using linear interpolation is given as,

$$y_{1j}(t) = \frac{(t_j - t)}{\text{h}} y_{j-1} + \frac{(t - t_{j-1})}{\text{h}} y_j. \tag{3.8}$$

Thus, v_k for $k = 1, 2...$ is given by

$$\begin{aligned} v_k &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{\omega(\tau)z'(\tau)y(\tau))}{(z(t_k) - z(\tau))^\alpha} d\tau = \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{\omega(\tau)z'(\tau)y(\tau))}{(z(t_k) - z(\tau))^\alpha} d\tau \\ &= \sum_{j=0}^{k} S_j^k y_j, \end{aligned} \tag{3.9}$$

where the coefficients S_j^k are given as under the transformation $\tau = t_{j-1} + p\text{h}$, as $\tau \in [t_{j-1}, t_j]$ so $p \in [0, 1]$

$$S_j^k = \frac{\mathrm{h}(\omega_j + \omega_{j-1})}{2\Gamma\left(1-\alpha\right)} \begin{cases} a_j^k & \text{for} \quad j = 0 \\ a_j^k + b_j^k & \text{for} \quad 1 \le j \le k-1 \\ b_j^k & \text{for} \quad j = k. \end{cases} \tag{3.10}$$

And

$$a_j^k = \int_0^1 \frac{z'(t_{j-1} + p\mathrm{h})(1-p)}{(z(t_k) - z(t_{j-1} + p\mathrm{h}))^\alpha}\, dp, \tag{3.11}$$

$$b_j^k = \int_0^1 \frac{z'(t_{j-1} + p\mathrm{h})p}{(z(t_k) - z(t_{j-1} + p\mathrm{h}))^\alpha}\, dp. \tag{3.12}$$

Since, $y(t)$ is bounded at $t = 0, v_0 = 0$. So,

$$V = [v_0, v_1, v_2, v_3, ..., v_{\mathcal{N}}]^T = FU, \tag{3.13}$$

where,

$$U = [y_0, y_1, y_2, y_3, ..., y_{\mathcal{N}}]^T, \tag{3.14}$$

and

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ S_0^1 & S_1^1 & 0 & 0 & 0 & \dots & 0 & 0 \\ S_0^2 & S_1^2 & S_2^2 & 0 & 0 & \dots & 0 & 0 \\ S_0^3 & S_1^3 & S_2^3 & S_3^3 & 0 & \dots & 0 & 0 \\ S_0^4 & S_1^4 & S_2^4 & S_3^4 & S_4^4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_0^{\mathcal{N}-1} & S_1^{\mathcal{N}-1} & S_2^{\mathcal{N}-1} & S_3^{\mathcal{N}-1} & S_4^{\mathcal{N}-1} & \dots & S_{\mathcal{N}-1}^{\mathcal{N}-1} & 0 \\ S_0^{\mathcal{N}} & S_1^{\mathcal{N}} & S_2^{\mathcal{N}} & S_3^{\mathcal{N}} & S_4^{\mathcal{N}} & \dots & S_{\mathcal{N}-1}^{\mathcal{N}} & S_{\mathcal{N}}^{\mathcal{N}} \end{bmatrix}_{(\mathcal{N}+1)\times(\mathcal{N}+1)} . \tag{3.15}$$

Using Eqs. (3.8) and (3.13), $J(y)$ can be re-written as

$$J \cong U^T AU, \text{ where } A = F^T EF. \tag{3.16}$$

Approximating $y(t)$ as a linear polynomial between two neighbouring nodes, Eq. (3.2) has the form:

$$K \cong U^T BU = 2, \tag{3.17}$$

where,

$$B = \frac{h}{6}\begin{bmatrix} 2(z_1-z_0) & z_1-z_0 & 0 & 0 & \dots & 0 & 0 \\ z_1-z_0 & 2(z_2-z_0) & z_2-z_1 & 0 & \dots & 0 & 0 \\ 0 & z_2-z_1 & 2(z_3-z_1) & z_3-z_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z_{\mathcal{N}-2}-z_{\mathcal{N}-3} & 2(z_{\mathcal{N}-1}-z_{\mathcal{N}-3}) & z_{\mathcal{N}}-z_{\mathcal{N}-1} \\ 0 & 0 & 0 & \dots & 0 & z_{\mathcal{N}}-z_{\mathcal{N}-1} & 2(z_{\mathcal{N}}-z_{\mathcal{N}-1}) \end{bmatrix}_{(\mathcal{N}+1)\times(\mathcal{N}+1)}. \tag{3.18}$$

Since, the given minimization problem reduces from a function space to a finite-dimensional space. Now we need to find the vector U which minimizes the functional J provided by Eq. (3.16) and fulfils the isoperimetric condition Eq. (3.17) with $y_0 = y_{\mathcal{N}} = 0$. From the Lagrange multiplier technique, we have

$$L = U^T AU - \lambda(U^T BU - 2). \tag{3.19}$$

Now, for obtaining the minimization condition, we differentiate Eq. (3.19) with respect to U and set the result as zero

$$AU = \lambda BU. \tag{3.20}$$

Now, if we differentiate Eq. (3.19) with respect to λ and equate it to zero, we get Eq. (3.17). The desired vector U is obtained by solving Eqs. (3.17) and (3.20) together with the BCs $y(0) = y(1) = 0$. Clearly, from Eq. (3.20), the problem is formulated as an eigenvalue problem. If we impose the condition $y_0 = y_{\mathcal{N}} = 0$, the matrices A and B become positive definite. The lowest eigenvalue of the eigenvalue problem and corresponding eigenvector is the desired solution for the problem [65]. Thus, by solving Eq. (3.20), we get the lowest eigenvalue λ_1 and corresponding eigenvector U_1. The amplitude of the eigenvector can be found using Eq. (3.17). The minimum value of the functional is given by $2\lambda_1$ [59].

3.2.3 The Quadratic Scheme(S2)

Following the similar steps as described for scheme S1, we discretize the interval $[0, 1]$ with equal step size h. At the node point $t = t_k$, we approximate Eq. (3.1)

$$J(y) = \int_0^1 \left[\left({}_0D^{\alpha}_{t;[z;\omega,1]}y(t)\right)\right]^2 z'(t)\,dt = \sum_{j=1}^{\mathcal{N}} \int_{t_{j-1}}^{t_j} \left[\frac{v'(t)}{z'(t)[\omega(t)]}\right]^2 z'(t)\,dt, \tag{3.21}$$

where, $v(t) = \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\omega(\tau)z'(\tau)y(\tau))}{(z(t)-z(\tau))^{\alpha}}\,d\tau\right]$, and at the node point t_j, $v(t_j) = v_j$. We approximate $v(t)$ on the first interval $[t_0, t_1]$, using linear interpolating polynomial and for the remaining subintervals ($j \geq 2$), the quadratic interpolating polynomial on three points $(t_{j-2}, v_{j-2}), (t_{j-1}, v_{j-1})$,

(t_j, v_j) is applied such that

$$\begin{aligned} v(t) &= \frac{(t - t_{j-1})(t - t_j)}{2\mathrm{h}^2} v_{j-2} - \frac{(t - t_{j-2})(t - t_j)}{\mathrm{h}^2} v_{j-1} + \frac{(t - t_{j-2})(t - t_{j-1})}{2\mathrm{h}^2} v_j, \\ v'(t) &= \frac{(2t - t_{j-1} - t_j)}{2\mathrm{h}^2} v_{j-2} - \frac{(2t - t_{j-2} - t_j)}{\mathrm{h}^2} v_{j-1} + \frac{(2t - t_{j-2} - t_{j-1})}{2\mathrm{h}^2} v_j. \end{aligned} \tag{3.22}$$

Again from Eqs. (3.21) and (3.22), we have

$$\sum_{j=1}^{\mathcal{N}} \int_{t_{j-1}}^{t_j} \left[\frac{v'(t)}{z'(t)[\omega(t)]}\right]^2 z'(t)\,dt = \int_{t_0}^{t_1} \left[\frac{v'(t)}{z'(t)[\omega(t)]}\right]^2 z'(t)\,dt$$
$$+ \sum_{j=2}^{\mathcal{N}} \int_{t_{j-1}}^{t_j} \left[\frac{v'(t)}{z'(t)[\omega(t)]}\right]^2 z'(t)dt$$
$$= 4[z_1 - z_0]^{-1}[\omega_1 + \omega_0]^{-2} \int_{t_0}^{t_1} \left[\frac{-v_0}{\mathrm{h}} + \frac{v_1}{\mathrm{h}}\right]^2 dt$$
$$+ 4\sum_{j=2}^{\mathcal{N}} [z_j - z_{j-1}]^{-1}[\omega_j + \omega_{j-1}]^{-2}$$
$$\int_{t_{j-1}}^{t_j} \left[\frac{(2t - t_{j-1} - t_j)}{2\mathrm{h}^2} v_{j-2} - \frac{(2t - t_{j-2} - t_j)}{\mathrm{h}^2} v_{j-1}\right.$$
$$\left. + \frac{(2t - t_{j-2} - t_{j-1})}{2\mathrm{h}^2} v_j\right]^2 dt. \tag{3.23}$$

Under the transformation, $t = t_{j-1} + \mathrm{h}p$, Eq. (3.23) becomes

$$\sum_{j=1}^{\mathcal{N}} \int_{t_{j-1}}^{t_j} \left[\frac{v'(t)}{z'(t)[\omega(t)]}\right]^2 z'(t)\,dt = 4[z_1 - z_0]^{-1}[\omega_1 + \omega_0]^{-2} \int_{t_0}^{t_1} \left[\frac{-v_0}{\mathrm{h}} + \frac{v_1}{\mathrm{h}}\right]^2 dt$$
$$+ 4\mathrm{h}\sum_{j=2}^{\mathcal{N}} [z_j - z_{j-1}]^{-1}[\omega_j + \omega_{j-1}]^{-2}$$
$$\int_0^1 \left[\frac{(2p-1)}{2\mathrm{h}} v_{j-2} - \frac{(2p)}{\mathrm{h}} v_{j-1} + \frac{(2p+1)}{\mathrm{h}} v_j\right]^2 dp. \tag{3.24}$$

Therefore, from Eqs. (3.21) and (3.24)

$$J(y) = V^T E V, \tag{3.25}$$

where,

$$V = [v_0, v_1, v_2, v_3, ..., v_{\mathcal{N}}]^T. \tag{3.26}$$

And

$$E = \begin{bmatrix} 13a_1b_1 & -14a_1b_1 & a_1b_1 & 0 & 0 & \dots & 0 & 0 \\ -14a_1b_1 & \frac{(a_1b_1+a_2b_2)29}{2} & -16a_2b_2 & a_2b_2 & 0 & \dots & 0 & 0 \\ a_1b_1 & -16a_2b_2 & \frac{(a_2b_2+a_3b_3)30}{2} & -16a_3b_3 & a_3b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{(a_{\mathcal{N}}b_{\mathcal{N}}+a_{\mathcal{N}+1}b_{\mathcal{N}+1})29}{2} & -14a_{\mathcal{N}+1}b_{\mathcal{N}+1} \\ 0 & 0 & 0 & 0 & 0 & \dots & -14a_{\mathcal{N}+1}b_{\mathcal{N}+1} & 13a_{\mathcal{N}+1}b_{\mathcal{N}+1} \end{bmatrix}, \tag{3.27}$$

where, $a_ib_j = \frac{1}{12\mathrm{h}}[z_j - z_{j-1}]^{-1}[\omega_j + \omega_{j-1}]^{-2}$ and the order of matrix is $(\mathcal{N}+1) \times (\mathcal{N}+1)$.

Now, we approximate $y(t)$ on the first interval $[t_0, t_1]$ using linear interpolating polynomial and for the other subintervals $(j \geq 2)$, the quadratic interpolating polynomial on three points $(t_{j-2}, y_{j-2}), (t_{j-1}, y_{j-1}), (t_j, y_j)$ is applied such that

$$y_{2j}(t) = \frac{(t - t_{j-1})(t - t_j)}{2\mathrm{h}^2} y_{j-2} - \frac{(t - t_{j-2})(t - t_j)}{\mathrm{h}^2} y_{j-1} + \frac{(t - t_{j-2})(t - t_{j-1})}{2\mathrm{h}^2} y_j. \tag{3.28}$$

Thus, v_k for $k = 1, 2...$ can be obtained using the formula

$$\begin{aligned} v_k &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{\omega(\tau) z'(\tau) y(\tau))}{(z(t_k) - z(\tau))^\alpha} \, d\tau = \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{\omega(\tau) z'(\tau) y(\tau))}{(z(t_k) - z(\tau))^\alpha} \, d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} \frac{\omega(\tau) z'(\tau)}{(z(t_k) - z(\tau))^\alpha} \left(\frac{y_1 - y_0}{\mathrm{h}} \right) d\tau \\ &\quad + \sum_{j=2}^{k} \frac{1}{\Gamma(1-\alpha)} \int_{t_{j-1}}^{t_j} \frac{\omega(\tau) z'(\tau)}{(z(t_k) - z(\tau))^\alpha} y_{2j}(\tau) \, d\tau \\ &= \sum_{j=0}^{k} S_j^k y_j, \end{aligned} \tag{3.29}$$

where the coefficients S_j^k are given as under the transformation $\tau = t_{j-1} + p\text{h}$

$$\text{for } k=1, \qquad \begin{cases} S_0^1 = a_0 \\ S_1^1 = -a_0 \end{cases}$$

$$\text{for } k=2, \qquad \begin{cases} S_0^2 = D_{2,2} \\ S_1^2 = C_{2,2} + a_1 \\ S_2^2 = B_{2,2} - a_1 \end{cases}$$

$$\text{for } k=3, \qquad \begin{cases} S_0^3 = D_{3,3} \\ S_1^3 = C_{3,3} + D_{3,2} \\ S_2^3 = B_{3,3} + C_{3,2} + a_2 \\ S_3^3 = B_{3,2} - a_2 \end{cases}$$

For $k \geq 4$,

$$\begin{cases} S_0^k = D_{k,k} \\ S_1^k = C_{k,k} + D_{k,k-1} \\ S_j^k = B_{k,k-j+2} + C_{k,k-j+1} + D_{k,k-j} \quad \text{for} \quad (2 \leq j \leq k-2) \\ S_{k-1}^k = B_{k,3} + C_{k,2} + a_{k-1} \\ S_k^k = B_{k,2} - a_{k-1} \end{cases} \tag{3.30}$$

And

$$a_{k-1} = \frac{\text{h}(\omega_1 + \omega_0)}{2\Gamma(1-\alpha)} \int_0^1 \frac{z'(t_0 + p\text{h})}{(z(t_k) - z(t_0 + p\text{h}))^\alpha} \, dp, \tag{3.31}$$

$$B_{k,j} = \frac{\text{h}(\omega_j + \omega_{j-1})}{2\Gamma(1-\alpha)} \int_0^1 \frac{z'(t_{j-1} + p\text{h})}{(z(t_k) - z(t_{j-1} + p\text{h}))^\alpha} \frac{p(p-1)}{2} \, dp, \tag{3.32}$$

$$C_{k,j} = \frac{\mathrm{h}(\omega_j + \omega_{j-1})}{2\Gamma(1-\alpha)} \int_0^1 \frac{z'(t_{j-1} + \mathrm{ph})}{(z(t_k) - z(t_{j-1} + \mathrm{ph}))^{\alpha}} (p+1)(p-1)\, dp, \tag{3.33}$$

$$D_{k,j} = \frac{\mathrm{h}(\omega_j + \omega_{j-1})}{2\Gamma(1-\alpha)} \int_0^1 \frac{z'(t_{j-1} + \mathrm{ph})}{(z(t_k) - z(t_{j-1} + \mathrm{ph}))^{\alpha}} \frac{p(p+1)}{2}\, dp. \tag{3.34}$$

Since the function $y(t)$ is bounded at $t = 0, v_0 = 0$. So,

$$V = [v_0, v_1, v_2, v_3, ..., v_{\mathcal{N}}]^T = FU, \tag{3.35}$$

where,

$$U = [y_0, y_1, u_2, u_3, ..., y_{\mathcal{N}}]^T, \tag{3.36}$$

and

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ S_0^1 & S_1^1 & 0 & 0 & 0 & \dots & 0 & 0 \\ S_0^2 & S_1^2 & S_2^2 & 0 & 0 & \dots & 0 & 0 \\ S_0^3 & S_1^3 & S_2^3 & S_3^3 & 0 & \dots & 0 & 0 \\ S_0^4 & S_1^4 & S_2^4 & S_3^4 & S_4^4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_0^{\mathcal{N}-1} & S_1^{\mathcal{N}-1} & S_2^{\mathcal{N}-1} & S_3^{\mathcal{N}-1} & S_4^{\mathcal{N}-1} & \dots & S_{\mathcal{N}-1}^{\mathcal{N}-1} & 0 \\ S_0^{\mathcal{N}} & S_1^{\mathcal{N}} & S_2^{\mathcal{N}} & S_3^{\mathcal{N}} & S_4^{\mathcal{N}} & \dots & S_{\mathcal{N}-1}^{\mathcal{N}} & S_{\mathcal{N}}^{\mathcal{N}} \end{bmatrix}_{(\mathcal{N}+1)\times(\mathcal{N}+1)} . \tag{3.37}$$

Now $J(y)$ can be re-written as using Eqs. (3.25) and (3.35),

$$J \cong U^T A U, \text{ where } A = F^T E F. \tag{3.38}$$

Following the similar concept, we approximate Eq. (3.2) using quadratic approximation and we get

$$K \cong U^T B U = 2, \tag{3.39}$$

where,

$$B = \begin{bmatrix} 40b_1+b_2 & 20b_1-7b_2 & -4b_2 & 0 & 0 & \dots & 0 & 0 \\ 20b_1-7b_2 & 40b_1+64b_2+b_3 & 23b_2-7b_3 & -4b_3 & 0 & \dots & 0 & 0 \\ -4b_2 & 23b_2-7b_3 & 31b_2+64b_3+b_4 & 23b_3-7b_4 & -4b_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \dots & 64b_{\mathcal{N}-1}+31b_{\mathcal{N}-2} & 23b_{\mathcal{N}} \\ 0 & 0 & 0 & 0 & 0 & \dots & 23b_{\mathcal{N}} & 31b_{\mathcal{N}} \end{bmatrix}, \tag{3.40}$$

is a matrix of order $(\mathcal{N}+1) \times (\mathcal{N}+1)$ and $b_j = \frac{120}{\mathrm{h}}[z_j - z_{j-1}]$.

We repeat the steps discussed for the linear scheme to obtain vector U using quadratic approximation.

3.2.4 The Quadratic-Linear Scheme(S3)

Here, first we partition the domain $[0, 1]$ into $\mathcal{N}$ sub-intervals and using a quadratic and linear interpolating function and approximate $v(t)$ and $y(t)$ over $[t_{j-1}, t_j]$, $j = 1, 2, ...\mathcal{N}$ as discussed in schemes S2 and S1, respectively. That is why this method is called the quadratic-linear technique. Now the functional $J(y)$ and isoperimetric constraint $K(y)$ are approximated by Eq. (3.25) and Eq. (3.17) respectively, where matrices, E, F and B are given by Eqs. (3.27), (3.15) and (3.18) respectively. Now, the GICFVP has become the generalized eigenvalue problem which is given as

$$AU = \lambda BU. \tag{3.41}$$

where, $A = F^T EF$. Solution of (3.41) provides us the desired minimal solution of functional, and $2\lambda_1$, gives us an approximate value of $J(y)$.

3.3 Error Analysis and Convergence of the Eigenvalues

Here, we derive the error bound of the approximated solution and convergence of the eigenvalues presented in Section 3.2.

3.3.1 Error Analysis of Linear Scheme(S1)

For the first scheme S1, we interpolate $y(t)$ by linear interpolating function $y_{1j}(t)$ on $[t_{j-1}, t_j]$ then we have,

$$y(t) - y_{1j}(t) = \frac{y''(\xi_j)}{2!}(t - t_j)(t - t_{j-1}), \qquad \xi_j \in [t_{j-1}, t_j], \tag{3.42}$$

where $y_{1j}(t)$ is given by the Eq. (3.8). To make our computation easier, we choose some notations such that $z(\tau) = s$ and $z(t_j) = z_j$.

Theorem 3.1. *For $0 < \alpha < 1$, and $y(t) \in C^2[0,1]$, the error bound is given by*

$$\|y - y_{1j}\|_{L^2[0,1]} \leq C_1 \mathrm{h}^{2-\alpha} \max_{0 \leq t \leq t_j} |\ y''(t)\ |, \tag{3.43}$$

where, C_1 is a constant.

Proof. First, we estimate the difference $\left|{}_0D^{\alpha}_{t;[z;\omega,1]}y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]}y_{1j}(t)\right|$, such that

$$\begin{aligned}\left|{}_0D^{\alpha}_{t;[z;\omega,1]}y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]}y_{1j}(t)\right| &= \left|{}_0D^{\alpha}_{t;[z;\omega,1]}(y - y_{1j})(t)\right| \\ &= \left|\frac{[\omega(t_j)]^{-1}}{\Gamma(1-\alpha)\, z'(t)} \frac{d}{dt} \int_0^t \frac{\omega(\tau) z'(\tau)(y - y_{1j})(\tau)}{(z(t) - z(\tau))^{\alpha}}\, d\tau\right|.\end{aligned} \tag{3.44}$$

Let $v(t) = \left[\frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\omega(\tau)z'(\tau)(y-y_{1j})(\tau)}{(z(t)-z(\tau))^\alpha}\, d\tau\right]$, and by approximating the first derivative of $v(t)$, we get

$$\frac{dv}{dt} = \frac{v(t_j) - v(t_{j-1})}{\mathrm{h}},$$

so that

$$\begin{aligned}
\mid v(t_j) \mid &= \left| \frac{1}{\Gamma(1-\alpha)} \int\limits_0^{t_j} \frac{\omega(t_j)z'(\tau)(y-y_{1j})(\tau)}{(z(t_j)-z(\tau))^\alpha}\, d\tau \right| \\
&= \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \int\limits_{t_{k-1}}^{t_k} \frac{\omega(t_j)z'(\tau)(y-y_{1j})(\tau)}{(z(t_j)-z(\tau))^\alpha}\, d\tau \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \left| \sum_{k=1}^{j} \frac{\mathrm{h}^2}{8} y''(\xi_k)\omega(t_k) \int\limits_{t_{k-1}}^{t_k} \frac{z'(\tau)}{(z(t_j)-z(\tau))^\alpha}\, d\tau \right| \\
&= \frac{1}{\Gamma(1-\alpha)} \left| \sum_{k=1}^{j} \frac{\mathrm{h}^2}{8} y''(\xi_k)\omega(t_k) \int\limits_{z_{k-1}}^{z_k} \frac{1}{(z(t_j)-s)^\alpha}\, ds \right| \\
&= \frac{1}{\Gamma(2-\alpha)} \left| \sum_{k=1}^{j} \frac{\mathrm{h}^2}{8} y''(\xi_k)\omega(t_k) \left[-(z(t_j)-z_k)^{-\alpha+1} + (z(t_j)-z_{k-1})^{-\alpha+1}\right] \right| \\
&\leq \frac{\mathrm{h}^2}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid \max_{0\leq t\leq t_j} \mid \omega(t_j) \mid \left| \sum_{k=1}^{j} \left[-(z(t_j)-z_k)^{-\alpha+1} \right.\right. \\
&\left.\left. +(z(t_j)-z_{k-1})^{-\alpha+1}\right] \right| \\
&\leq \frac{\mathrm{h}^2}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid \max_{0\leq t\leq t_j} \mid \omega(t_j) \mid \max_{0\leq t\leq t_j} \mid z_j - z_{j-1} \mid^{-\alpha+1}. \qquad (3.45)
\end{aligned}$$

Similarly, we can find $v(t_{j-1})$

$$\mid v(t_{j-1}) \mid \leq \frac{\mathrm{h}^2}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid \max_{0\leq t\leq t_j} \mid \omega(t_j) \mid \max_{0\leq t\leq t_j} \mid z_{j-1} - z_{j-2} \mid^{-\alpha+1}. \qquad (3.46)$$

Now, from the Eqs. (3.44-3.46), we have

$$
\begin{aligned}
\left|{}_0D^{\alpha}_{t;[z;\omega,1]}y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]}y_{1j}(t)\right| &= \left|\frac{[\omega(t_j)]^{-1}}{z'(t)}\frac{dv}{dt}\right| \\
&\leq \left|\frac{[\omega(t_j)]^{-1}}{z'(t)}\right| \frac{\mathrm{h}^2}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid \max_{0\leq t\leq t_j} \mid \omega(t_j) \mid \\
&\left(\max_{0\leq t\leq t_j} \mid z_j - z_{j-1} \mid^{-\alpha+1} + \max_{0\leq t\leq t_j} \mid z_{j-1} - y_{j-2} \mid^{-\alpha+1}\right) \\
&\leq (2L)^{1-\alpha}\frac{\mathrm{h}^{2-\alpha}}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid \max_{0\leq t\leq t_j} \mid \omega(t_j) \mid \\
&\left|\frac{[\omega(t_j)]^{-1}}{z'(t)}\right| \\
&\leq (2L)^{1-\alpha}\frac{\mathrm{h}^{2-\alpha}}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid \max_{0\leq t\leq t_j} \mid \omega(t_j) \mid \\
&\max_{0\leq t\leq t_j}\left|[\omega(t_j)]^{-1}\right| \max_{0\leq t\leq 1}\left|[z'(t)]^{-1}\right| \\
&= M(2L)^{1-\alpha}\frac{\mathrm{h}^{2-\alpha}}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid, \qquad (3.47)
\end{aligned}
$$

where $z(t)$ satisfies the Lipschitz condition on $[t_{j-1}, t_j]$ and L is Lipschitz constant and $M = \max_{0\leq t\leq 1}\left|[z'(t)]^{-1}\right|$.

Now, using the fractional Poincare-Friedrichs inequality [103] and Eq. (3.47), we have

$$
\begin{aligned}
\|y - y_{1j}\|_{L^2[0,1]} &\leq C\left\|{}_0D^{\alpha}_{t;[z;\omega,1]}y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]}y_{1j}(t)\right\|_{L^2[0,1]} \\
&= C\left(\int_0^1 \left|{}_0D^{\alpha}_{t;[z;\omega,1]}y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]}y_{1j}(t)\right|^2 dt\right)^{1/2} \\
&\leq MC(2L)^{1-\alpha}\frac{\mathrm{h}^{2-\alpha}}{8\Gamma(2-\alpha)} \max_{0\leq t\leq t_j} \mid y''(t) \mid \left(\int_0^1 1\, dt\right)^{1/2} \\
&= C_1\mathrm{h}^{2-\alpha} \max_{0\leq t\leq t_j} \mid y''(t) \mid.
\end{aligned}
$$

The proof is completed. $\square$

3.3.2 Error Analysis of Quadratic Scheme(S2)

For the second scheme S2, we interpolate $y(t)$ on $[t_0, t_1]$ by $y_{1j}(t)$ and on $[t_{j-2}, t_j]$ (for $j \geq 2$,) by $y_{2j}(t)$, then we have

$$
\begin{aligned}
(y - y_{1j})(t) &= \frac{y''(\xi_1)}{2!}(t - t_1)(t - t_0), \qquad \xi_1 \in [t_0, t_1],\\
(y - y_{2j})(t) &= \frac{y'''(\xi_j)}{3!}(t - t_j)(t - t_{j-1})(t - t_{j-2}), \qquad \xi_j \in [t_{j-2}, t_j].
\end{aligned}
$$

Theorem 3.2. *For $0 < \alpha < 1$, $y(t) \in C^3[0, 1]$, the error bound is given by*

1. *If $j = 1$, then*

$$\|y - y_{1j}\|_{L^2[0,1]} \leq C_1 \mathrm{h}^{2-\alpha} \max_{0 \leq t \leq t_1} \mid y''(t) \mid . \tag{3.48}$$

2. *If $j \geq 2$, then*

$$\|y - y_{2j}\|_{L^2[0,1]} \leq C_2 \mathrm{h}^{3-\alpha} \max_{t_1 \leq t \leq t_j} \mid y'''(t) \mid , \tag{3.49}$$

where, C_1, and C_2 are constants.

Proof. (1) For $j = 1$, scheme S2 becomes scheme S1, and its error bound is already proved in Theorem 3.1.

(2) For $j \geq 2$,

$$
\begin{aligned}
\left|{}_0D^{\alpha}_{t;[z;\omega,1]} y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]} y_{2j}(t)\right| &= \left|{}_0D^{\alpha}_{t;[z;\omega,1]} (y - y_{2j})(t)\right|\\
&= \left|\frac{[\omega(t_j)]^{-1}}{\Gamma(1-\alpha)\, z'(t)} \frac{d}{dt} \int_0^t \frac{\omega(\tau) z'(\tau)(y - y_{2j})(\tau)}{(z(t) - z(\tau))^{\alpha}}\, d\tau\right|.
\end{aligned}
\tag{3.50}
$$

Let $v(t) = \left[\frac{1}{\Gamma(1-\alpha)} \int\limits_{0}^{t} \frac{\omega(\tau)z'(\tau)(y-y_{2j})(\tau)}{(z(t)-z(\tau))^{\alpha}}\, d\tau\right]$, and by approximating the first derivative of $v(t)$, we get

$$\frac{dv}{dt} = \frac{v(t_j) - v(t_{j-1})}{\mathrm{h}},$$

so that

$$\begin{aligned}
\mid v(t_j) \mid &= \left|\frac{1}{\Gamma(1-\alpha)} \int\limits_{t_1}^{t_j} \frac{\omega(t_j)z'(\tau)(y-y_{2j})(\tau)}{(z(t_j)-z(\tau))^{\alpha}}\, d\tau\right| \\
&= \left|\frac{1}{\Gamma(1-\alpha)} \sum_{k=2}^{j} \int\limits_{t_{k-1}}^{t_k} \frac{\omega(t_k)z'(\tau)(y-y_{2j})(\tau)}{(z(t_j)-z(\tau))^{\alpha}}\, d\tau\right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \left|\sum_{k=2}^{j} \frac{\mathrm{h}^3}{12} y'''(\xi_k)\omega(t_k) \int\limits_{t_{k-1}}^{t_k} \frac{z'(\tau)}{(z(t_j)-z(\tau))^{\alpha}}\, d\tau\right| \\
&= \frac{1}{\Gamma(2-\alpha)} \left|\sum_{k=2}^{j} \frac{\mathrm{h}^3}{12} y'''(\xi_k)\omega(t_k) \left[-(z(t_j)-z_k)^{-\alpha+1} + (z(t_j)-z_{k-1})^{-\alpha+1}\right]\right| \\
&\leq \frac{\mathrm{h}^3}{12\Gamma(2-\alpha)} \max_{t_1\leq t\leq t_j} \mid y'''(t) \mid \max_{t_1\leq t\leq t_j} \mid \omega(t_j) \mid \\
&\quad \left|\sum_{k=2}^{j} \left[-(y(t_j)-y_k)^{-\alpha+1} + (z(t_j)-z_{k-1})^{-\alpha+1}\right]\right| \\
&\leq \frac{\mathrm{h}^3}{12\Gamma(2-\alpha)} \max_{t_1\leq t\leq t_j} \mid y'''(t) \mid \max_{t_1\leq t\leq t_j} \mid \omega(t_j) \mid \max_{t_1\leq t\leq t_j} \mid z_j - z_{j-1} \mid^{-\alpha+1}.
\end{aligned} \tag{3.51}$$

Similarly, we can find $v(t_{j-1})$

$$\mid v(t_{j-1}) \mid \leq \frac{\mathrm{h}^3}{12\Gamma(2-\alpha)} \max_{t_1\leq t\leq t_j} \mid y'''(t) \mid \max_{t_1\leq t\leq t_j} \mid \omega(t_j) \mid \max_{t_1\leq t\leq t_j} \mid z_{j-1} - y_{j-2} \mid^{-\alpha+1}. \tag{3.52}$$

Now, from the Eqs. (3.50-3.52), we have

$$
\begin{aligned}
\left|{}_0D^{\alpha}_{t;[z;\omega,1]}y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]}y_{1j}(t)\right| &= \left|\frac{[\omega(t_j)]^{-1}}{z'(t)}\frac{dv}{dt}\right| \\
&\leq \left|\frac{[\omega(t_j)]^{-1}}{z'(t)}\right| \frac{\mathrm{h}^3}{12\Gamma\left(2-\alpha\right)} \max_{t_1 \leq t \leq t_j} \mid y'''(t) \mid \\
&\max_{t_1 \leq t \leq t_j} \mid \omega(t_j) \mid \left(\max_{t_1 \leq t \leq t_j} \mid z_j - z_{j-1} \mid^{-\alpha+1} \right. \\
&\left. + \max_{t_1 \leq t \leq t_j} \mid z_{j-1} - z_{j-2} \mid^{-\alpha+1} \right) \\
&\leq (2L)^{1-\alpha} \frac{\mathrm{h}^{3-\alpha}}{12\Gamma\left(2-\alpha\right)} \max_{t_1 \leq t \leq t_j} \mid y'''(t) \mid \max_{t_1 \leq t \leq t_j} \mid \omega(t_j) \mid \\
&\left|\frac{[\omega(t_j)]^{-1}}{z'(t)}\right| \\
&\leq (2L)^{1-\alpha} \frac{\mathrm{h}^{3-\alpha}}{12\Gamma\left(2-\alpha\right)} \max_{t_1 \leq t \leq t_j} \mid y'''(t) \mid \max_{t_1 \leq t \leq t_j} \mid \omega(t_j) \mid \\
&\max_{t_1 \leq t \leq t_j} \left|[\omega(t_j)]^{-1}\right| \max_{t_1 \leq t \leq 1} \left|[z'(t)]^{-1}\right| \\
&= M_1 (2L)^{1-\alpha} \frac{\mathrm{h}^{3-\alpha}}{12\Gamma\left(2-\alpha\right)} \max_{t_1 \leq t \leq t_j} \mid y'''(t) \mid, \qquad (3.53)
\end{aligned}
$$

where $z(t)$ satisfies the Lipschitz condition on $[t_{j-1}, t_j]$, L is Lipschitz constant and $M_1 = \max\limits_{t_1 \leq t \leq 1} \mid [z'(t)]^{-1} \mid$.

Now following similar steps as done in Theorem 3.1, we get

$$
\begin{aligned}
\|y - y_{2j}\|_{L^2[0,1]} &\leq C \left\| {}_0D^{\alpha}_{t;[z;\omega,1]}y(t) - {}_0D^{\alpha}_{t;[z;\omega,1]}y_{2j}(t) \right\|_{L^2[0,1]} \\
&\leq C(2L)^{1-\alpha} M_1 \frac{\mathrm{h}^{3-\alpha}}{12\Gamma\left(2-\alpha\right)} \max_{t_1 \leq t \leq t_j} \mid y'''(t) \mid \left(\int_0^1 1\, dt \right)^{1/2} \\
&= C_2 \mathrm{h}^{3-\alpha} \max_{t_1 \leq t \leq t_j} \mid y'''(t) \mid .
\end{aligned}
$$

The proof is completed. □

3.3.3 Convergence of the Eigenvalues for Schemes S1 and S2

The considered GICFVP given by Eqs. (3.1) and (3.2) can be formulated as: Find $\lambda \in \mathbb{R}$ and $y \in V$, with $y \neq 0$ such that

$$B(y, x) = \lambda(y, x) \quad \forall x \in V, \tag{3.54}$$

where

$$B(y, x) = \int_0^1 \left[\left({}_0D^{\alpha}_{t;[z;\omega,1]} y(t) \right) \left({}_0D^{\alpha}_{t;[z;\omega,1]} x(t) \right) \right] z'(t)\, dt.$$

We approximate the solution $y \in V$ by $y_{\mathrm{h}} \in V_{\mathrm{h}}$ on a finite dimensional space $V_{\mathrm{h}} \subset V$ and the corresponding solution of the approximated problem becomes:

$$B(y_{\mathrm{h}}, x) = \lambda_{\mathrm{h}}(y_{\mathrm{h}}, x) \quad \forall x \in V_{\mathrm{h}}. \tag{3.55}$$

Let $\lambda^{(j)}, j \in \mathbb{N}$ be the sequence of monotonically increasing eigenvalues and $(y^{(j)}$ be the corresponding eigenfunctions of the Eq. (3.54). Let $W^{(j)}$ be the eigenspace spanned by eigenfunctions $\{(y^{(j)}\}$ and $V^{(j)}$ is the subspace of V with dimension equal to j, such that

$$W = \oplus_{i=1}^{j} W^{(j)},$$

where, $\oplus_{i=1}^{j} W^{(j)}$ denotes the direct sum of the eigenspaces.

Define the Rayleigh quotient for Eq. (3.54) as

$$\lambda^{(j)} = \min_{W \in V^{(j)}} \max_{x \in W} \frac{B(x, x)}{(x, x)}. \tag{3.56}$$

Let $W_{\mathrm{h}}^{(j)}$ =span$\{y_{\mathrm{h}}^{(j)}\}$ represents the associated eigenspace and $V_{\mathrm{h}}^{(j)}$ denotes the set of all subspaces of V_{h} with dimension equal to j, such that $W_{\mathrm{h}} = \oplus_{i=1}^{j} W_{\mathrm{h}}^{(j)}$. The analogous Rayleigh quotient can be defined for approximated eigenvalues as,

$$\lambda_{\mathrm{h}}^{(j)} = \min_{W_{\mathrm{h}} \in V_{\mathrm{h}}^{(j)}} \max_{x \in W_{\mathrm{h}}} \frac{B(x,x)}{(x,x)}. \tag{3.57}$$

Now, we obtain the bound for $\lambda_{\mathrm{h}}^{(j)}$ following [104]

$$\begin{aligned}
\lambda_{\mathrm{h}}^{(j)} &\leq \max_{y_{\mathrm{h}} \in W_{\mathrm{h}}} \frac{\left\| {}_0D^{\alpha}_{t;[z;\omega,1]} y_{\mathrm{h}} \right\|^2_{L^2[0,1]}}{\|y_{\mathrm{h}}\|^2_{L^2[0,1]}} = \max_{y \in V_{\mathrm{h}}^{(j)}} \frac{\left\| {}_0D^{\alpha}_{t;[z;\omega,1]} y_{\mathrm{h}} \right\|^2_{L^2[0,1]}}{\|y_{\mathrm{h}}\|^2_{L^2[0,1]}}, \\
&\leq \max_{y \in V_{\mathrm{h}}^{(j)}} \frac{\left\| {}_0D^{\alpha}_{t;[z;\omega,1]} y \right\|^2_{L^2[0,1]}}{\|y_{\mathrm{h}}\|^2_{L^2[0,1]}} = \max_{y \in V_{\mathrm{h}}^{(j)}} \frac{\left\| {}_0D^{\alpha}_{t;[z;\omega,1]} y_{\mathrm{h}} \right\|^2_{L^2[0,1]}}{\|y\|^2_{L^2[0,1]}} \frac{\|y\|^2_{L^2[0,1]}}{\|y_{\mathrm{h}}\|^2_{L^2[0,1]}}, \\
&\leq \lambda^{(j)} \max_{y \in V_{\mathrm{h}}^{(j)}} \frac{\|y\|^2_{L^2[0,1]}}{\|y_{\mathrm{h}}\|^2_{L^2[0,1]}},
\end{aligned} \tag{3.58}$$

also, we have

$$\|y_{\mathrm{h}}\|_{L^2[0,1]} \geq \|y\|_{L^2[0,1]} - \|y - y_{\mathrm{h}}\|_{L^2[0,1]}. \tag{3.59}$$

From Eqs. (3.43), and (3.59), we have

$$\|y_{\mathrm{h}}\|_{L^2[0,1]} \geq \left(1 - C_1 \mathrm{h}^{2-\alpha} \frac{\max_{0 \leq t \leq t_j} \mid y''(t) \mid}{\|y\|_{L^2[0,1]}} \right) \|y\|_{L^2[0,1]}. \tag{3.60}$$

The convergence order of the eigenvalues for the linear scheme (S1) is given by Eqs. (3.58) and (3.60)

$$\lambda_{\mathrm{h}}^{(j)} \leq \lambda^{(j)} \max_{y \in V_{\mathrm{h}}^{(j)}} \frac{1}{\left(1 - C_1 \mathrm{h}^{2-\alpha} \frac{\max_{0 \leq t \leq t_j} |y''(t)|}{\|y\|_{L^2[0,1]}} \right)^2} \approx \lambda^{(j)} \left(1 + 2M_2 C_1 \mathrm{h}^{2-\alpha} \frac{1}{\|y\|_{L^2[0,1]}} \right), \tag{3.61}$$

where, $M_2 = \max_{0\leq t\leq t_j} \mid y''(t) \mid$.

From Eqs. (3.49) and (3.59), we have

$$\|y_{\mathrm{h}}\|_{L^2[0,1]} \geq \left(1 - C_2 \mathrm{h}^{3-\alpha} \frac{\max_{t_1\leq t\leq t_j} \mid y'''(t) \mid}{\|y\|_{L^2[0,1]}}\right) \|y\|_{L^2[0,1]}. \tag{3.62}$$

Similarly, the convergence order of the eigenvalues for the scheme S2 is given by Eqs. (3.58) and (3.62)

$$\lambda_{\mathrm{h}}^{(j)} \leq \lambda^{(j)} \max_{y\in V_{\mathrm{h}}^{(j)}} \frac{1}{\left(1 - C_2 \mathrm{h}^{3-\alpha} \frac{\max_{t_1\leq t\leq t_j} |y'''(t)|}{\|y\|_{L^2[0,1]}}\right)^2} \approx \lambda^{(j)} \left(1 + 2M_3 C_2 \mathrm{h}^{3-\alpha} \frac{1}{\|y\|_{L^2[0,1]}}\right) \tag{3.63}$$

where, $M_3 = \max_{t_1\leq t\leq t_j} \mid y'''(t) \mid$,

which completes the proof.

3.4 Numerical Results and Discussion

Here, we discuss simulation results for the test cases varying α and the step size h to show the effectiveness of the three numerical schemes for different weight $\omega(t)$ and scale function $z(t)$. Matlab R2018b is used to execute all the numerical simulations. We implement three numerical methods and discuss the nature of the solution with respect to the exact solutions for integer-order $\alpha = 1$. In each cases, convergence rate is calculated by changing step size h. To calculate the convergence rate, we use the following formula.

Let y_e be the exact solution, y_{h} be the numerical solution at step size h and C be any constant. The convergence order R can be determined by [59]:

$$\frac{\Delta e_{\mathrm{h}}}{\Delta e_{\frac{\mathrm{h}}{2}}} = 2^R, \tag{3.64}$$

$$\text{or, } R = \log_2 \left[\frac{\Delta e_{\mathrm{h}}}{\Delta e_{\frac{\mathrm{h}}{2}}} \right], \tag{3.65}$$

where, $e_{\mathrm{h}} = y_e - y_{\mathrm{h}} = C\mathrm{h}^R$, $e_{\frac{\mathrm{h}}{2}} = y_e - y_{\frac{\mathrm{h}}{2}} = C(\mathrm{h}/2)^R$ and $\Delta e_{\mathrm{h}} = e_{\mathrm{h}} - e_{\frac{\mathrm{h}}{2}}$.

Tables 3.1, 3.5 and 3.9 show the values of the functional J at different step sizes h and α. The functional values are directly proportional to step size; as h decreases, J also decreases. We evaluate the convergence rate of the method by Eq. (3.64) because the exact solution for the order $\alpha \in (0, 1)$ is not available. Tables 3.2, 3.6, 3.10, and Tables 3.3, 3.7, 3.11 demonstrate the relative error calculation of the problem and ratios of these relative errors respectively. Excluding a pair of nodes towards the ends, the values of R demonstrate that the convergence order is about 2. From Tables 3.4, 3.8, and 3.12, we observe that the numerical convergence order validates the theoretical convergence order. The approximated solutions are plotted in Figures 3.1, for the schemes S1, S2, and S3 and α along with the corresponding integer-order solution $\alpha = 1$. For $z(t) = t$ and $\omega(t) = 1$, on comparing our results to those of Agrawal et al. [59] and Pandey et al. [65, 66], we found that they agreed with their results in particular case. Also, our numerical solutions are close to the exact solution when α tends to 1. The plotted Figures 3.2 demonstrate that the numerical solution converges to the analytical solution for schemes S1, S2, and S3 as the step size h decreases. In other case, we also obtain the approximate solution for $z(t) = t$, $\omega(t) = \exp(t)$ and there result are shown through Figures (3.3) and (3.4). Comparing all three schemes, we observe that scheme S2 perform better than S1 and S3. Further other choices of $z(t)$ and $w(t)$ can also be used to obtain the simulation results.

TABLE 3.1: Convergence of the functional J using S1 for $z(t) = t$, $\omega(t) = 1$.

$\alpha/\mathcal{N}$	0.5	0.6	0.7	0.8	0.9	0.99
4	3.9089	5.2273	7.2602	10.2604	14.6038	20.0580
8	3.8743	5.2244	7.2114	10.1065	14.2242	19.3276
16	3.8544	5.1614	7.1583	10.0579	14.1409	19.1512
32	3.7835	5.0880	7.1049	10.0283	14.1186	19.1079
64	3.7156	5.0208	7.0595	10.0072	14.1105	19.0972

TABLE 3.2: Relative error calculation for fixed $\alpha = 0.9$ and $z(t) = t$, $\omega(t) = 1$ using S1.

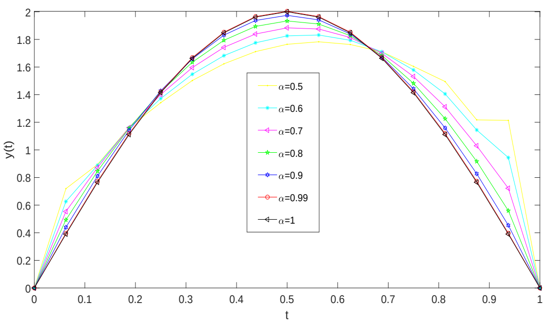
	$\Delta e_{\mathrm{h}} = e_{\mathrm{h}} - e_{\mathrm{h}/2}$		
t/h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
1/8	1.07×10^{-2}	2.7×10^{-3}	8.0×10^{-4}
2/8	1.5×10^{-2}	4.0×10^{-3}	1.2×10^{-3}
3/8	1.88×10^{-2}	5.0×10^{-3}	1.4×10^{-3}
4/8	1.99×10^{-2}	4.9×10^{-3}	1.4×10^{-3}
5/8	1.77×10^{-2}	4.2×10^{-3}	9.0×10^{-4}
6/8	1.28×10^{-3}	2.7×10^{-3}	4.0×10^{-4}
7/8	8.5×10^{-3}	1.0×10^{-3}	3.0×10^{-4}

TABLE 3.3: Ratios of the relative error for fixed $\alpha = 0.9$ and $z(t) = t$, $\omega(t) = 1$ based on the Table 3.2.

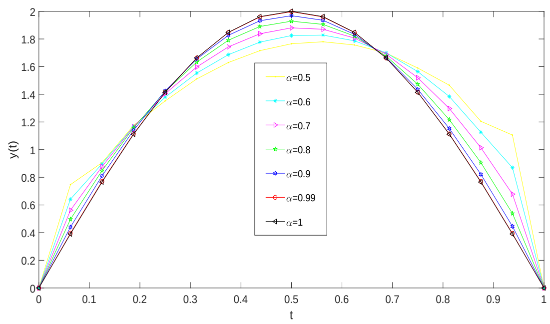
	$R = \Delta e_{\mathrm{h}}/e_{\mathrm{h}/2}$	
t/h	$\frac{1}{8}$	$\frac{1}{16}$
1/8	3.96	3.38
2/8	3.75	3.33
3/8	3.76	3.57
4/8	4.06	3.5
5/8	4.21	4.67
6/8	4.74	6.75
7/8	8.5	3.33

TABLE 3.4: Convergence order (CO) and maximum absolute error (MAE) for scheme S1 using different value of α and h and fixed $z(t) = t$, $w(t) = 1$.

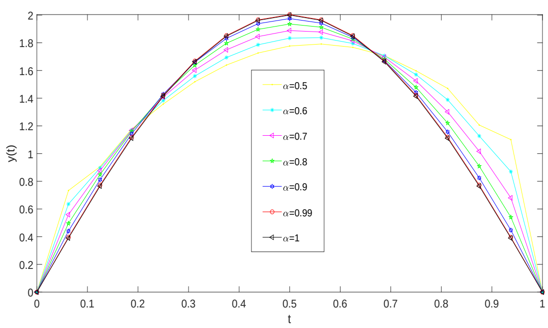
α	h	MAE	CO	CPU time(s)
0.2	1/4			22.036
	1/8	7.40×10^{-2}		45.718
	1/16	2.24×10^{-2}	1.724	118.020
	1/32	6.75×10^{-3}	1.730	410.278
0.4	1/4			30.181
	1/8	1.58×10^{-1}		55.290
	1/16	5.25×10^{-2}	1.585	121.453
	1/32	1.74×10^{-2}	1.593	512.277



S1

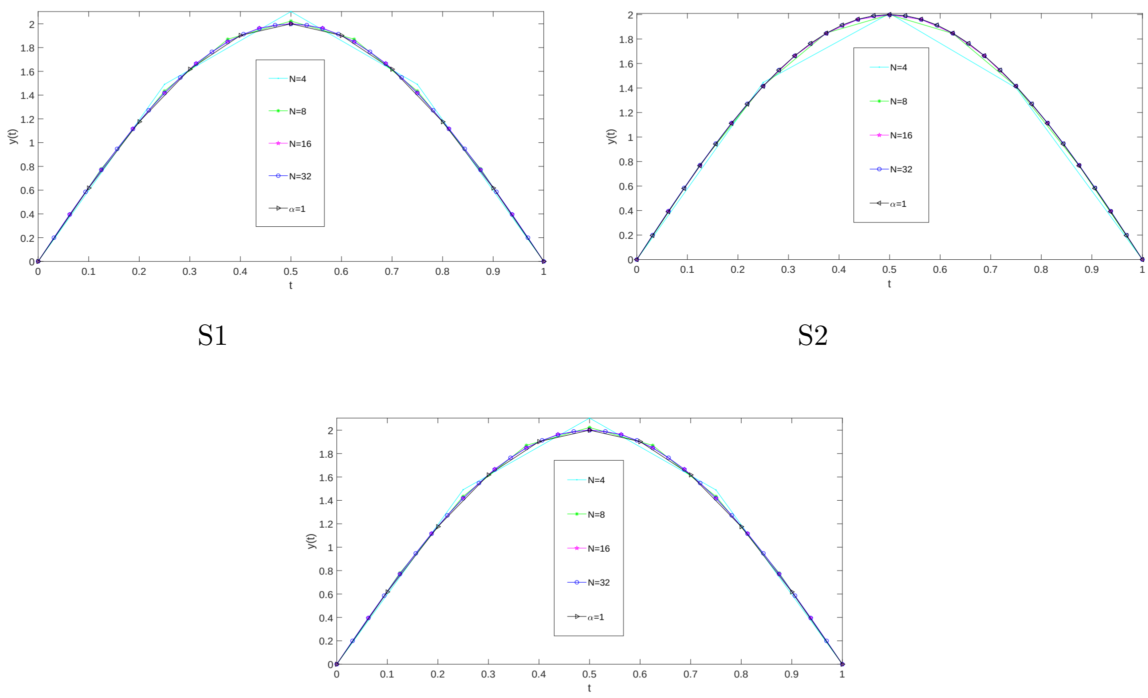


S2



S3

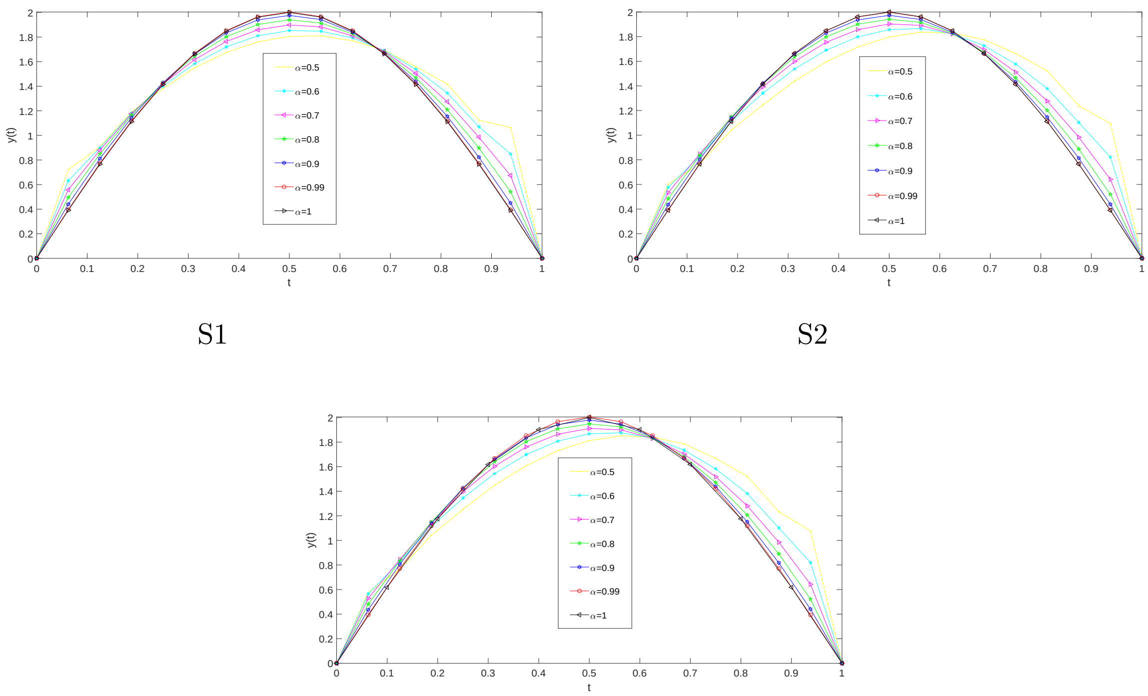
FIGURE 3.1: Comparison of the exact solution ($\alpha = 1$) and other numerical solution using S1, S2 and S3 for various α and $\mathcal{N} = 16, z(t) = t, \omega(t) = 1$.



S1 S2

S3

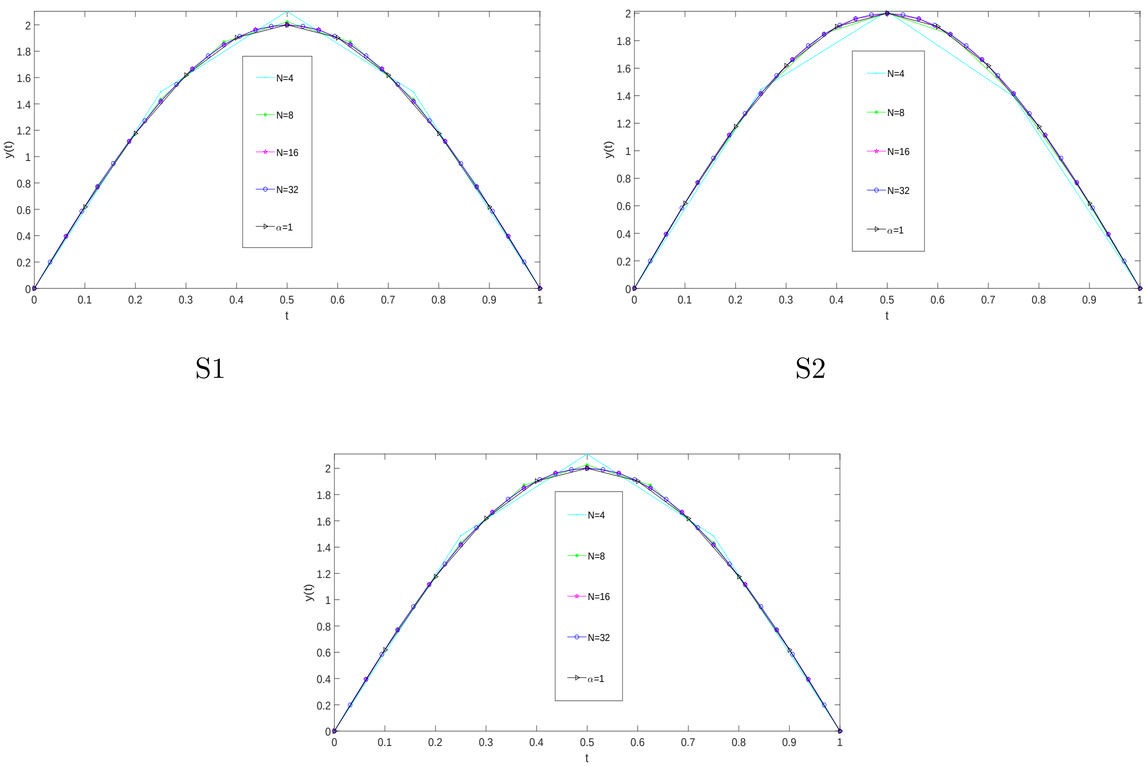
FIGURE 3.2: The plots of the numerical solution for $\alpha = 0.99$, $z(t) = t$, $w(t) = 1$ using S1, S2, and S3 at various values of $\mathcal{N}$ and the exact solution ($\alpha = 1$).



S1 S2

S3

FIGURE 3.3: Comparison of the exact solution ($\alpha = 1$) and other numerical solution using S1, S2 and S3 for various α and $\mathcal{N} = 16, z(t) = t, \omega(t) = \exp(t)$.



S1 S2 S3

FIGURE 3.4: The plots of the numerical solution for $\alpha = 0.99$ using S1, S2, and S3 at various values of $\mathcal{N}$ and the exact solution ($\alpha = 1$). $z(t) = t, \omega(t) = \exp(t)$.

TABLE 3.5: Convergence of the functional J using S2 for $z(t) = t$, $\omega(t) = 1$.

$\alpha/\mathcal{N}$	0.5	0.6	0.7	0.8	0.9	0.99
4	3.8974	5.2037	7.1653	10.0416	14.1098	19.1964
8	3.8802	5.1845	7.1594	10.0307	14.1092	19.0934
16	3.8137	5.1163	7.1213	10.0120	14.1004	19.0926
32	3.7431	5.0460	7.0744	10.0039	14.0561	19.0906
64	3.7065	5.0207	7.0395	10.0014	14.0285	19.0904

Table 3.6: Relative error calculation for fixed $\alpha = 0.9$ and $z(t) = t$, $\omega(t) = 1$ using S2.

$\Delta e_{\text{h}} = e_{\text{h}} - e_{\text{h}/2}$			
t/h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
1/8	1.11×10^{-2}	2.4×10^{-3}	7.0×10^{-4}
2/8	7.8×10^{-3}	2.0×10^{-3}	6.0×10^{-4}
3/8	5.2×10^{-3}	1.4×10^{-3}	5.0×10^{-4}
4/8	1.8×10^{-3}	4.0×10^{-4}	2.0×10^{-4}
5/8	2.3×10^{-3}	7.0×10^{-4}	2.0×10^{-4}
6/8	6.7×10^{-3}	2.0×10^{-3}	8.0×10^{-4}
7/8	1.16×10^{-2}	3.4×10^{-3}	1.3×10^{-3}

Table 3.7: Ratios of the relative error for fixed $\alpha = 0.9$ and $z(t) = t$, $\omega(t) = 1$ based on the Table 3.6.

$R = \Delta e_{\text{h}}/e_{\text{h}/2}$		
t/h	$\frac{1}{8}$	$\frac{1}{16}$
1/8	4.62	3.43
2/8	3.9	3.33
3/8	3.71	2.8
4/8	4.5	2
5/8	3.28	3.5
6/8	3.35	2.5
7/8	3.41	2.62

Table 3.9: Convergence of the cost functional J using S3 for $z(t) = t$, $\omega(t) = 1$.

$\alpha/\mathcal{N}$	0.5	0.6	0.7	0.8	0.9	0.99
4	4.0817	5.5110	7.6425	10.7799	15.3232	21.0372
8	4.0107	5.3440	7.3451	10.2590	14.4121	19.5729
16	3.9129	5.2227	7.2139	10.1071	14.1909	19.2125
32	3.8235	5.1267	7.1340	10.0473	14.1326	19.1233
64	3.7462	5.0490	7.0777	10.0162	14.1149	19.1011

TABLE 3.8: CO and MAE for scheme S2 using different value of α and h and fixed $z(t) = t$, $w(t) = 1$.

α	h	MAE	CO	CPU time(s)
0.2	1/4			44.104
	1/8	6.68×10^{-2}		91.098
	1/16	9.70×10^{-3}	2.784	235.297
	1/32	0.0014×10^{-3}	2.792	755.035
0.4	1/4			58.035
	1/8	8.91×10^{-2}		105.067
	1/16	1.49×10^{-2}	2.580	230.094
	1/32	1.74×10^{-2}	2.586	895.774

TABLE 3.10: Relative error calculation for fixed $\alpha = 0.9$ and $z(t) = t$, $\omega(t) = 1$ using S3.

$\Delta e_{\mathrm{h}} = e_{\mathrm{h}} - e_{\mathrm{h}/2}$			
t/h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
1/8	1.34×10^{-2}	3.4×10^{-3}	1.1×10^{-3}
2/8	1.75×10^{-2}	4.8×10^{-3}	1.5×10^{-3}
3/8	2.08×10^{-2}	5.6×10^{-3}	1.6×10^{-3}
4/8	2.08×10^{-2}	5.4×10^{-3}	1.4×10^{-3}
5/8	1.7×10^{-2}	4.1×10^{-3}	9.0×10^{-4}
6/8	9.9×10^{-3}	1.8×10^{-3}	2.0×10^{-4}
7/8	1.1×10^{-2}	2.0×10^{-3}	8×10^{-4}

TABLE 3.11: Ratios of the relative error for fixed $\alpha = 0.9$ and $z(t) = t$, $\omega(t) = 1$ based on the Table 3.10.

$R = \Delta e_{\mathrm{h}}/e_{\mathrm{h}/2}$		
t/h	$\frac{1}{8}$	$\frac{1}{16}$
1/8	3.94	3.09
2/8	3.64	3.2
3/8	3.71	3.5
4/8	3.85	3.86
5/8	4.15	4.56
6/8	5.5	9
7/8	5.5	2.5

TABLE 3.12: CO and MAE for scheme S3 using different value of α and h and fixed $z(t) = t$, $w(t) = 1$.

α	h	MAE	CO	CPU time(s)
0.2	1/4			33.256
	1/8	9.74×10^{-2}		75.258
	1/16	2.15×10^{-2}	2.180	170.097
	1/32	4.50×10^{-3}	2.256	612.067
0.4	1/4			36.102
	1/8	6.14×10^{-1}		58.626
	1/16	1.50×10^{-1}	2.034	128.012
	1/32	4.01×10^{-2}	2.067	598.450

3.5 Conclusions

We discussed three numerical methods such as linear, quadratic, and quadratic-linear approximation for a GICFVP. On approximation, the considered problem gets converted to an eigenvalue problem and by solving the eigenvalue problem, the minimum eigenvalue and corresponding eigenfunction were obtained. The eigenfunction corresponding to the minimum eigenvalue minimizes the considered cost functional. The achieved convergence order $2-\alpha$, $3-\alpha$ for S1 and S2 are also demonstrated through simulation results. Through provided Tables, one can conclude that the theoretical findings are established. Further, we observe from graphs that the numerical solutions regain the analytical solution of the associated integer-order problem as α approaches to 1. Additionally, in all three schemes, the approximations converge as h decreases, which indicates that schemes are convergent.

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