Chapter 2

On nonlinear singularly perturbed problems with integral boundary condition[1]

In this chapter, we consider the following class of nonlinear singularly perturbed problem with integral boundary condition

$$\begin{cases} \varepsilon u' + f(t,u) = 0 \ \text{ in } J = (0,T], \ T > 0, \\ u(0) = \mu u(T) + \displaystyle\int_0^T b(z)u(z)\,dz + d, \end{cases} \tag{2.1}$$

where μ and d are known constants, $0 < \varepsilon << 1$ is a parameter, called the 'perturbation parameter'. We assume that b and f are sufficiently smooth functions. Further, for any constant ν and δ, we assume that

$$0 < \nu \leq \frac{\partial f(t,u)}{\partial u} \leq \delta \ \text{ in } [0,T] \times \mathbb{R}. \tag{2.2}$$

Under these assumptions, problem (2.1) has a unique solution that exhibits a layer at $t = 0$ of width $\mathcal{O}(\varepsilon)$ (see [69, Section 2]). One may refer to [53–55, 74, 122, 123], for a discussion of existence and uniqueness results and applications of problems with integral boundary conditions.

The objective of this chapter is threefold: firstly, to propose a high-order scheme for the discretization of problem (2.1); secondly, to provide a general error analysis

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framework for the proposed scheme based on which we prove that the scheme is convergent of $\mathcal{O}(N^{-2}\ln N^2)$ on Shishkin meshes and of $\mathcal{O}(N^{-2})$ on Bakhvalov meshes; thirdly, to propose adaptive generation of meshes based on a suitable monitor function, for which we prove that the scheme is convergent of $\mathcal{O}(N^{-2})$.

In [2, 65, 68], a general classification of layer adapted meshes for various classes of singularly perturbed problems with non-integral boundary conditions is explored. Note that the analysis in [2, 65, 68] cannot be straightforward applied to the present scheme, because the integral boundary condition requires more technical arguments and thus a very fine analysis. Adaptive mesh methods based on mesh equidistribution have been previously studied for various classes of singularly perturbed problems with non-integral boundary/initial conditions (see, e.g. [41, 45, 124, 125] and the references therein), and for problem (2.1) in [70]. Note that the present adaptive mesh method based on mesh equidistribution is second-order convergent in comparison to the first-order convergent method of Liu et al. [70].

Following is the arrangement of this chapter. In Section 2.1, problem (2.1) is discretized by a hybrid scheme. A general framework for error analysis of the present scheme is developed in Section 2.2, and further ε-uniform convergence of the present scheme is established on Shishkin and Bakhvalov meshes using the general error estimates. Section 2.3 is devoted to the equidistribution mesh and ε-uniform convergence result of the present scheme on this mesh. In Section 2.4, the numerical results are presented. Finally, some conclusions are included in Section 2.5.

2.1 Discretization

Consider an arbitrary non-uniform mesh $\overline{\omega} = \{t_j \,|\, 0 = t_0 < t_1 < \cdots < t_N = T\}$ on $\overline{J}$. Let $\omega = \overline{\omega} \setminus t_0$. Suppose that the mesh sizes are defined by $h_j = t_j - t_{j-1}$, $1 \le j \le N$.

For any mesh function Y, we define

$$\left[D^{-}Y\right]_j := \frac{Y_j - Y_{j-1}}{h_j}.$$

We approximate the quasilinear singularly perturbed differential equation in (2.1) with a hybrid scheme that is a combination of the backward Euler scheme and the trapezoidal scheme. The hybrid scheme is defined as follows

$$[\mathcal{T}U]_j := \varepsilon \left[D^{-}U\right]_j + p_j^l f(t_{j-1}, U_{j-1}) + p_j^c f(t_j, U_j) = 0, \quad 1 \leq j \leq N, \tag{2.3}$$

where the values of the coefficients p_j^l and p_j^c depend on the relation between the local mesh size and the perturbation parameter: if $h_j\delta/2 \leq \varepsilon$, we consider the trapezoidal scheme, and the coefficients are defined by

$$p_j^l = 1/2, \quad p_j^c = 1/2,$$

and if $h_j\delta/2 > \varepsilon$, we consider the backward Euler scheme, and the coefficients are defined by

$$p_j^l = 0, \quad p_j^c = 1.$$

We approximate the integral term of the integral boundary condition in (2.1) using the composite trapezoidal rule as follows

$$U_0 = \mu U_N + \sum_{j=1}^{N} \tfrac{h_j}{2} \left[b_{j-1}U_{j-1} + b_jU_j\right] + d. \tag{2.4}$$

The backward Euler scheme (corresponding to the choice $p_j^l = 0, p_j^c = 1$) is considered in [69, 70] and shown to produce a first-order numerical method. Here, the coefficients p_j^l and p_j^c are considered such that the linearized operator corresponding to $\mathcal{T}$ satisfies a discrete maximum principle and the scheme is second order accurate. Using the arguments in Lemma 2.3, we can establish ε-uniform stability of the discrete scheme (2.3)–(2.4).

2.2 Convergence analysis

In this section, first, we derive a general error bound and then use it to prove ε-uniform convergence of the discrete scheme (2.3)–(2.4) on layer adapted Shishkin and Bakhvalov meshes. We shall use the following derivative bounds which can be proved using the arguments in [69].

Lemma 2.1. *The solution u of (2.1) satisfies*

$$|u^{(s)}(t)| \leq C\left(1 + \varepsilon^{-s} e^{-\nu t/\varepsilon}\right), \quad s = 0, 1, 2, 3, t \in \bar{J}.$$

We start our analysis by noting that the error function $\xi = u - U$ satisfies

$$\varepsilon\left[D^{-}\xi\right]_j + p_j^l\left[f(t_{j-1}, u_{j-1}) - f(t_{j-1}, U_{j-1})\right] + p_j^c\left[f(t_j, u_j) - f(t_j, U_j)\right]$$
$$= \varepsilon\left[D^{-}u - P(u')\right]_j, \; 1 \leq j \leq N, \quad (2.5)$$

with

$$\xi_0 - \mu\xi_N - \sum_{j=1}^{N} \tfrac{h_j}{2}\left[b_{j-1}\xi_{j-1} + b_j\xi_j\right] = -\sum_{j=1}^{N} \tfrac{h_j}{2}\left[b_{j-1}u_{j-1} + b_j u_j\right] + \int_0^T b(z)u(z)dz, \quad (2.6)$$

where $\left[P(z)\right]_j = p_j^l z_{j-1} + p_j^c z_j$.

Now, rewrite the error equation (2.5)–(2.6) in the form

$$\begin{cases} \varepsilon \left[D^{-}\xi\right]_j + \left[P(a\xi)\right]_j = \mathcal{R}_j, \quad 1 \le j \le N, & (2.7) \\ \xi_0 - \mu\xi_N - \sum_{j=1}^{N} \frac{h_j}{2}\left[b_{j-1}\xi_{j-1} + b_j\xi_j\right] = \widehat{\mathcal{R}}, & (2.8) \end{cases}$$

where

$$\begin{cases} a_j = \dfrac{\partial f}{\partial u}\left(t_j, u_j + \gamma_j(u_j - U_j)\right), \quad 0 < \gamma_j < 1, \\ \mathcal{R}_j = \varepsilon\left[D^{-}u - P(u')\right]_j, \quad 1 \le j \le N, \end{cases}$$

and

$$\widehat{\mathcal{R}} = -\sum_{j=1}^{N} \frac{h_j}{2}\left[b_{j-1}u_{j-1} + b_j u_j\right] + \int_0^T b(z)u(z)dz.$$

Lemma 2.2. *For the truncation errors* $\widehat{\mathcal{R}}$ *and* $\mathcal{R}_j$, $j = 1, \ldots, N$, *we have the following bound:*

$$\begin{cases} |\mathcal{R}_j| \le C\displaystyle\int_{t_{j-1}}^{t_j} (z - t_{j-1})(1 + \varepsilon^{-2}e^{-\nu z/\varepsilon})\,dz, \quad 1 \le j \le N, \\ |\widehat{\mathcal{R}}| \le C \displaystyle\max_{1\le j\le N}\int_{t_{j-1}}^{t_j} (z - t_{j-1})(1 + \varepsilon^{-2}e^{-\nu z/\varepsilon})\,dz. \end{cases}$$

Proof. We first analyse the truncation error $\mathcal{R}_j$, $j = 1, 2, \ldots, N$. Suppose $h_j\delta/2 \le \varepsilon$. Note that in this case $p_j^l = p_j^c = 1/2$. So, using Taylor expansions and Lemma 2.1, we obtain

$$\begin{aligned} |\mathcal{R}_j| &= \varepsilon\left|\left[D^{-}u - P(u')\right]_j\right| \\ &= \varepsilon\left|\left[\frac{u_j - u_{j-1}}{h_j} - \frac{u'_{j-1} + u'_j}{2}\right]\right| \\ &\underbrace{\le}_{\text{Taylor expansions}} C\varepsilon\int_{t_{j-1}}^{t_j} (z - t_{j-1})|u'''(z)|\,dz \\ &\underbrace{\le}_{\text{Lemma 2.1}} C\int_{t_{j-1}}^{t_j} (z - t_{j-1})(1 + \varepsilon^{-2}e^{-\nu z/\varepsilon})\,dz. \end{aligned}$$

Now suppose $h_j\delta/2 > \varepsilon$. Note that in this case $p_j^l = 0$, $p_j^c = 1$. Hence, using Taylor expansions and Lemma 2.1, we get

$$\begin{aligned}
|\mathcal{R}_j| &= \varepsilon \left| \left[D^- u - P(u') \right]_j \right| \\
&= \varepsilon \left| \left[\frac{u_j - u_{j-1}}{h_j} - u'_j \right] \right| \\
&\leq \frac{\varepsilon}{h_j} \int_{t_{j-1}}^{t_j} (z - t_{j-1}) u''(z)\, dz \\
&\leq C \int_{t_{j-1}}^{t_j} (z - t_{j-1})(1 + \varepsilon^{-2} e^{-\nu z/\varepsilon})\, dz.
\end{aligned}$$

Thus, for both the cases (i.e. $h_j\delta/2 \leq \varepsilon$ and $h_j\delta/2 > \varepsilon$), we have

$$|\mathcal{R}_j| \leq C \int_{t_{j-1}}^{t_j} (z - t_{j-1})(1 + \varepsilon^{-2} e^{-\nu z/\varepsilon})\, dz. \tag{2.9}$$

Next we use standard error estimate for the trapezoidal rule [126] and Lemma 2.1 to get

$$\begin{aligned}
|\widehat{\mathcal{R}}| &= \left| -\sum_{j=1}^{N} \tfrac{h_j}{2} \left[b_{j-1} u_{j-1} + b_j u_j \right] + \int_0^T b(z) u(z) dz \right| \\
&= \left| \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} b(z) u(z) dz - \sum_{j=1}^{N} \tfrac{h_j}{2} \left(b_{j-1} u_{j-1} + b_j u_j \right) \right| \\
&= \left| \sum_{j=1}^{N} \left\{ \int_{t_{j-1}}^{t_j} b(z) u(z) dz - \tfrac{h_j}{2} \left(b_{j-1} u_{j-1} + b_j u_j \right) \right\} \right| \\
&\leq \sum_{j=1}^{N} \left| \frac{1}{2} \int_{t_{j-1}}^{t_j} (z - t_{j-1})(z - t_j)(b(z) u(z))''\, dz \right| \\
&\leq C \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} |(z - t_{j-1})(z - t_j)|\, |(b(z) u(z))''|\, dz
\end{aligned}$$

$$\leq C\sum_{j=1}^{N} h_j \int_{t_{j-1}}^{t_j} (z-t_{j-1})\,|(b(z)u(z))''|\,dz$$
$$\leq C\sum_{j=1}^{N} h_j \int_{t_{j-1}}^{t_j} (z-t_{j-1})(1+\varepsilon^{-2}e^{-\nu z/\varepsilon})\,dz$$
$$\leq C\max_{1\leq j\leq N}\int_{t_{j-1}}^{t_j} (z-t_{j-1})(1+\varepsilon^{-2}e^{-\nu z/\varepsilon})dz\sum_{j=1}^{N} h_j$$
$$\leq C\max_{1\leq j\leq N}\int_{t_{j-1}}^{t_j} (z-t_{j-1})(1+\varepsilon^{-2}e^{-\nu z/\varepsilon})dz,$$

where we have used the fact that $\sum_{j=1}^{N} h_j = T$. □

Lemma 2.3. *For the error function ξ satisfying (2.7)–(2.8), it holds*

$$\max_{0\leq j\leq N}|\xi_j| \leq C\left(|\widehat{\mathcal{R}}| + \max_{1\leq j\leq N}|\mathcal{R}_j|\right). \tag{2.10}$$

Proof. Recalling (2.7), we have

$$\varepsilon\left[D^{-}\xi\right]_j + p_j^l a_{j-1}\xi_{j-1} + p_j^c a_j \xi_j = \mathcal{R}_j, \quad 1\leq j\leq N. \tag{2.11}$$

Thus,

$$\xi_j = \frac{(\varepsilon - h_j p_j^l a_{j-1})}{(\varepsilon + h_j p_j^c a_j)}\xi_{j-1} + \frac{\mathcal{R}_j h_j}{(\varepsilon + h_j p_j^c a_j)}. \tag{2.12}$$

We solve (2.12) recursively to get

$$\xi_j = \xi_0 \mathcal{P}_j + \sum_{k=1}^{j}\phi_k \mathcal{P}_{j-k}, \tag{2.13}$$

where,

$$\mathcal{P}_{j-k} = \begin{cases} \prod_{i=k+1}^{j} \frac{\varepsilon - h_i p_i^l a_{i-1}}{\varepsilon + h_i p_i^c a_i}, & 0\leq k\leq j-1, \\ 1, & k=j, \end{cases} \quad \text{and} \quad \phi_k = \frac{\mathcal{R}_k h_k}{\varepsilon + h_k p_k^c a_k}. \tag{2.14}$$

From (2.8) and (2.13), we obtain

$$\xi_0 = \frac{\mu \sum_{k=1}^{N} \phi_k \mathcal{P}_{N-k} + \sum_{j=1}^{N} \frac{h_j}{2}\left[b_{j-1}\left(\sum_{k=1}^{j-1} \phi_k \mathcal{P}_{j-k-1}\right) + b_j\left(\sum_{k=1}^{j} \phi_k \mathcal{P}_{j-k}\right)\right] + \widehat{\mathcal{R}}}{1-\mu\mathcal{P}_N - \sum_{j=1}^{N} \frac{h_j}{2}\left[b_{j-1}\mathcal{P}_{j-1} + b_j \mathcal{P}_j\right]}. \quad (2.15)$$

For a sufficient small ε, we make an obvious assumption that $\varepsilon < h_j$ for each j. Then

$$\mathcal{P}_N = \prod_{j=1}^{N} \frac{1-\rho_j p_j^l a_{j-1}}{1+\rho_j p_j^c a_j} \le \prod_{j=1}^{N} \frac{1-\rho_j p_j^l \nu}{1+\rho_j p_j^c \nu} << 1, \quad (2.16)$$

where, $\rho_j = h_j/\varepsilon$.

Suppose $\bar{b} = \max_{t\in[0,T]} b(t)$. When $\bar{b} > 0$ and $\mu > 0$, for a sufficiently small ε, there exists a positive constant $\bar{c}$, independent of ε and N, such that

$$1 - \mu\mathcal{P}_N - \sum_{j=1}^{N} \frac{h_j}{2}\left[b_{j-1}\mathcal{P}_{j-1} + b_j\mathcal{P}_j\right] \ge \bar{c} > 0. \quad (2.17)$$

The inequality (2.17) is obvious for other values of $\bar{b}$ and μ. Hence, from (2.15), we have

$$|\xi_0| \le C\left(|\widehat{\mathcal{R}}| + \max_{1\le j\le N}|\mathcal{R}_j|\right). \quad (2.18)$$

Thus, an application of the discrete maximum principle for (2.11) gives the desired result. □

Theorem 2.1. *For the solutions u and U of (2.1) and (2.3)–(2.4), respectively, we have*

$$\|u - U\|_{\varpi} \le C\left(\max_{1\le j\le N} \int_{t_{j-1}}^{t_j} \left(1 + \varepsilon^{-1} e^{-\nu z/2\varepsilon}\right) dz\right)^2. \quad (2.19)$$

Proof. From Lemmas 2.2 and 2.3, we have

$$\|u - U\|_{\overline{\omega}} \leq C \max_{1\leq j\leq N} \int_{t_{j-1}}^{t_j} (z - t_{j-1})(1 + \varepsilon^{-2} e^{-\nu z/\varepsilon}) dz. \tag{2.20}$$

For $s \in \mathbb{N}^+$ and for a positive decreasing function ψ on $[t_{j-1}, t_j]$, it holds [40]

$$\int_{t_{j-1}}^{t_j} (z - t_{j-1})^{(s-1)} \psi(z) dz \leq \frac{1}{s} \left(\int_{t_{j-1}}^{t_j} \psi(z)^{1/s} dz \right)^s. \tag{2.21}$$

So, using the above inequality in (2.20), we have the proof. □

2.2.1 Layer-adapted meshes

2.2.1.1 Shishkin meshes

These are piecewise-uniform meshes constructed with the help of a transition parameter σ_s defined by

$$\sigma_s = \min \left\{ \frac{T}{2}, 2\nu^{-1} \varepsilon \ln N \right\}. \tag{2.22}$$

The transition parameter σ_s divides $[0, T]$ into two subintervals $[0, \sigma_s]$ and $[\sigma_s, T]$. Further, each of $[0, \sigma_s]$ and $[\sigma_s, T]$ are again divided into $N/2$ uniform subintervals, with step sizes denoted by h and H, respectively. For simplicity, we consider $\sigma_s = 2\nu^{-1}\varepsilon \ln N$, as in the other case ε is large and standard arguments could be used for analysis.

Lemma 2.4. *Suppose Shishkin meshes are used to discretize* $[0, T]$*. Then*

$$\int_{t_{j-1}}^{t_j} \left(1 + \varepsilon^{-1} e^{-\nu z/2\varepsilon}\right) dz \leq C N^{-1} \ln N, \quad j = 1, 2, \ldots, N. \tag{2.23}$$

Proof. We first consider the fine region $[0, \sigma_s]$, i.e. for $1 \leq j \leq N/2$. By mean value theorem, we get

$$\begin{aligned}\int_{t_{j-1}}^{t_j} \left(1 + \varepsilon^{-1} e^{-\nu z/2\varepsilon}\right) dz &\leq h + \frac{2}{\nu}\left| e^{-\nu t_{j-1}/2\varepsilon} - e^{-\nu t_j/2\varepsilon} \right| \\ &\leq (1+\varepsilon^{-1})h \\ &\leq CN^{-1} \ln N. \end{aligned} \tag{2.24}$$

We next consider the coarse region $[\sigma_s, T]$, i.e. for $N/2 + 1 \leq j \leq N$. Using the transition point and the mesh size, we proceed as follows:

$$\begin{aligned}\int_{t_{j-1}}^{t_j} \left(1 + \varepsilon^{-1} e^{-\nu z/2\varepsilon}\right) dz \leq H + \frac{2}{\nu}\left| e^{-\nu t_{j-1}/2\varepsilon} - e^{-\nu t_j/2\varepsilon} \right| &\leq H + \frac{4}{\nu} e^{-\nu t_{j-1}/2\varepsilon} \\ &\leq C(N^{-1} + e^{-\nu \sigma_s/2\varepsilon}) \\ &\leq CN^{-1}. \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25), we are done with proof. □

Thus, using Lemma 2.4 and Theorem 2.1, we obtain

$$\|u - U\|_{\overline{\omega}} \leq CN^{-2} \ln^2 N. \tag{2.26}$$

2.2.1.2 Bakhvalov meshes

There are several ways one can construct Bakhvalov meshes. We follow the construction of [66] which is based on a transition parameter τ_b given by

$$\tau_b = \min\left\{\frac{T}{2}, 2\nu^{-1}\varepsilon|\ln \varepsilon|\right\}. \tag{2.27}$$

The transition parameter τ_b divides $[0,T]$ into two sub-intervals $[0,\tau_b]$ and $[\tau_b,T]$. The mesh points in $[\tau_b,T]$ are equidistant, but are graded in $[0,\tau_b]$. The mesh points $\{t_j\}_0^N$ are given by

$$t_j = \begin{cases} -\frac{2\varepsilon}{\nu}\ln\left(1-(1-\varepsilon)\frac{2j}{N}\right), & j = 0,1,\cdots,N/2, \\ \tau_b + \left(j-\frac{N}{2}\right)h, \quad j = N/2+1,\cdots,N, & h = \frac{2(T-\tau_b)}{N}. \end{cases}$$

Lemma 2.5. *Suppose Bakhvalov meshes are used to discretize* $[0,T]$*. Then*

$$\int_{t_{j-1}}^{t_j} \left(1+\varepsilon^{-1}e^{-\nu z/2\varepsilon}\right)dz \le CN^{-1},\ j = 1,2,\cdots,N. \tag{2.28}$$

Proof. Similar to the above proof, we only consider $\tau_b = 2\nu^{-1}\varepsilon \mid \ln\varepsilon \mid$. Inside the region $[\tau_b,T]$, we use the arguments in [66] to get $\int_{t_{j-1}}^{tj}(1+\varepsilon^{-1}e^{-\nu z/2\varepsilon})\,dz \le CN^{-1}$, $j = N/2+1,\cdots,N$. Now, inside the region $[0,\tau_b]$, i.e, for $j = 1,\ldots,N/2$,

$$\int_{t_{j-1}}^{tj}(1+\varepsilon^{-1}e^{-\nu z/2\varepsilon})\,dz \le h_j + \frac{2}{\nu}\left|\left(e^{-\nu t_{j-1}/2\varepsilon} - e^{-\nu t_j/2\varepsilon}\right)\right| \le CN^{-1},$$

where we have used $h_j \le CN^{-1}$ and $e^{-\frac{\nu t_{j-1}}{2\varepsilon}} - e^{-\frac{\nu t_j}{2\varepsilon}} = (1-\varepsilon)N^{-1}$ from [66]. □

Thus, using Lemma 2.5, Theorem 2.1 yields

$$\|u-U\|_{\overline{\omega}} \le CN^{-2}. \tag{2.29}$$

2.3 Mesh equidistribution and adaptivity

An equidistribution mesh is a special kind of non-uniform mesh formed by using a specially chosen positive function $M_E(u(t),t)$, called the monitor function. We say that a mesh $\overline{\omega} = \{t_j\,|\,0 = t_0 < t_1 < \cdots < t_N = T\}$ is equidistributed if it satisfies

[41]

$$\int_{t_{j-1}}^{t_j} M_E(u(z), z)dz = \int_{t_j}^{t_{j+1}} M_E(u(z), z)dz,\ j = 1, 2, \ldots, N-1.$$

This can also be written in the following form

$$\int_{t_{j-1}}^{t_j} M_E(u(z), z)dz = \frac{1}{N}\int_0^T M_E(u(z), z)dz, \quad j = 1, 2, \ldots, N. \tag{2.30}$$

Several monitor functions are proposed in research literature depending on the considered problem and its discretization (see [44–46, 124, 125] and the references therein). Some authors used a monitor function constructed on the basis of the arc-length of the numerical solution for various classes of singularly perturbed problems (see [124, 125] and the references therein). However, it has been pointed out in [46] that the arc length based monitor function will give at best a first-order method. In [45], a monitor function formed by using the combination of a specific constant and some power of the second-order derivative of the singular part of the solution is considered and shown to produce a second order method. Our choice of the monitor function is motivated from [45], but is different from [45] in the sense that we do not require to compute the singular part of the solution. We consider the following monitor function

$$M_E(u(t), t) = \alpha_c + |u''(t)|^{1/2}, \tag{2.31}$$

where α_c is a positive constant, chosen to avoid the scarcity of mesh points outside the boundary layer region. Similar to [45], we consider $\alpha_c = \int_0^T |u''(z)|^{1/2}\, dz$. Next, we bound step sizes and prove ε-uniform convergence of scheme (2.3)–(2.4) on the equidistribution mesh.

Lemma 2.6. *For the equidistribution mesh formed by using the monitor function (2.31), mesh widths satisfy* $h_j \le CN^{-1},\ j = 1, 2, ..., N.$

Proof. Taking the approximation of u'' from Lemma 2.1, we have

$$\begin{aligned}\mathcal{K} = \int_0^T M_E(u(z),z)dz &= \int_0^T \left(\alpha_c + |u''(z)|^{1/2}\right) dz \\ &\le \int_0^T \alpha_c dz + \int_0^T (1+\varepsilon^{-2}e^{-\nu z/\varepsilon})^{1/2} dz \\ &\le C + C\int_0^T (1+(\varepsilon^{-1}e^{-\nu z/2\varepsilon})^2)^{1/2} dz \\ &\le C + C\int_0^T (1+\varepsilon^{-1}e^{-\nu z/2\varepsilon}) dz \\ &= C + CT + \frac{2C}{\nu}(1-e^{-\nu T/2\varepsilon}) \le C.\end{aligned}$$

Thus, by the equidistribution principle (2.30), we get

$$\alpha_c h_j \le \int\limits_{t_{j-1}}^{t_j} M_E(u(z),z)dz = \frac{1}{N}\int\limits_0^T M_E(u(z),z)dz \le CN^{-1}.$$

Hence, $h_j \le CN^{-1}$. □

Theorem 2.2. *On adaptively generated equidistribution mesh using the monitor function (2.31), we have*

$$\|u-U\|_{\overline{\omega}} \le CN^{-2}.$$

Proof. Using the ideas in Section 2.2, we have

$$\|u-U\|_{\overline{\omega}} \le C\left(\max_{1\le j\le N}\int_{t_{j-1}}^{t_j}\left(1+\varepsilon^{-1}e^{-\nu z/2\varepsilon}\right)dz\right)^2.$$

Using the equidistribution principle (2.30) together with Lemma 2.6 and the arguments in [44, Theorem 21] and [45, Lemma 9], we proceed as follows

$$\int_{t_{j-1}}^{t_j}\left(1+\varepsilon^{-1}e^{-\nu z/2\varepsilon}\right)dz = \int_{t_{j-1}}^{t_j}(1+(\varepsilon^{-2}e^{-\frac{\nu z}{\varepsilon}})^{1/2})dz$$

$$\begin{aligned}
&\leq \; h_j + C\int_{t_{j-1}}^{t_j} \left(\alpha_c + |u''(z)|^{1/2}\right) dz \\
&= \; h_j + C\int_{t_{j-1}}^{t_j} M_E(u(z), z)dz \\
&\leq \; CN^{-1} + CN^{-1}\int_0^T M_E(u(z), z)dz \\
&\leq \; CN^{-1} + CN^{-1}\mathcal{K} \\
&\leq \; CN^{-1}.
\end{aligned}$$

Hence, we have the proof. □

2.4 Numerical results

We now provide numerical results obtained using the discrete scheme (2.3)–(2.4) on Shishkin and Bakhvalov meshes. Further, we provide numerical results obtained using the mesh equidistribution method. We consider the following test problem from [69, 70] for numerical experiments

$$\begin{cases} \varepsilon u' + 2u - e^{-u} + t^2 = 0, \quad t \in (0, 1], \\ u(0) = \frac{1}{2}u(1) - \frac{1}{4}\int_0^1 e^{-s}u(s)ds + 1. \end{cases} \tag{2.32}$$

The nonlinear equations in (2.3)–(2.4) are solved using the following iterative scheme

$$\begin{cases} U_j^{(n)} = U_j^{(n-1)} - \dfrac{(U_j^{(n-1)} - U_{j-1}^{(n)})\rho_j^{-1} + p_j^l f(t_{j-1}, U_{j-1}^{(n)}) + p_j^c f(t_j, U_j^{(n-1)})}{\rho_j^{-1} + p_j^c f_u(t_j, U_j^{(n-1)})}, & (2.33) \\ U_0^{(n)} = \mu U_N^{(n-1)} + \displaystyle\sum_{j=1}^{N} \frac{h_j}{2}[b_j U_j^{(n-1)} + b_{j-1}U_{j-1}^{(n-1)}] + d, \quad n = 1, 2, \ldots. & (2.34) \end{cases}$$

where $U_j^{(0)}$ is initial iteration that is given and $\rho_j = h_j/\varepsilon$. We consider the stopping criterion as $\|U^{(n)} - U^{(n-1)}\|_{\infty,\overline{\omega}} \leq 10^{-8}$. Further, the initial iteration is taken to be $U_j^{(0)} = 0.5$ (as considered in [69, 70]).

The numerical algorithm followed to find the non-uniform layer adaptive mesh in Section 2.3 and the numerical solution is given in Algorithm 1.

Figure 2.1 displays the numerical solution plots for $\varepsilon = 10^{-1}$, 10^{-3}, on the equidistribution mesh, which clearly confirms the presence of a layer near $t = 0$. We compute

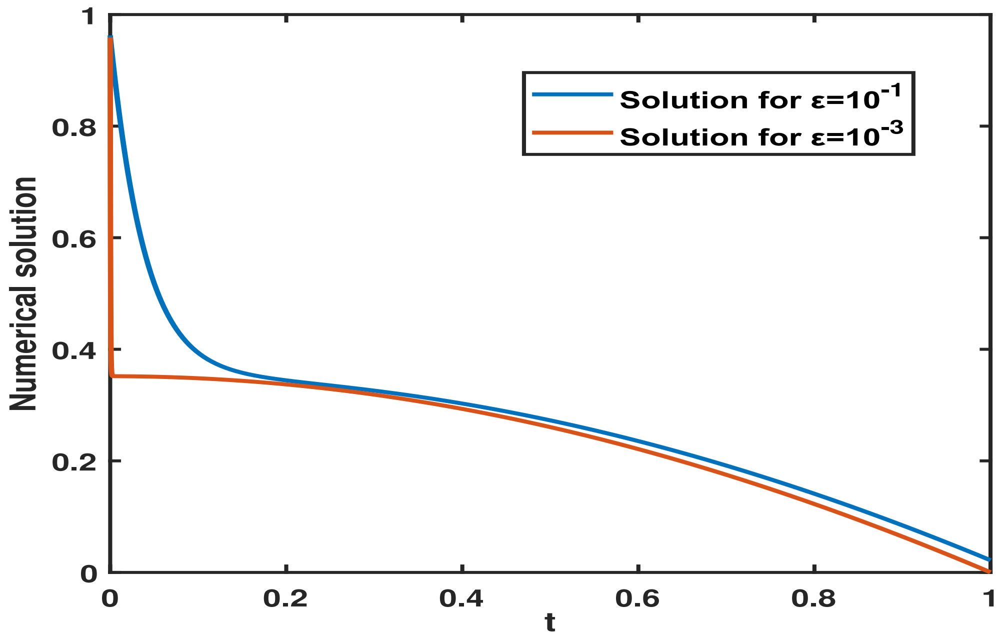


Figure 2.1: Numerical solution plots for $N = 128$ with $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-3}$ on the equidistribution mesh.

the numerical solutions for the values of ε from set $E_\varepsilon = \{10^{-1}, 10^{-2}, ..., 10^{-7}\}$. We use a variant of the double mesh principle to estimate the errors, since the exact solution of the test problem is not known to us. In this process we bisect the original mesh and compare the solutions obtained on the original mesh and the bisected

Algorithm 1: Adaptive mesh generation

Input: $N \in \mathbb{N},\ 0 < \varepsilon \leq 1, U_j^{(0)} = 0.5$ and $C_0 = 1.1$.
Output: Equidistribution mesh $\{t_j\}$ and the solution U_j.

Step 1. Start with $t_j^{(0)} = j/N,\ j = 0, ..., N$, as the initial iteration.

Step 2. Calculate $U_j^{(k)}$ satisfying (2.3)–(2.4), respectively on the mesh $\{t_j^{(k)}\}$ using the the iteration scheme (2.33)–(2.34).

Step 3. Compute the monitor function

$$M_j^{(k)} = \alpha_c^{(k)} + |D^2 U_j^{(k)}|^{1/2}, \quad 1 \leq j \leq N-1, \tag{2.35}$$

by defining

$$D^2 Z_j = \frac{1}{\hbar_i}\left(\frac{Z_{j+1} - Z_j}{h_{j+1}} - \frac{Z_j - Z_{j-1}}{h_j}\right) \quad \text{with} \quad \hbar_j = \frac{h_j + h_{j+1}}{2}. \tag{2.36}$$

Here,

$$\alpha_c^{(k)} = h_1 |D^2 U_1^{(k)}|^{1/2} + \sum_{j=2}^{N} \left\{ \frac{|D^2 U_{j-1}^{(k)}|^{1/2} + |D^2 U_j^{(k)}|^{1/2}}{2} \right\} + h_N |D^2 U_{N-1}^{(k)}|^{1/2}. \tag{2.37}$$

Step 4. For $\{t_j^{(k)}\}$ and $U_j^{(k)}$, set

$$H_j^{(k)} = \left(\frac{M_{j-1}^{(k)} + M_j^{(k)}}{2}\right)(t_j^{(k)} - t_{j-1}^{(k)}), \quad 1 \leq j \leq N,$$

where $M_j^{(k)}$ is calculated from (2.35) with $M_0^{(k)} = M_1^{(k)}$ and $M_N^{(k)} = M_{N-1}^{(k)}$.

Step 5. Set $B_0^{(k)} = 0$ and $B_j^{(k)} = \sum_{i=1}^{j} H_i^{(k)}$, $1 \leq j \leq N$. Define

$$\mathcal{C}^{(k)} := \frac{N}{B_N^{(k)}} \max_{1 \leq j \leq N} H_j^{(k)}.$$

Step 6. If $\mathcal{C}^{(k)} \leq \mathcal{C}_0$, then go to Step 9.

Step 7. Set $Y_j = jB_N^{(k)}/N$, $0 \leq j \leq N$. Generate new mesh $\{t_j^{(k+1)}\}$ by interpolating the points $(B_j^{(k)}, t_j)$ and evaluating the interpolant at Y_j, $0 \leq j \leq N$.

Step 8. Set $k = k + 1$ and return to Step 2.

Step 9. Take $\{t_j^{(k)}\}$ as the final adaptive mesh and $U_j^{(k)}$ as the adaptive solution. **Stop**.

mesh. We compute the error as follows

$$F_\varepsilon^N = \max_{0\le j\le N} \left| U_j^{\varepsilon,N} - U_{2j}^{\varepsilon,2N} \right|,$$

and ε-uniform error by

$$F^N = \max_{\varepsilon} \left\{ F_\varepsilon^N \right\}.$$

After that we compute ε-uniform convergence rate using

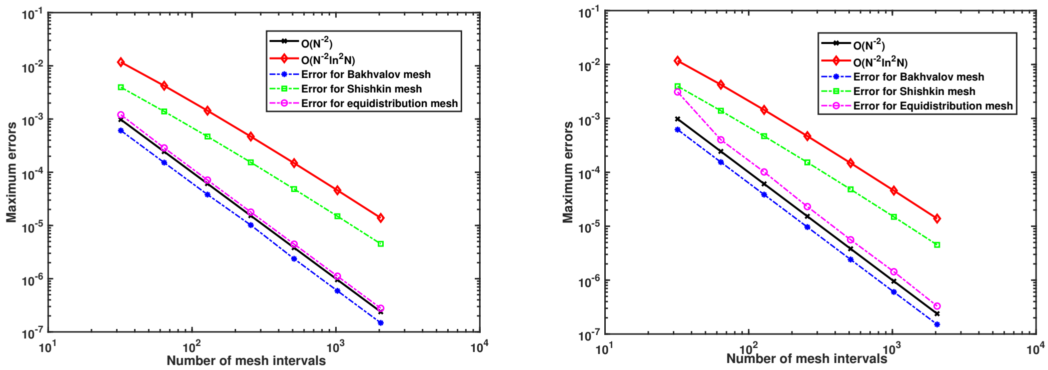
$$\varrho^N = \log_2 (F^N / F^{2N}).$$

We present ε-uniform errors and ε-uniform convergence rates obtained using the discrete scheme (2.3)–(2.4) on different layer-adaptive meshes in a tabular form in Table 2.1. The numerical results in Table 2.1 clearly confirms the theoretical outcomes.

	Uniform mesh		Shishkin mesh		Bakhvalov mesh		Equidistribution mesh	
N	F^N	ϱ^N	F^N	ϱ^N	F^N	ϱ^N	F^N	ϱ^N
2^5	4.1967e-02	−0.3472	3.9907e-03	1.5123	6.1968e-04	2.0042	3.7422e-03	2.1728
2^6	5.3387e-02	−0.1324	1.3989e-03	1.5651	1.5447e-04	2.0010	8.2992e-04	2.0315
2^7	1.3363e-01	1.8432	4.7277e-04	1.6214	3.8590e-05	2.0003	2.0299e-04	2.0451
2^8	3.7244e-02	−0.4519	1.5366e-04	1.6682	9.6458e-06	2.0000	4.9187e-05	2.1280
2^9	5.0944e-02	−1.4525	4.8349e-05	1.6961	2.4113e-06	2.0000	1.1253e-05	2.0870
2^{10}	1.3942e-01	2.1074	1.4922e-05	1.7231	6.0449e-07	2.0000	2.6487e-06	2.1123
2^{11}	3.2355e-02	–	4.5197e-06	–	1.5347e-07	–	6.1260e-07	–

TABLE 2.1: ε-uniform errors F^N and ε-uniform convergence ϱ^N for the test problem (2.32).

Further, one can see that on uniform meshes the errors are fluctuating and supporting the fact that uniform meshes are not suitable for problem (2.1). In addition, we plotted log-log graphs (in Figure 2.2) for maximum-pointwise errors vs number of



(A) For $\varepsilon = 10^{-2}$ (B) For $\varepsilon = 10^{-4}$

FIGURE 2.2: Log-log plots of the maximum pointwise errors for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-4}$.

mesh intervals for two different values of the perturbation parameter. This again authenticate the theoretical results.

Figure 2.3 displays the adaptive nature of the equidistribution mesh, where the mesh points are condensed towards the boundary layer in few iterations and finally adapts the solution behavior by itself. This confirms the adaptiveness of the equidistribution mesh.

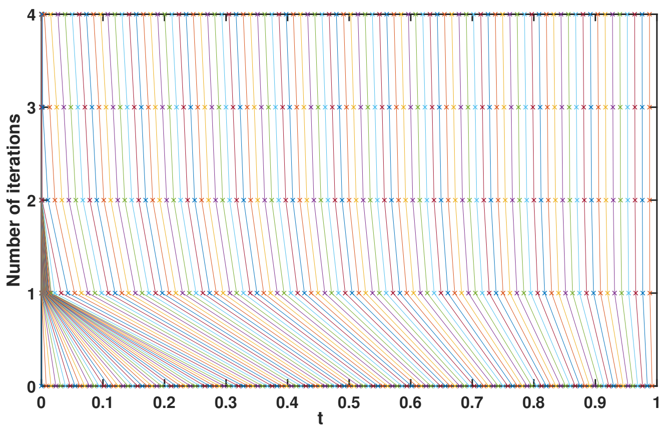


FIGURE 2.3: Movement of mesh points via equidistribution mesh for $\varepsilon = 10^{-3}$ and $N = 128$.

2.5 Conclusions

We have developed high-order convergent numerical methods on layer-adapted meshes for solving singularly perturbed nonlinear problems with integral boundary condition. A finite difference discretization of the problem is introduced on arbitrary nonuniform meshes and a general error analysis framework is developed. The method is shown to be uniformly convergent of $\mathcal{O}(N^{-2}\ln N^2)$ on Shishkin meshes and $\mathcal{O}(N^{-2})$ on Bakhvalov meshes. Further, a high-order numerical method on equidistributed meshes is developed and shown to be uniformly convergent of $\mathcal{O}(N^{-2})$. Numerical experiments are performed and presented results supported the theory.