# Chapter 2

# Notation and Preliminaries

Throughout this thesis, standard mathematical notation is adopted. The symbol $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{R}_+$ refers to the set of positive real numbers, i.e., $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z > 0\}$, and $\mathbb{R}_{\geq 0}$ indicates the set of non-negative real numbers. Likewise, $\mathbb{R}_{>k}$ and $\mathbb{R}_{\geq k}$ denote the sets of real numbers greater than $k$ and greater than or equal to $k$, respectively, where $k$ is a real constant. The set of non-negative integers is represented by $\mathbb{Z}_{\geq 0}$.

The notation $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. For any $p \geq 1$, the $\ell_p$-norm of a vector $y \in \mathbb{R}^n$ is defined as

$$\|y\|_p = \left( \sum_{i=1}^{n} |y_i|^p \right)^{1/p}.$$

If the norm type is not explicitly stated, it is assumed to be the Euclidean norm ($p = 2$), denoted by $|y|$ or $\|y\|_2$. For matrices, the transpose of a matrix $A$ is denoted by $A^\top$. Given a vector $a = [a_1, a_2, \dots, a_n] \in \mathbb{R}^n$, the notation $\mathrm{diag}(a)$ represents a diagonal matrix with diagonal entries $a_i$, where $i = 1, \dots, n$.

Given a set $S$, $\min\{S\}$ denotes the smallest element in $S$, and $\sup\{S\}$ denotes the supremum (least upper bound) of $S$. For functions $g_1 : S_1 \to S_2$ and $g_2 : S_2 \to S_3$, their composition is denoted as $g_2 \circ g_1 : S_1 \to S_3$, with $(g_2 \circ g_1)(\cdot) = g_2(g_1(\cdot))$.

An open $n$-dimensional ball of radius $\delta$ centered at a point $z_0 \in \mathbb{R}^n$ is denoted by

$$\mathbb{B}_\delta = \{z \in \mathbb{R}^n \mid \|z - z_0\| < \delta\}.$$

The absolute value function is represented as $|\cdot|$, and $\|\cdot\|_p$ is used for the $p$-norm of finite-dimensional vectors. The transpose of a vector is also written as $(\cdot)^\top$, and the ceiling function is denoted by $\lceil z \rceil$, representing the smallest integer greater than or equal to $z$. The sign function is defined as

$$\text{sign}(z) = \begin{cases} z/|z|, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

The space $\ell_\infty$ refers to the set of all bounded sequences.

To comprehend the results presented in this thesis, a foundational understanding of stability concepts in discrete-time dynamical systems is essential. The definitions and mathematical tools introduced in this section form a critical basis for the subsequent chapters.

## 2.1 Finite-Time Stability of DTS

The subsequent outcomes will prove beneficial for comprehending the notion of finite-time stability of DTS. Let us consider the following forced nonlinear DTS

$$z(k+1) = f(z(k), u(z(k))), \quad z(0) = z_0, \quad k \in \mathbb{Z}^+ \tag{2.1}$$

where $z(k) \in \mathbb{D} \subseteq \mathbb{R}^n$ denotes the system state, $u(k) \in \mathbb{R}^m$ represents the control input and $\mathbb{D}$ is an open set which includes the origin. The function $f(\cdot, \cdot)$ is in $\mathbb{R}^n$, and $z(k) = 0$ is the zero solution of (2.1). An equation of this form signifies an actuated system with specific inputs. If the system lacks an explicit input $u(k)$, it is considered autonomous.

**Definition 2.1** *[20]: Consider the system represented by* (2.1) *for an autonomous system, the nonlinear function defined as $f : \mathbb{D} \to \mathbb{D}$. The solution $z(k) = 0$ for* (2.1) *is deemed finite time stable (FTS) if there exists, $M$ containing origin such that $M \subseteq \mathbb{D}$ and an associated settling time function $K : M \setminus \{0\} \to \mathbb{Z}^+$ satisfying these conditions:*

1. *Finite-time convergence: For any initial condition $z(0)$ in $M \setminus \{0\}$ and $\forall\, k \geq K(z_0)$, the solution $z(k)$ lies in $M \cap \{0\}$;*

2. *Lyapunov stability: For any given $\epsilon > 0$, there exists $\delta > 0$ such that the ball of radius $\delta$ centered at the origin, $B_\delta(0)$, is a subset of $M$. Also, for any initial condition $z_o$ in $B_\delta(0) \setminus \{0\}$, the solution $z(k)$ resides in $B_\epsilon(0)$ for all time instances $k$ in the set $\{0, \ldots, K(z_0) - 1\}$.*

*If $z(k) = 0$ of (2.1) is FTS for $M = \mathbb{D} = \mathbb{R}^n$, it is considered globally finite-time stable.*

**Lemma 2.2** *[20]: Consider the case of autonomous system for (2.1), if there exists a neighborhood $\mathbb{N} \subseteq \mathbb{D}$ such that $0 \in \mathbb{N}$ and a continuous function $V : \mathbb{D} \to \mathbb{R}$ such that $V(0) = 0$, $V(z) > 0$ for $z \in \mathbb{N} \setminus \{0\}$, and*

$$\Delta V(z) \leq -c \min\left\{\frac{V(z)}{c}, V(z)^{\gamma}\right\}, \quad z \in \mathbb{N} \setminus \{0\} \tag{2.2}$$

*where $0 < \gamma < 1$ and $c > 0$ are real constants. Then, the zero solution $z(k) \equiv 0$ to (2.1) is FTS and the settling-time $K : M \to \mathbb{Z}^+$ can be represented as:*

$$K(z_0) \leq \left\lceil \log_{[1-cV(z_0)^{\gamma-1}]} \frac{c^{\frac{1}{1-\gamma}}}{V(z_0)} \right\rceil + 1, \quad V(z_0) > c^{\frac{1}{1-\gamma}}$$

*or*

$$K(z_0) = 1, \quad |z_0| \leq c^{\frac{1}{1-\gamma}}, \quad V(z_0) \leq c^{\frac{1}{1-\gamma}}$$

*where $M$ is an open ball containing the origin. Moreover, if $V(\cdot)$ is expanding infinitely outward and (2.5) is satisfied for $\mathbb{D} = \mathbb{R}^n$, then the solution $z(k) \equiv 0$ is globally FTS.*

**Lemma 2.3** *[20]: Consider the case of autonomous system for (2.1), if there exists a neighborhood $\mathbb{N} \subseteq \mathbb{D}$ such that $0 \in \mathbb{N}$ and a continuous function $V : \mathbb{D} \to \mathbb{R}$ such that $V(0) = 0$, $V(z) > 0$ for $z \in \mathbb{N} \setminus \{0\}$, and*

$$\Delta V(z) \leq \min\{V(z), c\}, \quad z \in \mathbb{N} \setminus \{0\} \tag{2.3}$$

*where $c > 0$ are real constants. Then, the zero solution $z(k) \equiv 0$ to (2.1) is FTS and the settling-time $K : M \to \mathbb{Z}^+$ can be represented as:*

$$K(z_0) \leq \left\lceil \frac{V(z_0)}{c} \right\rceil + 1, \quad V(z_0) > c$$

*or*

$$K(z_0) = 1, \quad |V(z_0)| \leq c,$$

*where $M$ is an open ball containing the origin. Moreover, if $V(\cdot)$ is expanding infinitely outward and (2.5) is satisfied for $\mathbb{D} = \mathbb{R}^n$, then the solution $z(k) \equiv 0$ is globally FTS.*

## 2.2 Comparison Functions

We begin by recalling several standard definitions of comparison functions that are pivotal in the stability analysis.

**Definition 2.4 (Class $\mathcal{K}$ function [48])** *A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing, and satisfies $\alpha(0) = 0$. If additionally $\lim_{s\to\infty} \alpha(s) = \infty$, it is said to be a class $\mathcal{K}_\infty$ function.*

**Definition 2.5 (Class $\mathcal{L}$ function [48])** *A function $\iota : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{L}$ if it is continuous, strictly decreasing, and $\lim_{s\to\infty} \iota(s) = 0$.*

**Definition 2.6 (Class $\mathcal{KL}$ function [48])** *A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is of class $\mathcal{L}$.*

## 2.3 Minimum Operator Based Reaching Law for DSMC

In this subsection, we introduce two minima-based Reaching Law (RL) for DSMC [2] applicable to both unperturbed and perturbed DTS. These laws utilize a difference equation approach with minima to ensure the FTS of discrete dynamical systems [20].

### 2.3.1 RL1 for DTS

Considering the following system [2]:

$$\aleph^i_{k+1} = \aleph^i_k - \text{sign}[\aleph^i_k] \min(|\aleph^i_k|, \varsigma) + \delta^i{}_k \tag{2.4}$$

where $\varsigma \in \mathbb{R}^+$ is selected such that $\delta^i_L \leq \delta^i_k \leq \delta^i_U$ and $\varsigma \geq \max(\|\delta_L\|, \|\delta_U\|)$. We define $\max(\|\delta_L\|, \|\delta_U\|) = \delta_m$, where $\delta_m \in \mathbb{R}^+$. Furthermore, we define the following diagonal matrices: $\aleph_{(k)} = \text{diag}\{\aleph^1_{(k)}, \aleph^2_{(k)}, \ldots, \aleph^i_{(k)}\}, \delta_{(k)} = \text{diag}\{\delta^1_{(k)}, \delta^2_{(k)}, \ldots, \delta^i_{(k)}\}, \text{sign}\,\aleph_{(k)} = \text{diag}\{\text{sign}\,\aleph^1_{(k)}, \ldots, \text{sign}\,\aleph^i_{(k)}\}$, $i = 1, 2, \ldots, n$. We define the switching function as

$$s_k := \alpha(z_k).$$

where, $z_k$ is the state of system. We define the sliding manifold by the level set

$$\alpha^{-1}(0) := \{z \in D : \alpha(z) = 0\}.$$

**Definition 1.** We confirm the presence of QSM in the neighborhood of the set $\alpha^{-1}(0)$ when $|\aleph_k^i| \leq \omega_m = \delta_m$, where $\omega_m$ is QSM band.

**Definition 2.** The system described by equation (2.4) satisfies the QSM reachability in the vicinity of $\alpha^{-1}(0)$ if, for some $k \geq 0$, the following inequalities are satisfied:

$$\begin{aligned} |\aleph_k^i| > \varsigma &\Rightarrow |\aleph_{k+1}^i| < |\aleph_k^i| - \varepsilon, \\ |\aleph_k^i| \leq \varsigma &\Rightarrow 0 \leq |\aleph_{k+1}^i| \leq \omega_m. \end{aligned} \tag{2.5}$$

where $\varepsilon \in \mathbb{R}^+$ is minutely small.

### 2.3.2 RL2 for DTS

Consider the following RL for DTS [2]:

$$\aleph_{k+1}^i = \aleph_k^i - \varsigma \text{sign}[\aleph_k^i] \min\left(\frac{|\aleph_k^i|}{\varsigma}, |\aleph_k^i|^\beta\right) + \delta^i{}_k. \tag{2.6}$$

where $\varsigma \in \mathbb{R}^+$ as defined for RL1, and $\beta \in (0, 1)$.

**Definition 3.** The DTS (2.6) satisfies the QSM reaching condition in the vicinity of the set $\alpha^{-1}(0)$ if, for some $k \geq 0$, the following conditions hold:

$$\begin{aligned} \frac{|\aleph_k^i|}{\varsigma} > |\aleph_k^i|^\beta &\Rightarrow |\aleph_{k+1}^i| < |\aleph_k^i| - \varepsilon, \\ \frac{|\aleph_k^i|}{\varsigma} \leq |\aleph_k^i|^\beta &\Rightarrow 0 \leq |\aleph_{k+1}^i| \leq \delta_m. \end{aligned}$$

where $\varepsilon \in \mathbb{R}^+$ is arbitrarily small.

**Remark 1.** Definitions 2 and 3 resemble the ideal sliding mode definition for continuous-time systems in the case $\delta_k = 0$ as $\aleph_k$ precisely converges to $\alpha^{-1}(0)$ and remains there indefinitely.

**Remark 2.** The significance of the minimum operator in the control loop is to ensure finite-time reachability to the sliding surface. The role of minimum operator in the control loop can be further emphasized using the example given in [20] and further illustrated for DSMC in [2]. Consider scalar system given by $x(k+1) = x(k) - \text{sign}(x(k)) \min\{|x(k)|, c\}, \quad x(0) = x_0$, The right-hand side of system is continuous everywhere and, for every initial condition in $\mathbb{R} \setminus \{0\}$, and it has a unique solution in forward time. Furthermore, if $|x(k)| \leq c$, $k \in \mathbb{Z}^+$, then $x(k+1) = 0$, which which insures finite time reachability with settling time function given by $K(x_0) = \left\lceil \frac{|x_0|}{c} \right\rceil$.

It is now explicitly stated that the notions of FTS and the associated comparison functions are included solely to complete the formal definitions and notations used later; a comprehensive tutorial on these topics is beyond the scope of the thesis. The central contributions are: (i) a *data-driven* sliding-mode framework employing minimum-operator reaching laws, (ii) a *prescribed-performance data-driven* design ensuring predefined transient bounds, (iii) a *multirate* output-feedback control architecture, (iv) a finite-time discrete control strategy for a two-DOF helicopter system, and (v) an adaptive switching scheme using a ceiling function for discrete-time sliding-mode control. Each of these contributions addresses finite-time behavior through the proposed control laws and synthesis procedures, while the preliminaries on FTS and comparison functions serve only as foundational terminology for these results.