Chapter 2

Fixed time synchronization of quaternion-valued neural networks with time varying delay

2.1 Introduction

Quaternion-valued neural network (QVNN) is an extension of the real-valued neural network (RVNN). The QVNNs are explored these days at a rapid pace because of their wide range of applications in information processing, biological systems, secure communication and so on [50, 51]. There are some problems which can be solved by QVNNs, but not by RVNNs. A quaternion back propagation algorithm gets the correct geometrical transformation for the image compression, whereas a real valued neural network fails here [52]. An element of QVNN stores four times the amount of information stored by an element of RVNN. All the states and activation functions of QVNNs are quaternions. Therefore, dynamical properties of quaternion-valued neural networks are interesting topics to be explored. Furthermore, the fixed time stability of the QVNNs is explored in the current chapter. Fixed-time stability is one in which the trajectories of the solution curve reaches the equilibrium point in finite and this time is independent of the initial conditions. The eariler works on fixed-time synchronization includes [50, 53, 54, 55]. But these are mainly on the complex-valued neural networks (CVNNs). Motivated by the above fact and inspired by the work given in [56] on fixed time synchronization of QVNNs without time delay, it is made an endeavour to find the synchronization of QVNNs with time varying delay using the lemma given in [56]. This lemma is more conservative than the lemma used in previous articles on fixed-time synchronization. A comparison of the settling time using the two lemmas is also included in the chapter.

2.2 Model Description and Preliminaries

In this chapter, a delayed Quaternion valued neural network is considered as

$$\dot{w}(t) = -Cw(t) + Af(w(t)) + Bg(w(t-\tau_1(t))) + I(t), \tag{2.1}$$

where $w(t) = [w_1(t), w_2(t), ..., w_n(t)]^T \in \mathbb{Q}^n$ refers to the state vector of n neurons in the network; $C = \text{diag}\{c_1, c_2, ..., c_n\} \in \mathbb{R}^n$ denotes the self-feedback connection weights matrix with $c_i > 0$; $A = [a_{ij}]_{n\times n} \in \mathbb{Q}^{n\times n} and\ B = [b_{ij}]_{n\times n} \in \mathbb{Q}^{n\times n}$ represent connection weight matrices; $f(.) = [f_1(.), f_2(.), ..., f_n(.)]^T : \mathbb{Q}^n \to \mathbb{Q}^n and\ g(.) = [g_1(.), g_2(.), ..., g_n(.)]^T : \mathbb{Q}^n \to \mathbb{Q}^n,$ are quaternion-valued nonlinear activation functions for time t and $t - \tau_1(t)$ respectively; $\tau_1(t)$ is the time-varying delay; $I(t) = [I_1(t), I_2(t), ..., I_n(t)]^T$ denotes an external input vector.

Since, the state vector consists of quaternion numbers, so these can be denoted as $w_i = w_i^R + w_i^I \bar{i} + w_i^J \bar{j} + w_i^K \bar{k}$, for $i = 1, 2, ..., n$. Similarly, the activation functions $f(.)$ and $g(.)$ can be written as

$$f(w) = f^R(w^R) + f^I(w^I)\bar{i} + f^J(w^J)\bar{j} + f^K(w^K)\bar{k}, \tag{2.2}$$

$$g(w) = g^R(w^R) + g^I(w^I)\bar{i} + g^J(w^J)\bar{j} + g^K(w^K)\bar{k}, \tag{2.3}$$

where $f^R(.) = [f_1^R, f_2^R, ..., f_n^R]^T; f_i^R : \mathbb{R} \to \mathbb{R}$, for $i = 1, 2, ...n$.
Similarly, $f_i^j : \mathbb{R} \to \mathbb{R}$, for $j = I, J, K$ and $i = 1, 2, ...n$; $g_i^j : \mathbb{R} \to \mathbb{R}$, for $j = R, I, J, K$ and $i = 1, 2, ...n$;
All of the above real valued constituents of each of the activation functions satisfy the following assumptions.

Assumption 1. For any $w \in \mathbb{Q}$, $w = w^R + w^I\bar{i} + w^J\bar{j} + w^K\bar{k}$, where w^R, w^I, w^J, w^K are all real numbers. Therefore, each quaternion number can be written as an equivalent four real numbers. In the same way, matrices can be written as $A = A^R + A^I\bar{i} + A^J\bar{j} + A^K\bar{k}, B = B^R + B^I\bar{i} + B^J\bar{j} + B^K\bar{k}$.

Assumption 2. For all $w^R,\ w^I,\ w^J,\ w^K \in \mathbb{R}^n$, the four constituents of each of the activation functions satisfy the following Lipschitz conditions.

$$|f_k^m(x) - f_k^m(y)| \leq M_k|x - y|,$$

$$|g_k^m(x) - g_k^m(y)| \leq N_k|x - y|,$$

where $m = R, I, J, K$ and M_k, N_k are constants for $k = 1, 2, ..., n$.

Now, the system (2.1) can be separated as

$$\begin{aligned}
\dot{w}^R(t) =& - Cw^R(t) + A^R f^R(w^R(t)) - A^I f^I(w^I(t)) - A^J f^J(w^J(t)) - A^K f^K(w^K(t)) \\
&+ B^R g^R(w^R(t-\tau_1(t))) - B^I g^I(w^I(t-\tau_1(t)))) - B^J g^J(w^J(t-\tau_1(t))) \\
&- B^K g^K(w^K(t-\tau_1(t))) + I^R(t), \\
\dot{w}^I(t) =& - Cw^I(t) + A^I f^R(w^R(t)) + A^R f^I(w^I(t)) - A^K f^J(w^J(t)) + A^J f^K(w^K(t)) \\
&+ B^I g^R(w^R(t-\tau_1(t))) + B^R g^I(w^I(t-\tau_1(t))) - B^K g^J(w^J(t-\tau_1(t))) \\
&+ B^J g^K(w^K(t-\tau_1(t))) + I^I(t), \\
\dot{w}^J(t) =& - Cw^J(t) + A^J f^R(w^R(t)) + A^K f^I(w^I(t)) + A^R f^J(w^J(t)) - A^I f^K(w^K(t)) \\
&+ B^J g^R(w^R(t-\tau_1(t))) + B^K g^I(w^I(t-\tau_1(t))) + B^R g^J(w^J(t-\tau_1(t))) \\
&- B^I g^K(w^K(t-\tau_1(t))) + I^J(t), \\
\dot{w}^K(t) =& - Cw^K(t) + A^K f^R(w^R(t)) - A^J f^I(w^I(t)) + A^I f^J(w^J(t)) + A^R f^K(w^K(t)) \\
&+ B^K g^R(w^R(t-\tau_1(t))) - B^J g^I(w^I(t-\tau_1(t))) + B^I g^J(w^J(t-\tau_1(t))) \\
&+ B^R g^K(w^K(t-\tau_1(t))) + I^K(t),
\end{aligned} \tag{2.4}$$

where $w^R(t) = [w_1^R(t), w_2^R(t), ..., w_n^R(t)]^T,\ w^I(t) = [w_1^I(t), w_2^I(t), ..., w_n^I(t)]^T,\ w^J(t) = [w_1^J(t), w_2^J(t), ..., w_n^J(t)]^T,\ w^K(t) = [w_1^K(t), w_2^K(t), ..., w_n^K(t)]^T,\ f^R(w^R(t)) = [f_1^R(w_1^R(t)), f_2^R(w_2^R(t)), ..., f_n^R(w_n^R(t))]^T,\ f^I(w^I(t)) = [f_1^I(w_1^I(t)), f_2^I(w_2^I(t)), ..., f_n^I(w_n^I(t))]^T,\ f^J(w^J(t)) = [f_1^J(w_1^J(t)), f_2^J(w_2^J(t)), ..., f_n^J(w_n^J(t))]^T,\ f^K(w^K(t)) = [f_1^K(w_1^K(t)), f_2^K(w_2^K(t)), ..., f_n^K(w_n^K(t))]^T,\ g^R(w^R(t-\tau_1(t))) = [g_1^R(w_1^R(t-\tau_1(t))), g_2^R(w_2^R(t-\tau_1(t))), ..., g_n^R(w_n^R(t-\tau_1(t)))]^T,\ g^I(w^I(t-\tau_1(t))) = [g_1^I(w_1^I(t-\tau_1(t))), g_2^I(w_2^I(t-\tau_1(t))), ..., g_n^I(w_n^I(t-\tau_1(t)))]^T,\ g^J(w^J(t-\tau_1(t))) = [g_1^J(w_1^J(t-\tau_1(t))), g_2^J(w_2^J(t-\tau_1(t))), ..., g_n^J(w_n^J(t-\tau_1(t)))]^T,\ g^K(w^K(t-\tau_1(t))) = [g_1^K(w_1^K(t-\tau_1(t))), g_2^K(w_2^K(t-\tau_1(t))), ..., g_n^K(w_n^K(t-\tau_1(t)))]^T,\ A^R + A^I\bar{i} + A^J\bar{j} + A^K\bar{k} = A,\ B^R + B^I\bar{i} + B^J\bar{j} + B^K\bar{k} = B.$

The slave system corresponding to master system (2.4) is given by

$$\begin{aligned}
\dot{s}^R(t) =& - Cs^R(t) + A^R f^R(s^R(t)) - A^I f^I(s^I(t)) - A^J f^J(s^J(t)) - A^K f^K(s^K(t)) \\
& + B^R g^R(s^R(t-\tau_1(t))) - B^I g^I(s^I(t-\tau_1(t)))) - B^J g^J(s^J(t-\tau_1(t))) \\
& - B^K g^K(s^K(t-\tau_1(t))) + I^R(t) + u^R(t), \\
\dot{s}^I(t) =& - Cs^I(t) + A^I f^R(s^R(t)) + A^R f^I(s^I(t)) - A^K f^J(s^J(t)) + A^J f^K(s^K(t)) \\
& + B^I g^R(s^R(t-\tau_1(t))) + B^R g^I(s^I(t-\tau_1(t))) - B^K g^J(s^J(t-\tau_1(t))) \\
& + B^J g^K(s^K(t-\tau_1(t))) + I^I(t) + u^I(t), \\
\dot{s}^J(t) =& - Cs^J(t) + A^J f^R(s^R(t)) + A^K f^I(s^I(t)) + A^R f^J(s^J(t)) - A^I f^K(s^K(t)) \\
& + B^J g^R(s^R(t-\tau_1(t))) + B^K g^I(s^I(t-\tau_1(t))) + B^R g^J(s^J(t-\tau_1(t))) \\
& - B^I g^K(s^K(t-\tau_1(t))) + I^J(t) + u^J(t), \\
\dot{s}^K(t) =& - Cs^K(t) + A^K f^R(s^R(t)) - A^J f^I(s^I(t)) + A^I f^J(s^J(t)) + A^R f^K(s^K(t)) \\
& + B^K g^R(s^R(t-\tau_1(t))) - B^J g^I(s^I(t-\tau_1(t))) + B^I g^J(s^J(t-\tau_1(t))) \\
& + B^R g^K(s^K(t-\tau_1(t))) + I^K(t) + u^K(t).
\end{aligned} \tag{2.5}$$

Let us define the error function as $e(t) = s(t) - w(t) = e^R(t) + e^I(t)\bar{i} + e^J(t)\bar{j} + e^K(t)\bar{k}$, where the expressions of $e^R(t), e^I(t), e^J(t)$ and $e^K(t)$ are obtained by subtracting equations (2.5) from equations (2.4) as

$$
\begin{aligned}
\dot{e}^R(t) =& - Ce^R(t) + A^R(f^R(s^R(t)) - f^R(w^R(t))) - A^I(f^I(s^I(t)) - f^I(w^I(t))) \\
& - A^J(f^J(s^J(t)) - f^J(w^J(t))) - A^K(f^K(s^K(t)) - f^K(w^K(t))) \\
& + B^R(g^R(s^R(t-\tau_1(t))) - g^R(w^R(t-\tau_1(t)))) - B^I(g^I(s^I(t-\tau_1(t))) \\
& - g^I(w^I(t-\tau_1(t)))) - B^J(g^J(s^J(t-\tau_1(t))) - g^J(w^J(t-\tau_1(t)))) \\
& - B^K(g^K(s^K(t-\tau_1(t))) - g^K(w^K(t-\tau_1(t)))) + u^R(t), \\
\dot{e}^I(t) =& - Ce^I(t) + A^I(f^R(s^R(t)) - f^R(w^R(t))) + A^R(f^I(s^I(t)) - f^I(w^I(t))) \\
& - A^K(f^J(s^J(t)) - f^J(w^J(t))) + A^J(f^K(s^K(t)) - f^K(w^K(t))) \\
& + B^I(g^R(s^R(t-\tau_1(t))) - g^R(w^R(t-\tau_1(t)))) + B^R(g^I(s^I(t-\tau_1(t))) \\
& - g^I(w^I(t-\tau_1(t)))) - B^K(g^J(s^J(t-\tau_1(t))) - g^J(w^J(t-\tau_1(t)))) \\
& + B^J(g^K(s^K(t-\tau_1(t))) - g^K(w^K(t-\tau_1(t)))) + u^I(t), \\
\dot{e}^J(t) =& - Ce^J(t) + A^J(f^R(s^R(t)) - f^R(w^R(t))) + A^K(f^I(s^I(t)) - f^I(w^I(t))) \\
& + A^R(f^J(s^J(t)) - f^J(w^J(t))) - A^I(f^K(s^K(t)) - f^K(w^K(t))) \\
& + B^J(g^R(s^R(t-\tau_1(t))) - g^R(w^R(t-\tau_1(t)))) + B^K(g^I(s^I(t-\tau_1(t))) \\
& - g^I(w^I(t-\tau_1(t)))) + B^R(g^J(s^J(t-\tau_1(t))) - g^J(w^J(t-\tau_1(t)))) \\
& - B^I(g^K(s^K(t-\tau_1(t))) - g^K(w^K(t-\tau_1(t)))) + u^J(t), \\
\dot{e}^K(t) =& - Ce^K(t) + A^K(f^R(s^R(t)) - f^R(w^R(t))) - A^J(f^I(s^I(t)) - f^I(w^I(t))) \\
& + A^I(f^J(s^J(t)) - f^J(w^J(t))) + A^R(f^K(s^K(t)) - f^K(w^K(t))) \\
& + B^K(g^R(s^R(t-\tau_1(t))) - g^R(w^R(t-\tau_1(t)))) - B^J(g^I(s^I(t-\tau_1(t))) \\
& - g^I(w^I(t-\tau_1(t)))) + B^I(g^J(s^J(t-\tau_1(t))) - g^J(w^J(t-\tau_1(t)))) \\
& + B^R(g^K(s^K(t-\tau_1(t))) - g^K(w^K(t-\tau_1(t))) + u^K(t).
\end{aligned} \tag{2.6}
$$

Let the controller be defined as

$$u_p^\mu = -\lambda_{1p} e_p^\mu(t) - sgn(e_p^\mu(t))\big(\lambda_{2p}^\mu |e_p^\mu(t-\tau_1(t))| + \lambda_{3p}|e_p^\mu(t)|^\alpha + \lambda_{4p}|e_p^\mu(t)|^\beta\big), \quad (2.7)$$

where $\lambda_{1p} > 0$, $0 < \alpha < 1$, $\beta > 1$, $\mu = R, I, J, K$ and $p = 1, 2, ..., n$.

Following definitions and lemmas will be used in this chapter as and when required.

Definition 2.1. [57, 58] If $f(t)$ be differentiable on $\mathbb{R}$, then the upper right Dini derivative of $|f(t)| = f(t).\ sgn(f(t))$ is given by $D^+(|f(t)|) = \dot{f}(t).\ sgn(f(t))$ for $f(t) \neq 0$.

Lemma 2.2. *([59]) If there is a continuous and radially unbounded function* $V(.) : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ *and e(t) is any solution of* (2.6)*, such that*

(i) $V(e(t)) = 0$ *iff* $e(t) = 0$;

(ii) for $a, b, c > 0, 0 < \alpha < 1$ *and* $\beta > 1$,

$$\dot{V}(t) \leq -aV^\alpha(e(t)) - bV^\beta(e(t)) - cV(e(t)),$$

then the origin of the system (2.6) *achieves fixed time stability. The settling time is given by* $T_{set} = \frac{1}{c(1-\alpha)} ln(1 + \frac{c}{a}) + \frac{1}{c(\beta-1)} ln(1 + \frac{c}{b})$.

Lemma 2.3. *([56]) If there is a continuous and radially unbounded function* $V(.) : \mathbb{R}^n \to \mathbb{R}_+$ *and e(t) is any solution of* (2.6) *, such that*

(i) $V(e(t)) = 0$ *iff* $e(t) = 0$,

(ii) for $a, b > 0, 0 < \alpha < 1$ *and* $\beta > 1$,

$$\dot{V}(t) \leq -aV^\alpha(e(t)) - bV^\beta(e(t)),$$

then the origin of the system (2.6) achieves fixed time stability with

$$V(t) = 0,\ t \geq T(e_0)$$

and the settling time is given by $T_{set} = \frac{1}{a(1-\alpha)} + \frac{1}{b(\beta-1)}$.

Lemma 2.4. *([56]) Let* $w_i \geq 0$ *for* $i = 1, 2, ..., n$, *and* $0 < p \leq 1, q > 1$, *then*

$$\sum_{i=1}^{n} w_i^p \geq (\sum_{i=1}^{n} w_i)^p, \sum_{i=1}^{n} w_i^q \geq n^{1-q}(\sum_{i=1}^{n} w_i)^q.$$

2.3 Fixed time synchronization criteria for QVNNs

In this section, we will ensure the fixed time synchronization of the master-slave systems (2.4)-(2.5).

Theorem 2.5. *The master system (2.4) & slave system (2.5) achieve fixed time synchronization with the help of the Assumption 2 and the controller (2.7) if*

$$c_p + \lambda_{1p} - \sum_{q=1}^{n} \big(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\big) M_p > 0, \tag{2.8}$$

$$\lambda_{2p}^{\mu} - \sum_{q=1}^{n} \big(|b_{qp}^R| + |b_{qp}^I| + |b_{qp}^J| + |b_{qp}^K|\big) N_p > 0, \tag{2.9}$$

$$\min_p \lambda_{3p} > K \geq \max_{1\leq p\leq n} c_p \text{ and } \min_p(\lambda_{4p})(4n)^{1-\beta} > K. \tag{2.10}$$

Also, the settling time is estimated as

$$T_{set} \leq T(e_0) \leq \frac{1}{c(1-\alpha)} ln\left(1 + \frac{c}{\min_p(\lambda_{3p}) - K}\right) + \frac{1}{c(\beta-1)} ln\left(1 + \frac{c}{\min_p(\lambda_{4p})(4n)^{1-\beta} - K}\right), \tag{2.11}$$

where $c = \sum_{p=1}^{n} \big(c_p + \lambda_{1p} - \sum_{q=1}^{n} \big(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\big) M_p\big)$.

Proof. Let us consider Lyapunov function as

$$V(t) = \sum_{i=1}^{4} V_i(t).$$

where

$$V_1(t) = \sum_{p=1}^{n} |e_p^R(t)|, \; V_2(t) = \sum_{p=1}^{n} |e_p^I(t)|, \; V_3(t) = \sum_{p=1}^{n} |e_p^J(t)|, \; V_4(t) = \sum_{p=1}^{n} |e_p^K(t)|.$$

Now, the Dini derivative of $V_1(t)$ is given by

$$\begin{aligned}
D^+(V_1(t)) =& \sum_{p=1}^{n} \dot{e}_p^R(t).sgn(e_p^R(t)) \\
=& \sum_{p=1}^{n} sgn(e_p^R(t))\Bigg(- c_p e_p^R(t) + \sum_{q=1}^{n} a_{pq}^R(f_q^R(s_q^R(t)) - f_q^R(w_q^R(t))) - a_{pq}^I(f_q^I(s_q^I(t)) \\
&- f_q^I(w_q^I(t))) - a_{pq}^J(f_q^J(s_q^J(t)) - f_q^J(w_q^J(t))) - a_{pq}^K(f_q^K(s_q^K(t)) - f_q^K(w_q^K(t))) \\
&+ b_{pq}^R(g_q^R(s_q^R(t-\tau_1(t))) - g_q^R(w_q^R(t-\tau_1(t)))) - b_{pq}^I(g_q^I(s_q^I(t-\tau_1(t))) \\
&- g_q^I(w_q^I(t-\tau_1(t)))) - b_{pq}^J(g_q^J(s_q^J(t-\tau_1(t))) - g_q^J(w_q^J(t-\tau_1(t)))) \\
&- b_{pq}^K(g_q^K(s_q^K(t-\tau_1(t))) - g_q^K(w_q^K(t-\tau_1(t)))) + u_p^R(t)\Bigg) \\
\leq& - \sum_{p=1}^{n} c_p|e_p^R(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\Bigg(|a_{pq}^R|M_q|e_q^R(t)| + |a_{pq}^I|M_q|e_q^I(t)| + |a_{pq}^J|M_q|e_q^J(t)| \\
&+ |a_{pq}^K|M_q|e_q^K(t)| + |b_{pq}^R|N_q|e_q^R(t-\tau_1(t))| + |b_{pq}^I|N_q|e_q^I(t-\tau_1(t))| \\
&+ |b_{pq}^J|N_q|e_q^J(t-\tau_1(t))| + |b_{pq}^K|N_q|e_q^K(t-\tau_1(t))|\Bigg) + \sum_{p=1}^{n} sgn(e_p^R(t)) \\
&\times \Bigg(- \lambda_{1p}e_p^R(t) - sgn(e_p^R(t))\big(\lambda_{2p}^R|e_p^R(t-\tau_1(t))| + \lambda_{3p}|e_p^R(t)|^\alpha + \lambda_{4p}|e_p^R(t)|^\beta\big)\Bigg)
\end{aligned}$$

$$\begin{aligned}
D^+(V_1(t)) \leq & -\sum_{p=1}^{n}\left(c_p - \sum_{q=1}^{n}|a_{qp}^R|M_p + \lambda_{1p}\right)|e_p^R(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\Bigg(|a_{qp}^I|M_p|e_p^I(t)| \\
& + |a_{qp}^J|M_p|e_p^J(t)| + |a_{qp}^K|M_p|e_p^K(t)| + |b_{qp}^R|N_p|e_p^R(t-\tau_1(t))| \\
& + |b_{qp}^I|N_p|e_p^I(t-\tau_1(t))| + |b_{qp}^J|N_p|e_p^J(t-\tau_1(t))| + |b_{qp}^K|N_p|e_p^K(t-\tau_1(t))|\Bigg) \\
& - \sum_{p=1}^{n}\lambda_{2p}^R|e_p^R(t-\tau_1(t))| - \sum_{p=1}^{n}\big(\lambda_{3p}|e_p^R(t)|^\alpha + \lambda_{4p}|e_p^R(t)|^\beta\big).
\end{aligned}$$

In the similar way, the Dini derivatives of $V_2(t), V_3(t), V_4(t)$ are calculated as

$$\begin{aligned}
D^+(V_2(t)) \leq & -\sum_{p=1}^{n}\left(c_p - \sum_{q=1}^{n}|a_{qp}^R|M_p + \lambda_{1p}\right)|e_p^I(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\Bigg(|a_{qp}^I|M_p|e_p^R(t)| \\
& + |a_{qp}^J|M_p|e_p^K(t)| + |a_{qp}^K|M_p|e_p^J(t)| + |b_{qp}^R|N_p|e_p^I(t-\tau_1(t))| \\
& + |b_{qp}^I|N_p|e_p^R(t-\tau_1(t))| + |b_{qp}^J|N_p|e_p^K(t-\tau_1(t))| + |b_{qp}^K|N_p|e_p^J(t-\tau_1(t))|\Bigg) \\
& - \sum_{p=1}^{n}\lambda_{2p}^I|e_p^I(t-\tau_1(t))| - \sum_{p=1}^{n}\big(\lambda_{3p}|e_p^I(t)|^\alpha + \lambda_{4p}|e_p^I(t)|^\beta\big).
\end{aligned}$$

$$\begin{aligned}
D^+(V_3(t)) \leq & -\sum_{p=1}^{n}\left(c_p - \sum_{q=1}^{n}|a_{qp}^R|M_p + \lambda_{1p}\right)|e_p^J(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\Bigg(|a_{qp}^J|M_p|e_p^R(t)| \\
& + |a_{qp}^K|M_p|e_p^I(t)| + |a_{qp}^I|M_p|e_p^K(t)| + |b_{qp}^R|N_p|e_p^J(t-\tau_1(t))| \\
& + |b_{qp}^J|N_p|e_p^R(t-\tau_1(t))| + |b_{qp}^K|N_p|e_p^I(t-\tau_1(t))| + |b_{qp}^I|N_p|e_p^K(t-\tau_1(t))|\Bigg) \\
& - \sum_{p=1}^{n}\lambda_{2p}^J|e_p^J(t-\tau_1(t))| - \sum_{p=1}^{n}\big(\lambda_{3p}|e_p^J(t)|^\alpha + \lambda_{4p}|e_p^J(t)|^\beta\big).
\end{aligned}$$

$$
\begin{aligned}
D^{+}(V_4(t)) \leq & -\sum_{p=1}^{n}\Big(c_p-\sum_{q=1}^{n}|a_{qp}^{R}|M_p+\lambda_{1p}\Big)|e_p^{K}(t)|+\sum_{p=1}^{n}\sum_{q=1}^{n}\Big(|a_{qp}^{I}|M_p|e_p^{J}(t)| \\
& +|a_{qp}^{K}|M_p|e_p^{R}(t)|+|a_{qp}^{J}|M_p|e_p^{I}(t)|+|b_{qp}^{R}|N_p|e_p^{K}(t-\tau_1(t))| \\
& +|b_{qp}^{I}|N_p|e_p^{J}(t-\tau_1(t))|+|b_{qp}^{K}|N_p|e_p^{R}(t-\tau_1(t))|+|b_{qp}^{J}|N_p|e_p^{I}(t-\tau_1(t))|\Big) \\
& -\sum_{p=1}^{n}\lambda_{2p}^{K}|e_p^{K}(t-\tau_1(t))|-\sum_{p=1}^{n}\big(\lambda_{3p}|e_p^{J}(t)|^{\alpha}+\lambda_{4p}|e_p^{J}(t)|^{\beta}\big).
\end{aligned}
$$

Combining the above four inequalities, we get

$$
\begin{aligned}
D^{+}(V(t)) = & D^{+}(V_1(t))+D^{+}(V_2(t))+D^{+}(V_3(t))+D^{+}(V_4(t)) \\
\leq & -\sum_{p=1}^{n}\Big(c_p+\lambda_{1p}-\sum_{q=1}^{n}\big(|a_{qp}^{R}|+|a_{qp}^{I}|+|a_{qp}^{J}|+|a_{qp}^{K}|\big)M_p\Big)|e_p^{R}(t)| \\
& -\sum_{p=1}^{n}\Big(\lambda_{2p}^{R}-\sum_{q=1}^{n}\big(|b_{qp}^{R}|+|b_{qp}^{I}|+|b_{qp}^{J}|+|b_{qp}^{K}|\big)N_p\Big)|e_p^{R}(t-\tau_1(t))| \\
& -\sum_{p=1}^{n}\Big(c_p+\lambda_{1p}-\sum_{q=1}^{n}\big(|a_{qp}^{R}|+|a_{qp}^{I}|+|a_{qp}^{J}|+|a_{qp}^{K}|\big)M_p\Big)|e_p^{I}(t)| \\
& -\sum_{p=1}^{n}\Big(\lambda_{2p}^{I}-\sum_{q=1}^{n}\big(|b_{qp}^{R}|+|b_{qp}^{I}|+|b_{qp}^{J}|+|b_{qp}^{K}|\big)N_p\Big)|e_p^{I}(t-\tau_1(t))| \\
& -\sum_{p=1}^{n}\Big(c_p+\lambda_{1p}-\sum_{q=1}^{n}\big(|a_{qp}^{R}|+|a_{qp}^{I}|+|a_{qp}^{J}|+|a_{qp}^{K}|\big)M_p\Big)|e_p^{J}(t)| \\
& -\sum_{p=1}^{n}\Big(\lambda_{2p}^{J}-\sum_{q=1}^{n}\big(|b_{qp}^{R}|+|b_{qp}^{I}|+|b_{qp}^{J}|+|b_{qp}^{K}|\big)N_p\Big)|e_p^{J}(t-\tau_1(t))| \\
& -\sum_{p=1}^{n}\Big(c_p+\lambda_{1p}-\sum_{q=1}^{n}\big(|a_{qp}^{R}|+|a_{qp}^{I}|+|a_{qp}^{J}|+|a_{qp}^{K}|\big)M_p\Big)|e_p^{K}(t)| \\
& -\sum_{p=1}^{n}\Big(\lambda_{2p}^{K}-\sum_{q=1}^{n}\big(|b_{qp}^{R}|+|b_{qp}^{I}|+|b_{qp}^{J}|+|b_{qp}^{K}|\big)N_p\Big)|e_p^{K}(t-\tau_1(t))| \\
& -\sum_{p=1}^{n}\lambda_{3p}\big(|e_p^{R}(t)|^{\alpha}+|e_p^{I}(t)|^{\alpha}+|e_p^{J}(t)|^{\alpha}+|e_p^{K}(t)|^{\alpha}\big) \\
& -\sum_{p=1}^{n}\lambda_{4p}\big(|e_p^{R}(t)|^{\beta}+|e_p^{I}(t)|^{\beta}+|e_p^{J}(t)|^{\beta}+|e_p^{K}(t)|^{\beta}\big).
\end{aligned}
$$

By using Lemma 2.2 and Lemma 2.4, we obtain

$$\begin{aligned}
D^+(V(t)) \leq &- \sum_{p=1}^{n} \left(c_p + \lambda_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\right) M_p \right) V(t) + KV(t) \\
&- \sum_{p=1}^{n} \lambda_{3p} \left(|e_p^R(t)|^\alpha + |e_p^I(t)|^\alpha + |e_p^J(t)|^\alpha + |e_p^K(t)|^\alpha\right) \\
&- \sum_{p=1}^{n} \lambda_{4p} \left(|e_p^R(t)|^\beta + |e_p^I(t)|^\beta + |e_p^J(t)|^\beta + |e_p^K(t)|^\beta\right) \\
\leq &- \sum_{p=1}^{n} \left(c_p + \lambda_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\right) M_p \right) V(t) + KV(t) \\
&- \sum_{p=1}^{n} \lambda_{3p} \left(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|\right)^\alpha \\
&- \sum_{p=1}^{n} 4^{1-\beta} \lambda_{4p} \left(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|\right)^\beta \\
\leq &- \sum_{p=1}^{n} \left(c_p + \lambda_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\right) M_p \right) V(t) + KV(t) \\
&- \min_p(\lambda_{3p}) \left(\sum_{p=1}^{n} \left(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|\right)^\alpha \right) \\
&- \min_p(\lambda_{4p})(4n)^{1-\beta} \left(\sum_{p=1}^{n} \left(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|\right)^\beta \right) \\
\leq &- \left(\min_p(\lambda_{3p}) - K\right) V^\alpha(t) - \left(\min_p(\lambda_{4p})(4n)^{1-\beta} - K\right) V^\beta(t) \\
&- \sum_{p=1}^{n} \left(c_p + \lambda_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\right) M_p \right) V(t).
\end{aligned}$$

□

Remark 2.3.1. The above result is obtained for time varying delay, which is generalization of the result proved in [56]. Also, the lemma used to prove the results in the present chapter is less conservative than the lemma used in [56] for proving fixed time synchronization.

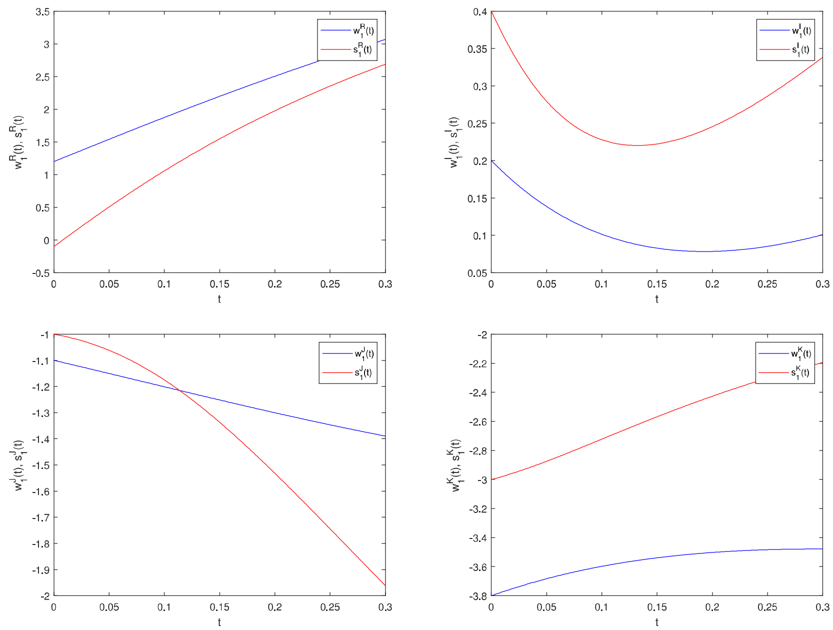

FIGURE 2.1: Plots of the trajectories (a): $w_1^R(t)$ and $s_1^R(t)$, (b): $w_1^I(t)$ and $s_1^I(t)$, (c): $w_1^J(t)$ and $s_1^J(t)$, (d): $w_1^K(t)$ and $s_1^K(t)$ vs. time t of the master and slave systems (2.4) and (2.5) without controllers.

Remark 2.3.2. The quaternion numbers are non-commutative. In order to tackle its non-commutative nature, QVNN is segregated into four RVNNs. Also mathematical model considered for the study contains time varying delay. Therefore, the mathematical computation towards achieving synchronization of QVNNs with time delays assumes more complexity. However, the complexity can be reduced by establishing fixed time stability of error functions with the help of a flexible mathematical lemma and proper design of controllers which include time varying delay.

Remark 2.3.3. The settling time using lemma 2.2 is less than the lemma 2.3 as

$$\frac{1}{c(1-\alpha)}ln(1+\frac{c}{a})+\frac{1}{c(\beta-1)}ln(1+\frac{c}{b})-\frac{1}{a(1-\alpha)}-\frac{1}{b(\beta-1)}$$
$$=\frac{1}{(1-\alpha)}\left[\frac{log(1+\frac{c}{a})}{c}-\frac{1}{a}\right]+\frac{1}{(\beta-1)}\left[\frac{log(1+\frac{c}{b})}{c}-\frac{1}{b}\right]<0.$$

2.4 Example

Here, the synchronization of two identical QVNNs with periodic time delay through a numerical example is discussed for different particular cases.

Example 2.1. Consider the QVNN given as

$$\dot{w}(t) = -Cw(t) + Af(w(t)) + Bg(w(t - \tau_1(t))) + I(t), \tag{2.12}$$

where $w = w^1 + w^2 i + w^3 j + w^4 k$,

$$A = \begin{pmatrix} 2 + 1.2\bar{i} + 1.5\bar{j} + 1.3\bar{k} & -1 - 4.5\bar{i} - 2.3\bar{j} - 3.2\bar{k} \\ 1 + 1.7\bar{i} + 1.4\bar{j} + 1.6\bar{k} & 0.8 + 0.9\bar{i} + 0.6\bar{j} + 0.5\bar{k} \end{pmatrix},$$

$$B = \begin{pmatrix} 1 - 0.9\bar{i} + 2.1\bar{j} + \bar{k} & 2 + 2\bar{i} - \bar{j} + 1.1\bar{k} \\ 3 - \bar{i} + 0.4\bar{j} + 0.6\bar{k} & 0.2 - \bar{i} + 2\bar{j} + \bar{k} \end{pmatrix},$$

$C = diag[1.7, 2.8], I = (1.2 - 1.2\bar{i} + 1.3\bar{j} + 1.4\bar{k}, 1.4 + 1.3\bar{i} + 1.4\bar{j} - 1.3\bar{k})^T, \tau_1(t) = 0.45\sin(t) + 0.55$

and the activation functions are

$$f_j(w_j^\mu) = tanh(w_j^\mu),\ g_j(w_j^\mu) = tanh(w_j^\mu),$$

for j=1,2,3 and $\mu = 1, 2, 3, 4$.

Taking the initial conditions as

$w_1(0) = 1.2 + 0.2\bar{i} - 1.1\bar{j} - 3.8\bar{k}$, $w_2(0) = 5.3 + 0.1\bar{i} + 0.2\bar{j} - 1.4\bar{k}$,

$s_1(0) = -0.1 + 0.4\bar{i} - 1.0\bar{j} - 3.0\bar{k}, s_2(0) = 0.8 + 2.0\bar{i} + 1.0\bar{j} + 0.5\bar{k}$,

Figure 2.1(a)-(d) depict that the trajectories of master and slave systems without controllers are not synchronized. The divergence tendencies of the first elements of master and slave systems $w_1^R(t)$ and $s_1^R(t)$, $w_1^I(t)$ and $s_1^I(t)$, $w_1^J(t)$ and $s_1^J(t)$, $w_1^K(t)$ and $s_1^K(t)$ against time are clearly found in Fig.2.1(a), Fig.1(b), Fig.1(c), Fig.1(d)

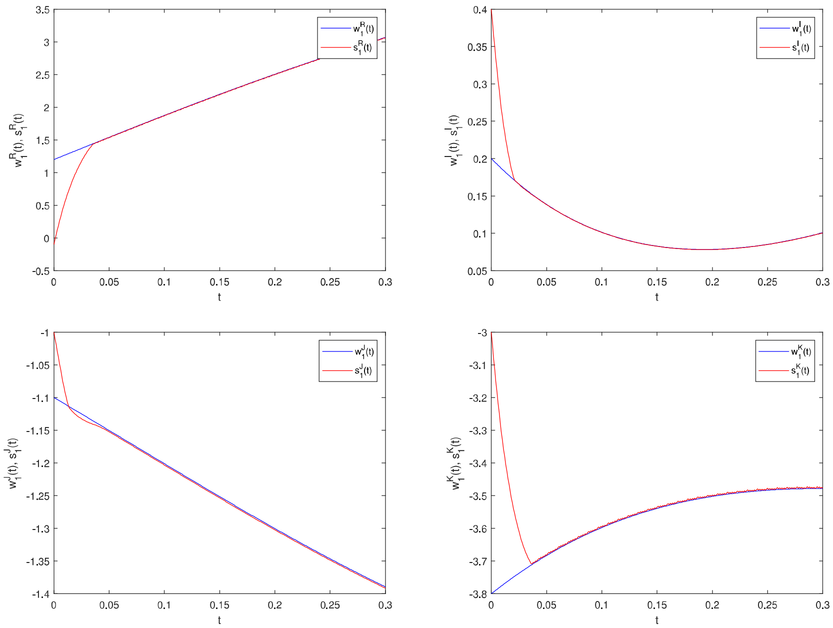

FIGURE 2.2: Plots of the trajectories (a): $w_1^R(t)$ and $s_1^R(t)$, (b): $w_1^I(t)$ and $s_1^I(t)$, (c): $w_1^J(t)$ and $s_1^J(t)$, (d): $w_1^K(t)$ and $s_1^K(t)$ vs. time t of the master and slave systems (2.4) and (2.5) in presence of controllers.

respectively.

From the given conditions, Assumption 1 holds for $M_p = 1 = N_p$. Taking $\lambda_{11} = 11, \lambda_{12} = 12, \alpha = 0.5, \beta = 1.5, \lambda_{2p}^{\mu} = 11, \lambda_{3p} = 20, \lambda_{4p}^{\mu} = 10$, the controllers can be written as

$$u_1^{\mu}(t) = -11e_1^{\mu}(t) - sgn(e_1^{\mu}(t))[11e_1^{\mu}(t-\tau_1(t)) + 20|e_1^{\mu}(t)|^{0.5} + 10|e_1^{\mu}(t)|^{1.5}],$$
$$u_2^{\mu}(t) = -12e_1^{\mu}(t) - sgn(e_1^{\mu}(t))[11e_1^{\mu}(t-\tau_1(t)) + 20|e_1^{\mu}(t)|^{0.5} + 10|e_1^{\mu}(t)|^{1.5}].$$

Based on the choices of the controllers and also for the choices of the various parameters mentioned above, all the required conditions of Theorem 1 are satisfied. Thus the master and slave systems (2.4) and (2.5) are synchronized, which are pictorially shown through Figure 2.2(a)-(d). The figures clearly show that the first elements of the master and slave systems are synchronized with the use of controllers. Here the settling time given in equation (2.11) is calculated as $T_{set} = 2.66$.

Remark 2.4.1. Settling time in the above example is $T_{set} = 2.66$. If lemma 2.3 is used to prove the above theorem, then settling time would be $T_{set} = 3.85$. Therefore, lemma 2.2 gives a better time estimate as compared to lemma 2.3, which computationally justifies the fact mentioned in Remark 2.3.3 that the proposed results obtained by using lemma 2.2 are better as compared to the results obtained by using lemma 2.3.

2.5 Conclusion

The present study is on fixed time synchronization of master-slave systems with time varying delay. The state vector is quaternion number. The QVNNs are decomposed into four real valued networks, and then by constructing the Lyapunov function and using some already proved lemmas, the results have been obtained. The proposed results are validated through an example of synchronizing two QVNNs with periodic time delay. For future endeavour, the key challenges will be to find the fixed time synchronization of QVNNs with different kinds of time delays like mixed time varying delays and also with different kinds of controllers like impulsive controller, intermittent controller etc. Anti synchronization of the above systems will also be studied in future. An extension of fixed time stability viz., predefined stability of the QVNNs can add more significant knowledge about QVNNs. The results obtained in the present contribution can be applied to the CVNN, RVNN, QVMNN (Quaternion valued memristor neural network) with time varying delays.

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