Chapter 1

Introduction

1.1 Nonlinear Waves and Hyperbolic Equations

The aspects of mathematics, which relate to wave motion, occupy a position of considerable importance in the discipline of science and engineering from the point of the mathematical modelling.

The most familiar physical notion of a wave is the one involving the propagation of a disturbance or variation that transfers energy progressively from point to point in a medium and that may take the form of an elastic deformation or of a variation of pressure, electric or magnetic intensity, electric potential, or temperature. This form of wave is thus inherently connected with motion of some kind involving the space $\mathbb{R}^n$ and the time t, so that it gives rise naturally to problems of an evolutionary nature with respect to time. For this reason the time variable will always need to be distinguished from the other independent variables.

Although, wave propagation can be described by partial differential equations that may be classified either as hyperbolic or as parabolic type; in present investigation we shall confine ourselves to quasilinear hyperbolic partial differential equations only. Further, if the governing system of equations are non-linear, it is not possible to apply the principle of superposition of solutions as in case of linear partial differential equations. In most physical situations hyperbolic partial differential equations provide the basic mathematical tool to describe the wave propagation most naturally. The mathematical theory of hyperbolic equations is dominated by the concept of characteristic hypersurfaces and their geometry. Across these hypersurfaces a continuous solution may exhibit Lipschitz discontinuities in its first or higher order normal derivatives. These hypersurfaces act as transporters of these discontinuities, when they exist, just as they also transport elements of a solution hypersurface when it is differentiable (smooth). In the case of one space dimension and time these characteristic hypersurfaces reduce to the families of characteristic curves in the $(x,\ t)$-plane, along each of which may be transported a Lipschitz discontinuity in a derivative of the solution normal to the characteristics. The solution hypersurface itself then reduces to an ordinary smooth surface on which a Lipschitz discontinuity in the first derivative of the solution normal to a characteristic curve manifests itself in the form of a crease on the surface. This crease in the solution surface, or its analogue in $\mathbb{R}^n \times t$, may be interpreted as representing a clearly defined propagating wavefront. The solution on the side of the wavefront towards which propagation takes place may then be regarded as being the 'undisturbed solution' ahead of the wavefront, whilst the solution on the other side regarded as a propagating 'disturbance wave' which is entering a region occupied by the undisturbed solution. The refinement of concept of the wavefront above comes about when the first Lipschitz discontinuity to occur is one for which $n > 1$. The solution surface will then appear smooth across the wavefront, which will coincide with the line in the solution surface across which a discontinuous change of curvature takes place. Therefore, a wavefront separating an undisturbed state from a propagating disturbance wave will still exist, but it will be appropriate to call such a disturbance a smooth fronted wave.

Most of the physical problems, arising in gasdynamics, lead to the formulation of a quasilinear system of first order partial differential equations. These equations are linear in the first derivative of dependent variables, but the coefficients may be functions of dependent variables. In order to present the mathematical description let us consider a general quasilinear system of first order equations in $\mathbb{R}^n \times t$ written as

$$A_0(\mathcal{U}, \bar{x},\ t)\mathcal{U}_t + \sum_{i=1}^{n} A_i(\mathcal{U},\ \bar{x},\ t)\,\mathcal{U}_{x_i} + B(\mathcal{U},\ \bar{x},\ t) = 0, \tag{1.1.1}$$

where $\mathcal{U}(\bar{x},\ t)$ is the column vector with the m elements $u_1(\bar{x},\ t)$, $u_2(\bar{x},\ t)$,..., $u_m(\bar{x},\ t)$, $\bar{x} = (x_1,\ x_2,\ .....,\ x_n)$ is a vector in $\mathbb{R}^n$, the $A_i(\mathcal{U}, \bar{x},\ t)$ are $m \times m$ matrices with elements dependent on $\mathcal{U}$, $\bar{x}$ and t and $B\,(\mathcal{U},\ \bar{x},\ t)$ is the column vector with m elements $b_1\,(\mathcal{U},\ \bar{x},\ t)$, $b_2\,(\mathcal{U},\ \bar{x},\ t), \ldots\ldots, b_m\,(\mathcal{U},\ \bar{x},\ t)$. The subscripts t and x_i denote partial differentiation.

The basic idea underlying the hyperbolicity of a system is that the Cauchy problem should be well posed for it. For the first order system (1.1.1) the Cauchy problem amounts to specifying $\mathcal{U}$ at points on some initial manifold $\boldsymbol{S}$ in $\mathbb{R}^{n-1} \times t$, so that the system will be hyperbolic when this data is sufficient to determine a unique solution that depends continuously on the data specified at points of $\boldsymbol{S}$.

With this idea in mind, and in keeping with the geometrical approach to wavefronts that has been adopted so far, let us now seek to determine when it is possible to group terms of (1.1.1) that they express the derivative of $\mathcal{U}$ normal to $\boldsymbol{S}$ in terms of derivatives of $\mathcal{U}$ in $\boldsymbol{S}$.

A new coordinate system $(\bar{\xi},\ t')$ is introduced, where $\bar{\xi} = (\xi_1, \xi_{2,\ldots\ldots\ldots}\xi_n)$ with $\xi_i = \xi_i(\bar{x}, t)$ being differentiable functions of their arguments and $t = t'$. The manifold S is taken to be associated with the coordinates ξ_k and to have the equation $\xi_k(\bar{x}, t) =$

a (constant), and aside from this restriction, the other ξ_i's will be chosen arbitrarily. The transformation thus becomes

$$t = t', \qquad \xi_i(\bar{x}, t) = \text{Constant}, \qquad for \quad i = 1, 2, ...n. \tag{1.1.2}$$

Also, it is assumed that initially the transformation is non-singular in the vicinity of $\boldsymbol{S}$. Using the transformation (1.1.2), Eq. (1.1.1) may be recast as

$$A_0(U, \bar{x},\ t)\left(\frac{\partial \mathcal{U}}{\partial t'} + \sum_{j=1}^{n} \frac{\partial \xi_j}{\partial t}\frac{\partial \mathcal{U}}{\partial \xi_j}\right) + \sum_{i,j=1}^{n} A_i(\mathcal{U},\ \bar{x},\ t)\frac{\partial \xi_j}{\partial x_i}\frac{\partial \mathcal{U}}{\partial \xi_j} + B(\mathcal{U},\ \bar{x},\ t) = 0. \tag{1.1.3}$$

As the derivative of $\mathcal{U}$ is to be expressed normal to $\boldsymbol{S}$, and $\boldsymbol{S}$ has been embedded in the family of coordinate manifolds $\xi_k(\bar{x}, t) = \text{Constant}$, it follows that the required derivative is $\partial \mathcal{U}/\partial \xi_k$, which may be separated from (1.1.3) and may be written in the following form:

$$\Lambda \partial \mathcal{U}/\partial \xi_k + \mathcal{R} = 0, \tag{1.1.4}$$

where

$$\Lambda = \left(A_0(\mathcal{U}, \bar{x}, t)\frac{\partial \xi_k}{\partial t} + \sum_{i=1}^{n} A_i(\mathcal{U}, \bar{x}, t)\frac{\partial \xi_k}{\partial x_i}\right), \tag{1.1.5}$$

and $\mathcal{R}$ is column vector with its m elements dependent upon $\mathcal{U}$, $\bar{x}$, t and $\partial \mathcal{U}/\partial \xi_k$ with $i \neq k$. Consequently, the derivative $\partial \mathcal{U}/\partial \xi_k$ normal to $\boldsymbol{S}$ is to be determined from equation (1.1.4) provided Λ^{-1} exists, which implies the condition

$$\det \Lambda \neq 0. \tag{1.1.6}$$

Dividing $\det \Lambda$ by $|\nabla_x \xi_k| = \left(\sum_{i=1}^{n} (\partial \xi_k/\partial x_i)^2\right)^{1/2}$ and setting

$$-\lambda = \frac{\partial \xi_k/\partial t}{|\nabla_x \xi_k|}, \qquad \upsilon_i = \frac{\partial \xi_k/\partial x_i}{|\nabla_x \xi_k|}, \qquad for \quad i = 1, 2, ..., n. \tag{1.1.7}$$

So that the unit vector $\bar{\upsilon} = (\upsilon_1, \upsilon_2, ...\upsilon_n)$ is then the normalized spatial gradient $\nabla_x \xi_k$ of ξ_k.

Using (1.1.7), the condition (1.1.6) is written as

$$F(P; \bar{\upsilon}, \lambda) \neq 0, \tag{1.1.8}$$

where

$$F(P; \bar{\upsilon}, \lambda) = \left| \sum_{i}^{n} \upsilon_i A_i(P) - \lambda A_0(P) \right|. \tag{1.1.9}$$

Here, the notation $A_i(P)$ has been employed to signify the value of $A_i(\mathcal{U},\ \bar{x},\ t)$ at point P of the manifold $\boldsymbol{S}$. The expression $F(P; \bar{\upsilon}, \lambda)$ is a homogeneous polynomial of degree $m$in the quantities $(-\lambda, \upsilon_1, \upsilon_2, ...\upsilon_n)$, which is also called characteristic polynomial of the system (1.1.1) with respect to $\boldsymbol{S}$.

It may be noted here that the normal derivative $\partial U / \partial \xi_k$ will be indeterminate at any point P of a manifold $\boldsymbol{S}$ for which

$$F(P; \bar{\upsilon}, \lambda) = 0. \tag{1.1.10}$$

The manifold $\boldsymbol{S}$ on which the condition (1.1.10) is satisfied are called characteristic manifolds; and the manifolds for which (1.1.8) is satisfied are called non-characteristic.

The system (1.1.1) is said to be strictly hyperbolic in t-direction at point P if the zeros $\lambda^{(1)}, \lambda^{(2)}, ..., \lambda^{(m)}$ of the characteristic polynomial $F(P; \bar{\upsilon}, \lambda)$ are all real and distinct for all choices of the unit vectors $\bar{\upsilon}$ and if the right eigenvectors $r^{(1)}, r^{(2)}, ..., r^{(m)}$ satisfying

$$\sum_{i=1}^{m} \left[\upsilon_i A_i(P) - \lambda^{(j)} A_0(P) \right] r^{(j)} = 0, \tag{1.1.11}$$

span the space E^m occupied by the m element eigenvectors. The system (1.1.1) will merely be said to be hyperbolic in t-direction if the eigenvectors span the space E^m but the eigenvalues, although all real, are not all distinct (Jeffery (1976), Courant and Friedrichs (1999)).

1.2 Ideal and Non-Ideal Gases

The equation of state of an ideal gas is written as $PV = n\Re T$, where n is the number of molecules of the gas, $\Re$ is the gas constant, T is the absolute temperature, P is the pressure and V is the volume of the gas. It is a good description of most gases in the low density regime, where on average molecules are far apart.

The equation of real gases is $\lim\limits_{P\to 0} PV/\Re T = 1$ with compressibility factor Z defined as

$$Z(P,T) = PV/\Re T.$$

It may be remarked that the deviation from unity indicates the degree of departure from ideal behavior.

In theoretical derivations of the ideal gas law it is necessary to make two assumptions i.e. the gas molecules are very small (they have no volume) and the molecules are non-interacting.

However, if the temperature of the gas is very high and density is too low the assumption that the gas is ideal is no longer valid. The popular alternative to the ideal gas is a simplified van der Waals model. Dutch Physicist van der Waals derived an equation of state without the assumptions of ideal gas, which is known as the van der Waals equation of state, written as

$$\left(P + \frac{a}{V^2}\right)(V - b) = \Re T,$$

where a and b are constants and determined experimentally. Since the specific volume of a gas is reciprocal of the density of the gas. So, van der Waals equation of state in term of density ρ may be given as

$$\left(P + a\rho^2\right)(1 - b\rho) = \rho \Re T,$$

The van der Waals model is very near to the behavior of real gases for wide range of temperature and pressure. In case of compressible flows, the pressure of gas is taken to be very high consequently the term $a\varrho^2$ is very small compared to the gas pressure p, The study of shock waves through a non-ideal gas is of great technical interest in many industrial applications such as chemical, nuclear and aerospace.

An understanding of the properties of the shock waves both in the near-field and the far-field is useful with regard to the characteristics such as shock strength, shock overpressure, shock speed, and impulse. Roberts and Wu (1996) examined the similarity solutions and determined the conditions for the stability of strong spherical imploding shock wave for both ideal and van der Waals gases. They found that when the van der Waals excluded volume is sufficiently large, a new type of solution is found and the shock may be linearly stable. Wu and Roberts (1996) investigated the problem of structure and stability of strong spherical shock waves whereas Somogyi and Roberts (2007) analyzed numerical stability of an imploding spherical shock wave in van der Waals gas. Arora and Siddiqui (2013) investigated the behavior of weak shocks in a non-ideal gas. Vishwakarma and Nath (2009) have used the similarity method to discuss the propagation of shock wave in non-ideal dusty gas.

Jena (2009) and Oliveri and Speciale (2002) obtained the solution of weak shock wave in different material media by using the Lie group of transformations. Arora et al. (2012) investigated the behaviour of strong shock wave in non-ideal gas using similarity transformation technique. Recently Bira et al. (2018) have studied the Collision of characteristic shock with weak discontinuity in non-ideal magnetogas-dynamics.

1.3 Dusty Gas

In case of a dusty gas we consider that solid particles present in the gas are of uniform size and uniformly distributed in the mixture and solid particles are spherical with same mass and same radius. Also, volume fraction of small solid particles does not exceed 5% of the total volume of the gas Pai(1977). In case of the propagation of shock wave the velocity of the mixture is very high so, the dust particles present in the mixture are assumed to be a pseudo fluid. Here it is assumed that the dust particles are small solid sphere of identical mass m_{sp}, radius r_{sp} and specific heat c_{sp}. Consider an element of dusty gas with mass $M = M_{sp} + M_g$ and volume $V = V_{sp} + V_g$, where the subscripts g and sp stand for gas and dust particles respectively unless it is specified. The volume of the solid particles in the mixture may be given as

$$V_{sp} = n_{sp} V \tau_{sp},$$

where $\tau_{sp} = \frac{4}{3}\Pi r_{sp}^3$, is the volume of solid particles in dusty gas and n_{sp} is the number of solid dust particles per unit volume of dusty gas in an element. The mass of solid dust particles in the dusty gas may be given as

$$M_{sp} = n_{sp} m_{sp} V.$$

Also, the species density of solid particles may be given as

$$\rho_{sp} = \frac{M_{sp}}{V_{sp}} = \frac{m_{sp}}{\tau_{sp}}.$$

The specific heat of dusty gas at constant pressure is given by

$$c_{pd} = k_p c_{sp} + (1 - k_p)\, c_p,$$

where c_p and c_{sp} stands for specific heat of gas and solid particle respectively. If $c_{\upsilon d}$ denotes the specific heat of dusty gas at constant volume, the ratio of specific heats for dusty gas is given by (Pai 1977) as

$$\Gamma = \frac{c_{pd}}{c_{\nu d}} = \frac{\gamma + \delta\beta}{1 + \delta\beta},$$

where $\delta = k_p / (1 - k_p)$, $\beta = c_{sp}/c_p$, $\gamma = c_p/c_\nu$ with c_ν as specific heat of gas at constant volume. The equation of state for adiabatic dusty gas flow is given by Chadha and Jena (2014)

$$p = \frac{(1 - k_p)}{(1 - Z)} \rho \Re T,$$

where Z is the volume fraction of dusty gas and $\Re$ is the gas constant. The relation between Z and k_p is also given by

$$Z = \frac{k_p}{(1 - k_p)\,\Omega + k_p},$$

where $\Omega = \rho_{sp}/\rho_g$ is the ratio of the density of the solid particles to the species density of the gas.

1.4 Riemann Problem

A Riemann problem, named after Bernhard Riemann, is an initial value problem for conservation equations under a very particular initial condition that consists of a piecewise constant initial data with a single jump discontinuity.

To understand the solution of Riemann problem let us consider the system of partial differential equations in conservative form

$$U_t + F(U)_x = 0, \tag{1.4.1}$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_m \end{bmatrix}, \qquad F(U) = \begin{bmatrix} f_1 \\ f_2 \\ . \\ . \\ . \\ f_m \end{bmatrix}.$$

U is called the vector of conserved variables, F is the vector of fluxes and each of its components f_i is a function of the components u_j of U. The system is hyperbolic if the Jacobian matrix

$$A(U) = \frac{\partial F}{\partial U}, \tag{1.4.2}$$

has real eigenvalues $\lambda_i(U)$ and a complete set of linearly independent eigenvectors $k_i(U)$, $i = 1, 2, ..., m$. Moreover, if all eigenvalues $\lambda_i(U)$ are different then system is said to be strictly hyperbolic. Riemann problem for the system (1) is the initial value problem with initial data

$$U(x,0) = \begin{cases} U_l = (u_1^l, u_2^l, ...u_m^l) & \text{if } x < x_0 \\ U_r = (u_1^r, u_2^r, ...u_m^r) & \text{if } x > x_0 \end{cases}, \tag{1.4.3}$$

Let us take Euler equations for one dimensional compressible flow

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0, \tag{1.4.4}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(p + (\rho u)^2\right) = 0, \tag{1.4.5}$$

$$\frac{\partial}{\partial t}\left(\rho\left(\frac{1}{2}u^2 + e\right)\right) + \frac{\partial}{\partial x}\left(\rho u\left(\frac{1}{2}u^2 + e\right) + p\rho\right) = 0, \tag{1.4.6}$$

where ρ, u, p and $e = \dfrac{p}{(\gamma - 1)\rho}$ are density, flow velocity pressure and specific internal energy of the perfect gas respectively and t, x stand for time and spatial coordinates respectively. The equations (1.4.4 - 1.4.6) may be written as

$$U_t + F(U)_x = 0 \tag{1.4.7}$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho\left(\frac{1}{2}u^2 + e\right) \end{bmatrix},$$

and

$$F = \begin{bmatrix} \rho u \\ p + (\rho u)^2 \\ \rho u\left(\frac{1}{2}u^2 + e\right) + p\rho \end{bmatrix},$$

$$U(x,0) = \begin{cases} U_l = (\rho_l, u_l, p_l) & \text{if } x < x_0 \\ U_r = (\rho_r, u_r, p_r) & \text{if } x > x_0 \end{cases}, \tag{1.4.8}$$

where U_l and U_R are respectively left and right constant states. The equation (1.4.7) together with equation (1.4.8) is known as Riemann problem for Eular equation. The solution to the Riemann problem depends on the physical variables of the left and right states and is always similar, namely the values of the quantities are all constant on any ray issuing from the initial position of the jump. Moreover, the solution of the Riemann problem is composed of three waves, with always a contact discontinuity as the middle one while the other two are either a rarefaction or shock wave or one is rarefaction wave and remaining other is shock wave, while for isentropic and isothermal Euler equations the solution composed of only two waves, shock wave and rarefaction wave. Each of the waves might be of zero intensity. Finally, when both left and right constant states are connected with rarefaction waves, the formation of vacuum region may occur between the two parts of the gas receding from each other. The study of the solutions to Riemann problem is therefore very important for understanding the nonlinear waves which propagate within the compressible fluids. The contact discontinuity separates two states that can be indicated by (ρ_l^*, u^*, p^*) and (ρ_r^*, u^*, p^*), since velocity and pressure are equal on the two sides and only density is discontinuous as shown in the fig.1.1. The solution of the Riemann problem consists of therefore in determing the values ρ_l^*, ρ_r^*, u^* and p^* and at the same time finding the the type of waves, shock or rarefaction, generated at the left and right of the contact discontinuity. Smoller (2009) developed an analytical approach for solving the Riemann problem and described the iteration of shock wave for gasdynamics. Toro (1994) has discussed some modern shock capturing numerical methods for solving linear and non-linear hyperbolic conservation laws, with smooth

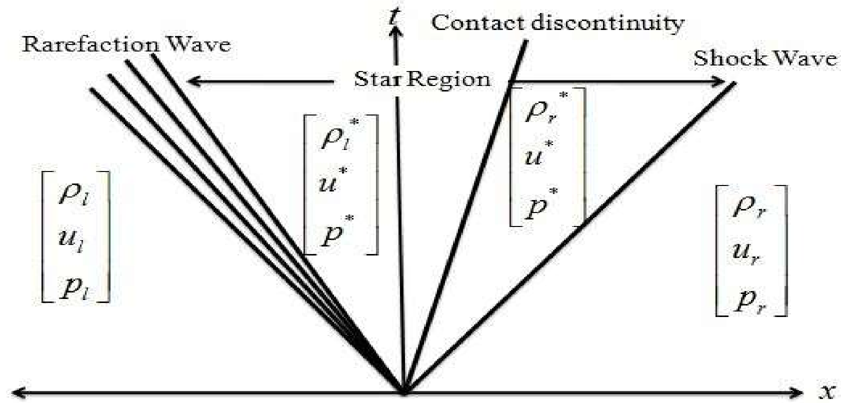

FIGURE 1.1: Typical Solution of Riemann Problem

and discontinuous solutions, in multidimensional geometry. All methods studied by him are illustrated through practical numerical examples, numerical solutions are compared with exact solutions and in some problems with reliable experimental data. LeVeque (2002) has discussed some powerful numerical techniques for approximating their solution, for both linear and non-linear conservation laws. Lax and Wenroff (1964) derived the difference scheme for solving the conservation laws and also, presented the detailed analysis of accuracy and stability of difference scheme. The numerical techniques derived were used at various levels of approximation for the solution of the Riemann problem. Sharma and Raja Shekhar (2010) studied the solution of Riemann problem for isentropic magnetogasdynamics and have discussed all possible elementary wave interactions.

1.5 Review of Literature

When there is a relative motion between a body and a fluid, the disturbance (if infinitely small) caused by the body is propagated through the fluid with the speed of sound. The speed of sound in such cases is the speed with which rarefaction waves or compression waves of very small amplitude propagate. When the compressions in the flow are of finite amplitude, there usually occurs a discontinuous rise of pressure leading to a shock wave. In addition to discontinuities in pressure, there occur discontinuous increase in temperature, density, entropy and other fluid properties. If initially, the fluid is still and shock wave is moving then, after the passage of the shock, the fluid will move in the direction of the shock. Gas compressions, which have finite amplitude, travel faster than the speed of sound, as in the case of strong explosions. The instance of formation of shock waves was largely studied by various authors. Macpherson (1971) studied the formation of a shock wave in dense argon by applying a molecular-dynamic approach. Shifrin (1970) studied the formation of a shock wave in a plane flow of a perfect gas. Saldatov (1970) studied a symmetric two-way traffic flow and determined the instance of formation of a shock wave by using the Riemann method. During the past decades many authors have studied the problem of growth and decay of shock waves propagating in a variety of media. Ardavan-Rhad (1970) studied the propagation of plane shock wave into a non-isentropic, non-viscous and non-heat conducting media. Flack and Wittig (1971) presented the general solution for the case of normal shock wave moving through a medium in which all flow properties vary arbitrarily. By applying numerical methods Sod (1977) studied the propagation of one-dimensional shock wave with cylindrical and spherical symmetry. Chen and Gurtin (1970) and Colemann and Gurtin (1967) studied the growth and decay of shock waves with internal state variables. Chen (1971) studied the propagation of shock waves in elastic nonconductors. Thermodynamic influences on the propagation of shock waves have been studied by Chen (1973). Bowen and Chen (1974) studied the same problem in the ideal mixture with several temperature layers. One of the interesting properties of the shock waves is the problem of determining the differential effects of the shock fronts on the rear flow field. To this problem Thomas (1947) developed a tensorial approach which was further extended by Kanwal (1958) for three dimensional shocks in stationary, pseudo-stationary and unsteady flows of non-conducting gases. The problem of vorticity generation by a shock has also been solved by several authors like Truesdell (1952), Hayes (1957), Kanwal (1960) and Ram (1978).

A considerable amount of work has also been drawn on the shock structure. A lot of work on the shock structure was carried out by Kuznetsov (1979), Goldman and Sirovich (1969). Wave fronts which are concave in the direction of propagation exhibit different kinds of behaviour depending on the strength of the wave-front. Generally, wave front propagates normal to itself and therefore has a tendency to converge. The shocks of weak strength are called weak shocks. Focusing of weak shock is an important problem. This problem of focusing of weak shocks was studied by Wanner et al. (1972). Observers of atomic explosion are also known to have seen shock waves of strong strength, called blast wave. Ram (1981) provided a closed form self similar solution to a MHD flow disturbed by propagating blast waves. Further, in the final stages of collapse, the shock becomes very strong and the pressure ahead is neglected in comparison to the pressure behind the shock wave. This leads to similarity formulation of the problem. In the problem, the ratio of distance to a particular power of time is known as similarity exponent, which is not known a priori. Several numerical and analytical methods have been developed for the determination of similarity exponent of the problem e.g. Taylor (1950), Butler (1954), Sedov (1959), Stanyukovich (1960), Welsh (1967), Zel'dovich & Raiser (1967), Lazarus (1981), Chisnell (1998). Zen'kevich and Stepanov (2007) provided analytical solution of self similar equations in Lagrangian mass coordinate, describing the dynamics of the explosion and the propagation of a strong shock wave.

Branover (1978) analyzed the magnetohydrodynamic flow in ducts. Taylor and

Cargill (2001) studied the problem of self-similar expansion waves in magnetogasdynamics flows. Lock and Mestel (2008) discussed the possibility of self-similar, imploding, finite annular z-pinch solutions to the equations of ideal MGD (magnetogasdynamics for a perfect gas at infinite magnetic Reynolds number).

The study of fluid flow containing solid particles has been the interest of many engineering and scientific research such as fluidized beds, centrifugal separation of particular matter from fluids, many chemical processes, solid particle motion in rocket exhaust and dust flow in geophysical and astrophysical problems. The physical situations involved in the process is very complicated and it is not possible to develop an analytical approach for these problems as it depends on various factors such as Reynold's number, Mach number, shape and size of the particles, interaction between particles etc.

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