Chapter 4

Quasi-projective synchronization of memristor-based complex valued recurrent neural network with time-varying delay and mismatched parameters

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The present chapter is concerned with the quasi-projective synchronization of memristor-based complex valued recurrent neural network in the presence of time-varying delay and mismatched parameters. We have considered two non-identical memristor-based complex valued recurrent neural networks with time-varying delays as master and response systems. In main results, we have obtained two results on quasi-projective synchronization of memristor-based CVRNNs with time-varying delay and mismatched parameters. One is based on matrix measure approach and second one is based on Lyapunov stability theory. The optimal synchronization error bound has been discussed by using fundamental calculus. Two numerical examples to verify the obtained results are given at the end of the chapter.

4.1 Preliminaries and Model Formulation

Let us assume the memristor-based CVRNN model with time-varying delay as

$$\dot{\gamma}_l(t) = -\, r_l\gamma_l(t) + \sum_{m=1}^{n} p_{lm}(\gamma_m(t))f_m(\gamma_m(t)) + \sum_{m=1}^{n} q_{lm}(\gamma_m(t))f_m(\gamma_m(t-\tau(t))) + I_l(t), t \geq 0, \quad l = 1, 2, ...n. \tag{4.1.1}$$

which can be defined in another form as

$$\dot{\gamma}(t) = -R\gamma(t) + P(\gamma(t))f(\gamma(t)) + Q(\gamma(t))f(\gamma(t-\tau(t))) + I(t),$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t), ..., \gamma_n(t))^T \in C^n$ represents state of n-th neurons at time t, $R = diag(r_1, r_2, ..., r_n) \in R^{n\times n}$, $r_k > 0$ represents n-th self-inhibition. $P(\gamma(t)) = (p_{lm}(\gamma_m(t)))_{n\times n}$ and $Q(\gamma(t)) = (q_{lm}(\gamma_m(t)))_{n\times n}$, where $P(\gamma(t)), Q(\gamma(t)) \in C^{n\times n}$, denote complex-valued connection weight matrix without and with time-varying delay, respectively. $I(t) = (I_1(t), I_2(t), ..., I_n(t))^T$ denotes external input vector. $f(\gamma(t)) = (f_1(\gamma_1(t)), f_2(\gamma_2(t)), ..., f_n(\gamma_n(t)))^T : C^n \to C^n$ and $g(\gamma(t-\tau(t))) = (g_1(\gamma_1(t-\tau(t))), g_2(\gamma_2(t-\tau(t))), ..., g_n(\gamma_n(t-\tau(t))))^T : C^n \to C^n$ denote complex-valued activation functions without and with time-varying delay, respectively, where $\tau(t)$ represents time-varying delay, which satisfies $0 \leq \tau(t) \leq \tau$, and $t - \tau(t) \in T$, where $\tau > 0$ is a known constant.

The initial condition of the system (4.1.1) is given as $\gamma(s) = \phi(s), s \in [t_0 - \tau, t_0]$, where $\phi(s) = (\phi_1(s), \phi_2(s), .., \phi_n(s)) \in C^n$.

Assumption 3. *Suppose $p(t)$ can be separated into real and imaginary parts as*

$$p(t) = p^R(t) + ip^I(t),$$

where $p^R(t)$ *and* $p^I(t)$ *satisfy the constraints, and* $p^{\underline{R}}, p^{\overline{R}}, p^{\underline{I}}$ *and* $p^{\overline{I}}$ *are real constants, with* $p^{\underline{R}} < p^{\overline{R}}$*, and* $p^{\underline{I}} < p^{\overline{I}}$*.*

Assumption 4. *[106] Suppose* $w = u + iv$*, where* $u, v \in R$*.* $f_l(w)$ *and* $g_l(w)$ *are described by* $f_l(w) = f_l^R(u) + if_l^I(v)$ *and* $g_l(w) = g_l^R(u) + ig_l^I(v)$*, where* $l = 1, 2, ..., n$ *and* $f_l^R(.), f_l^I(.), g_l^R(.), g_l^I(.) : R \to R$ *satisfy the following Lipschitz conditions:*
$\|f_l^R(\nu) - f_l^R(\eta)\|_w \leq r_l\|\nu - \eta\|_w,$
$\|f_l^I(\nu) - f_l^I(\eta)\|_w \leq s_l\|\nu - \eta\|_w,$
$\|g_l^R(\nu) - g_l^R(\eta)\|_w \leq m_l\|\nu - \eta\|_w,$
$\|g_l^I(\nu) - g_l^I(\eta)\|_w \leq n_l\|\nu - \eta\|_w,$
where $r_l, s_l, m_l,$ *and* n_l*, are the Lipschitz constants,* ν *and* $\eta \in R^n$*.*

Furthermore, based on the pinched hysteretic property for memristor, the connection weights $p_{lm}(w_m(t))$, $q_{lm}(w_m(t))$ become functions of $w_l, w_m \in C$, satisfying

$$Z_{lm}(w_l - w_m) = \begin{cases} \underline{Z}_{lm}, & \dot{w_l} - \dot{w_m} > 0, \\ \overline{Z}_{lm}, & \dot{w_l} - \dot{w_m} < 0, \\ lim_{s\to t^-} Z_{lm}(w_l - w_m), & \dot{w_l} - \dot{w_m} = 0, \end{cases}$$

for some constants $\underline{Z}_{lm}$ and $\overline{Z}_{lm}$, with $\underline{Z}_{lm} \neq \overline{Z}_{lm}$. Besides, for two constants, $\underline{p} < p(t) < \overline{p}$.

Assumption 5. *[106] There exist positive constants* $r_l, s_l, m_l,$ *and* k_l *such that*

$$|f_l^R(\nu) - f_l^R(\eta)| \leq r_l^1|\nu - \eta|,$$
$$|f_l^I(\nu) - f_l^I(\eta)| \leq s_l^1|\nu - \eta|,$$
$$|g_l^R(\nu) - g_l^R(\eta)| \leq m_l^1|\nu - \eta|,$$
$$|g_l^I(\nu) - g_l^I(\eta)| \leq k_l^1|\nu - \eta|,$$

for all $\nu, \eta \in R$, *with* $\nu \neq \eta$, *where* $l = 1, 2, ..., n$.

Definition 4.1.1. Suppose that the set $F(\mu) \subset R^n$ is non-empty for any $\mu \in R^n$, then μ to $F(\mu)$ is said to be a set-valued map.

Definition 4.1.2. Suppose $\frac{d\mu}{dt} = f(\mu)$, $\mu \in C^n$, be a right-hand discontinuous system. Let,

$$\phi(\mu) = \bigcap_{\delta > 0} \bigcap_{\nu(F)=0} \overline{co}[f(B(x,\delta))\ E]$$

be a set-valued map, where $\overline{co}[\Omega]$ denotes closure of Ω, which is a convex hull set and $B(x,\delta) = \{y : \|y - x\| \leq \delta\}$. $\nu(F)$ denotes Lebesgue measure of the set F. Then, a solution of this system is an absolutely continuous function $\mu(t), t \in [0,T]$ with the initial condition $\mu(0) = \mu_0$. This solution also satisfies the following so-called differential inclusion:
$\frac{d\mu}{dt} \in \phi(\mu)$, for $t \in [0,T]$.

In view of Assumption 4, suppose $\gamma(t) = \upsilon(t) + i\varrho(t)$, where $\upsilon(t), \varrho(t) \in R^n$, then from equation (4.1.1), we have

$$\begin{aligned}
\dot{\upsilon}_l(t) =& - r_l \upsilon_l(t) + \sum_{m=1}^{n} co(p_{lm}^{\underline{R}}, p_{lm}^{\overline{R}}) f_m^R(\gamma_m(t)) - \sum_{m=1}^{n} co(p_{lm}^{\underline{I}}, p_{lm}^{\overline{I}}) f_m^I(\gamma_m(t)) \\
&+ \sum_{m=1}^{n} co(q_{lm}^{\underline{R}}, q_{lm}^{\overline{R}}) g_m^R(\gamma_m(t - \tau(t))) - \sum_{m=1}^{n} co(q_{lm}^{\underline{I}}, q_{lm}^{\overline{I}}) g_m^I(\gamma_m(t - \tau(t))) + I_l^R(t), \\
\dot{\varrho}_l(t) =& - r_l \varrho_l(t) + \sum_{m=1}^{n} co(p_{lm}^{\underline{I}}, p_{lm}^{\overline{I}}) f_m^R(\gamma_m(t)) + \sum_{m=1}^{n} co(p_{lm}^{\underline{R}}, p_{lm}^{\overline{R}}) f_m^I(\gamma_m(t)) \\
&+ \sum_{m=1}^{n} co(q_{lm}^{\underline{I}}, q_{lm}^{\overline{I}}) g_m^R(\gamma_m(t - \tau(t))) + \sum_{m=1}^{n} co(q_{lm}^{\underline{R}}, q_{lm}^{\overline{R}}) g_m^I(\gamma_m(t - \tau(t))) + I_l^I(t).
\end{aligned} \tag{4.1.2}$$

The initial conditions of equation (4.1.2) will be

$$\begin{cases} \upsilon(s) = \phi^R(s), \\ \varrho(s) = \phi^I(s), -\tau \leq s \leq 0, \end{cases}$$

where

$$co(\underline{p^R_{lm}}, \overline{p^R_{lm}}) = \begin{cases} \hat{p}^R_{lm}, & \dot{\upsilon}_l(t) > \dot{\upsilon}_{l+1}(t) \\ [\underline{p^R}_{lm}, \overline{p^R}_{lm}], & \dot{\upsilon}_l(t) = \dot{\upsilon}_{l+1}(t) \\ \check{p}^R_{lm}, & \dot{\upsilon}_l(t) < \dot{\upsilon}_{l+1}(t) \end{cases} ; co(\underline{p^I_{lm}}, \overline{p^I_{lm}}) = \begin{cases} \hat{p}^I_{lm}, & \dot{\upsilon}_l(t) > \dot{\upsilon}_{l+1}(t) \\ [\underline{p^I}_{lm}, \overline{p^I}_{lm}], & \dot{\upsilon}_l(t) = \dot{\upsilon}_{l+1}(t) \\ \check{p}^I_{lm}, & \dot{\upsilon}_l(t) < \dot{\upsilon}_{l+1}(t), \end{cases}$$

$$co(\underline{q^R_{lm}}, \overline{q^R_{lm}}) = \begin{cases} \hat{q}^R_{lm}, & \dot{\varrho}_l(t) > \dot{\varrho}_{l+1}(t) \\ [\underline{q^R}_{lm}, \overline{q^R}_{lm}], & \dot{\varrho}_l(t) = \dot{\varrho}_{l+1}(t) \\ \check{q}^R_{lm}, & \dot{\varrho}_l(t) < \dot{\varrho}_{l+1}(t) \end{cases} ; co(\underline{q^I_{lm}}, \overline{q^I_{lm}}) = \begin{cases} \hat{q}^I_{lm}, & \dot{\varrho}_l(t) > \dot{\varrho}_{l+1}(t) \\ [\underline{q^I}_{lm}, \overline{q^I}_{lm}], & \dot{\varrho}_l(t) = \dot{\varrho}_{l+1}(t) \\ \check{q}^I_{lm}, & \dot{\varrho}_l(t) < \dot{\varrho}_{l+1}(t), \end{cases}$$

with $\overline{p^R}_{lm} = max\{\hat{p}_{lm}, \check{p}_{lm}\}$, $\underline{p^R}_{lm} = min\{\hat{p}_{lm}, \check{p}_{lm}\}$, $\overline{q^R}_{lm} = max\{\hat{q}_{lm}, \check{q}_{lm}\}$, $\underline{q^R}_{lm} = min\{\hat{q}_{lm}, \check{q}_{lm}\}$.

The real part $\phi^R(s)$ and the imaginary part $\phi^I(s)$ of $\phi(s)$ are continuous in $[t_0 - \tau, t_0]$ with the norm defined as $\|\phi^R(s)\|_l = sup_{t_0-\tau \leq s \leq t_0} \|\phi^R(s)\|_l$ and $\|\phi^I\|_l = sup_{t_0-\tau \leq s \leq t_0} \|\phi^I(s)\|_l$.

Further, the system (4.1.2) can be written as

$$\begin{aligned} \dot{\upsilon}(t) \in & - R\upsilon(t) + co\Big(\underline{P^R}, \overline{P^R}\Big) f^R(\gamma(t)) - co\Big(\underline{P^I}, \overline{P^I}\Big) f^I(\gamma(t)) \\ & + co\Big(\underline{Q^R}, \overline{Q^R}\Big) g^R(\gamma(t - \tau(t))) - co\Big(\underline{Q^I}, \overline{Q^I}\Big) g^I(\gamma(t - \tau(t))) + I^R(t), \\ \dot{\varrho}(t) \in & - R\varrho(t) + co\Big(\underline{P^I}, \overline{P^I}\Big) f^R(\gamma(t)) + co\Big(\underline{P^R}, \overline{P^R}\Big) f^I(\gamma(t)) \\ & + co\Big(\underline{Q^I}, \overline{Q^I}\Big) g^R(\gamma(t - \tau(t))) + co\Big(\underline{Q^R}, \overline{Q^R}\Big) g^I(\gamma(t - \tau(t))) + I^I(t). \end{aligned} \quad (4.1.3)$$

For some constant matrices $P^R \in co\Big(P^{\underline{R}}, P^{\overline{R}}\Big)$, $P^I \in co\Big(P^{\underline{I}}, P^{\overline{I}}\Big)$, $Q^R \in co\Big(Q^{\underline{R}}, Q^{\overline{R}}\Big)$, and $Q^I \in co\Big(Q^{\underline{I}}, Q^{\overline{I}}\Big)$, the system (4.1.3) can be written as

$$\begin{aligned}\dot{\upsilon}(t) =& -R\upsilon(t) + P^R f^R(\gamma(t)) - P^I f^I(\gamma(t)) + Q^R g^R(\gamma(t-\tau(t)))\\ &- Q^I g^I(\gamma(t-\tau(t))) + I^R(t),\\ \dot{\varrho}(t) =& -R\varrho(t) + P^I f^R(\gamma(t)) + P^R f^I(\gamma(t)) + Q^I g^R(\gamma(t-\tau(t)))\\ &+ Q^R g^I(\gamma(t-\tau(t))) + I^I(t).\end{aligned} \tag{4.1.4}$$

Let us construct the corresponding response system as

$$\begin{aligned}\dot{\tilde{\gamma}}_l(t) =& -r'_l\tilde{\gamma}_l(t) + \sum_{m=1}^{n} p'_{lm}(\tilde{\gamma}_m(t))f_m(\tilde{\gamma}_m(t)) + \sum_{m=1}^{n} q'_{lm}(\tilde{\gamma}_m(t))f_m(\tilde{\gamma}_m(t-\tau(t)))\\ &+ I'_l(t) + \Xi_l(t),\end{aligned} \tag{4.1.5}$$

where $\tilde{\gamma}(t) = (\tilde{\gamma}_1(t), \tilde{\gamma}_2(t), ..., \tilde{\gamma}_n(t))^T \in C^n$ represents the state of nth neurons at time t and $\Xi_l(t) = (\Xi_1(t), \Xi_2(t), .., \Xi_n(t))^T \in C^n$ denotes the coupling control.

The initial condition of the response system (4.1.5) is
$\tilde{\gamma}(s) = \tilde{\phi}(s), s \in [t_0 - \tau, t_0]$,

where $\tilde{\phi}(s) = (\tilde{\phi}_1(s), \tilde{\phi}_2(s), ..., \tilde{\phi}_n(s)) \in C^n$, $\tilde{\phi}^R(s)$ and $\tilde{\phi}^I(s)$ denote continuous functions on interval $[t_0 - \tau, t_0]$.

By real decomposition method, the equation (4.1.5), is reduced to

$$\begin{aligned}\dot{\tilde{\upsilon}}_l(t) =& -r'_l\tilde{\upsilon}_l(t) + \sum_{m=1}^{n} co(p'^{\underline{R}}_{lm}, p'^{\overline{R}}_{lm})f^R_m(\tilde{\gamma}_m(t)) - \sum_{m=1}^{n} co(p'^{\underline{I}}_{lm}, p'^{\overline{I}}_{lm})f^I_m(\tilde{\gamma}_m(t))\\ &+ \sum_{m=1}^{n} co(q'^{\underline{R}}_{lm}, q'^{\overline{R}}_{lm})g^R_m(\tilde{\gamma}_m(t-\tau(t))) - \sum_{m=1}^{n} co(q'^{\underline{I}}_{lm}, q'^{\overline{I}}_{lm})g^I_m(\tilde{\gamma}_m(t-\tau(t)))\\ &+ I'^R_p(t) + \Xi^R_p(t),\end{aligned}$$

$$\dot{\tilde{\varrho}}_l(t) = - r'_l \tilde{\varrho}_l(t) + \sum_{m=1}^{n} co(p'^{\underline{I}}_{lm}, p'^{\overline{I}}_{lm}) f^R_m(\tilde{\gamma}_m(t)) + \sum_{m=1}^{n} co(p'^{\underline{R}}_{lm}, p'^{\overline{R}}_{lm}) f^I_m(\tilde{\gamma}_m(t))$$
$$+ \sum_{m=1}^{n} co(q'^{\underline{I}}_{lm}, q'^{\overline{I}}_{lm}) g^R_m(\tilde{\gamma}_m(t-\tau(t))) + \sum_{m=1}^{n} co(q'^{\underline{R}}_{lm}, q'^{\overline{R}}_{lm}) g^I_m(\tilde{\gamma}_m(t-\tau(t)))$$
$$+ I'^I_l(t) + \Xi^I_l(t), \tag{4.1.6}$$

and the corresponding initial conditions are

$$\begin{cases} \tilde{\upsilon}(s) = \tilde{\phi}^R(s), \\ \tilde{\varrho}(s) = \tilde{\phi}^I(s), -\tau \leq s \leq 0, \end{cases}$$

where

$$co(p'^{\underline{R}}_{lm}, p'^{\overline{R}}_{lm}) = \begin{cases} \hat{p}'^R_{lm}, & \dot{\tilde{\upsilon}}_l(t) > \dot{\tilde{\upsilon}}_{l+1}(t) \\ [p'^{\underline{R}_{lm}}, p'^{\overline{R}_{lm}}], & \dot{\tilde{\upsilon}}_l(t) = \dot{\tilde{\upsilon}}_{l+1}(t) \\ \check{p}'^R_{lm}, & \dot{\tilde{\upsilon}}_l(t) < \dot{\tilde{\upsilon}}_{l+1}(t) \end{cases} ; co(p'^{\underline{I}}_{lm}, p'^{\overline{I}}_{lm}) = \begin{cases} \hat{p}'^I_{lm}, & \dot{\tilde{\upsilon}}_l(t) > \dot{\tilde{\upsilon}}_{l+1}(t) \\ [p'^{\underline{I}_{lm}}, p'^{\overline{I}_{lm}}], & \dot{\tilde{\upsilon}}_l(t) = \dot{\tilde{\upsilon}}_{l+1}(t) \\ \check{p}'^I_{lm}, & \dot{\tilde{\upsilon}}_l(t) < \dot{\tilde{\upsilon}}_{l+1}(t), \end{cases}$$

$$co(q'^{\underline{R}}_{lm}, q'^{\overline{R}}_{lm}) = \begin{cases} \hat{q}'^R_{lm}, & \dot{\tilde{\varrho}}_l(t) > \dot{\tilde{\varrho}}_{l+1}(t) \\ [q'^{\underline{R}_{lm}}, q'^{\overline{R}_{pq}}], & \dot{\tilde{\varrho}}_l(t) = \dot{\tilde{\varrho}}_{l+1}(t) \\ \check{q}'^R_{lm}, & \dot{\tilde{\varrho}}_l(t) < \dot{\tilde{\varrho}}_{l+1}(t) \end{cases} ; co(q'^{\underline{I}_{lm}}, q'^{\overline{I}}_{lm}) = \begin{cases} \hat{q}'^I_{lm}, & \dot{\tilde{\varrho}}_l(t) > \dot{\tilde{\varrho}}_{l+1}(t) \\ [q'^{\underline{I}_{lm}}, q'^{\overline{I}_{lm}}], & \dot{\tilde{\varrho}}_l(t) = \dot{\tilde{\varrho}}_{l+1}(t) \\ \check{q}'^I_{lm}, & \dot{\tilde{\varrho}}_l(t) < \dot{\tilde{\varrho}}_{l+1}(t), \end{cases}$$

with $p'^{\overline{R}_{lm}} = max\{\hat{p}'_{lm}, \check{p}'_{lm}\}$, $p'^{\underline{R}_{lm}} = min\{\hat{p}'_{lm}, \check{p}'_{lm}\}$, $q'^{\overline{R}_{lm}} = max\{\hat{q}'_{lm}, \check{q}'_{lm}\}$, $q'^{\underline{R}_{lm}} = min\{\hat{q}'_{lm}, \check{q}'_{lm}\}$.

Further, the system (4.1.6) can be written as

$$\dot{\tilde{\upsilon}}(t) \in - R'\tilde{\upsilon}(t) + co\Big(P'^{\underline{R}}, P'^{\overline{R}}\Big) f^R(\tilde{\gamma}(t)) - co\Big(P'^{\underline{I}}, P'^{\overline{I}}\Big) f^I(\tilde{\gamma}(t))$$
$$+ co\Big(Q'^{\underline{R}}, Q'^{\overline{R}}\Big) g^R(\tilde{\gamma}(t-\tau(t))) - co\Big(Q'^{\underline{I}}, Q'^{\overline{I}}\Big) g^I(\tilde{\gamma}(t-\tau(t))) + I'^R(t) + \Xi^R(t),$$

$$\dot{\tilde{\varrho}}(t) \in -R'\tilde{\varrho}(t) + co\Big(P'^{\underline{I}}, P'^{\overline{I}}\Big) f^R(\tilde{\gamma}(t)) + co\Big(P'^{\underline{R}}, P'^{\overline{R}}\Big) f^I(\tilde{\gamma}(t))$$
$$+ co\Big(Q'^{\underline{I}}, Q'^{\overline{I}}\Big) g^R(\tilde{\gamma}(t-\tau(t))) + co\Big(Q'^{\underline{R}}, Q'^{\overline{R}}\Big) g^I(\tilde{\gamma}(t-\tau(t))) + I'^I(t) + \Xi^I(t). \tag{4.1.7}$$

For some constant matrices $P'^R \in co\Big(P^{\underline{R}}, P^{\overline{R}}\Big)$, $P'^I \in co\Big(P^{\underline{I}}, P^{\overline{I}}\Big)$, $Q'^R \in co\Big(Q^{\underline{R}}, Q^{\overline{R}}\Big)$, and $Q'^I \in co\Big(Q^{\underline{I}}, Q^{\overline{I}}\Big)$, the system (4.1.7) can be written as

$$\begin{aligned}
\dot{\tilde{\upsilon}}(t) =& -R'\tilde{\upsilon}(t) + P'^R f^R(\tilde{\gamma}(t)) - P'^I f^I(\tilde{\gamma}(t)) + Q'^R g^R(\tilde{\gamma}(t-\tau(t))) \\
& - Q'^I g^I(\tilde{\gamma}(t-\tau(t))) + I'^R(t) + \Xi^R(t), \\
\dot{\tilde{\varrho}}(t) =& -R'\tilde{\varrho}(t) + P'^I f^R(\tilde{\gamma}(t)) + P'^R f^I(\tilde{\gamma}(t)) + Q'^I g^R(\tilde{\gamma}(t-\tau(t))) \\
& + Q'^R g^I(\tilde{\gamma}(t-\tau(t))) + I'^I(t) + \Xi^I(t),
\end{aligned} \tag{4.1.8}$$

where $\Xi^R(t)$ and $\Xi^I(t)$ denote the control input vectors with control inputs are linear combinations of the difference between the state vectors of the drive-response systems (4.1.4) and (4.1.8), respectively.

Now, the control coupling vectors are taken as

$$\Xi^R(t) = \Omega\theta^R(t), \quad \Xi^I(t) = \Omega\theta^I(t), \tag{4.1.9}$$

where $\Xi^R(t) = [\Xi_1^R(t), \Xi_2^R(t), ..., \Xi_n^R(t)]^T, \Xi^I(t) = [\Xi_1^I(t), \Xi_2^I(t), ..., \Xi_n^I(t)]^T$, $\theta^R(t) = [\theta_1^R(t), \theta_2^R(t), ..., \theta_n^R(t)]^T, \theta^I(t) = [\theta_1^I(t), \theta_2^I(t), ..., \theta_n^I(t)]^T$ and $\Omega = \begin{pmatrix} \alpha_{11} \ldots \alpha_{1n} \\ \vdots \ddots \vdots \\ \alpha_{n1} \ldots \alpha_{nn} \end{pmatrix}$ represents the controller gain matrix.

Assumption 6. *Let us consider that the solution of the system is bounded for any initial function* $v(t) \in C([-\tau, 0],\ R^n)$*, i.e.,* $\exists$ *positive constant* $K \in R^+$ *and time instant* t_0 *s.t.* $\|v(t)\|_w \leq K, \quad \forall\ t \geq t_0$.

Let us define the projective errors as $\theta^R(t) = \widetilde{v}(t) - rv(t), \theta^I(t) = \widetilde{\varrho}(t) - r\varrho(t)$. From equations (4.1.4) and (4.1.8), we can describe the error systems as

$$
\begin{aligned}
\dot{\theta}^R(t) =& -R'\theta^R(t) + P'^R f^R(\theta(t)) - P'^I f^I(\theta(t)) + Q'^R g^R(\theta(t-\tau(t))) \\
& - Q'^I g^I(\theta(t-\tau(t))) + H(t) + \Omega\theta^R(t), \\
\dot{\theta}^I(t) =& -R'\theta^I(t) + P'^R f^I(\theta(t)) + P'^I f^R(\theta(t)) + Q'^R g^I(\theta(t-\tau(t))) \\
& + Q'^I g^R(\theta(t-\tau(t))) + H'(t) + \Omega\theta^I(t),
\end{aligned}
\tag{4.1.10}
$$

where $\theta^R(t) = (\theta_1^R(t), \theta_2^R(t)...\theta_n^R(t))^T \in R^n, \theta^I(t) = (\theta_1^I(t), \theta_2^I(t)...\theta_n^I(t))^T \in R^n$,
$f^R(\theta(t)) = f^R(\widetilde{\gamma}(t)) - f^R(\gamma(t)), f^I(\theta(t)) = f^I(\widetilde{\gamma}(t)) - f^I(\gamma(t))$,
$g^R(\theta(t-\tau(t))) = g^R(\widetilde{\gamma}(t-\tau(t))) - g^R(\gamma(t-\tau(t)))$,
$g^I(\theta(t-\tau(t))) = g^I(\widetilde{\gamma}(t-\tau(t))) - g^I(\gamma(t-\tau(t)))$,
and

$$
\begin{aligned}
H(t) =& -r(R'-R)v(t) + r(P'^R - P^R)f^R(\gamma(t)) - r(P'^I - P^I)f^I(\gamma(t)) \\
& + r(Q'^R - Q^R)g^R(\gamma(t-\tau(t))) - r(Q'^I - Q^I)g^I(\gamma(t-\tau(t))) + I'^R(t) - rI^R(t), \\
H'(t) =& -r(R'-R)\varrho(t) + r(P'^I - P^I)f^R(\gamma(t)) + r(P'^R - P^R)f^I(\gamma(t)) \\
& + r(Q'^I - Q^I)g^R(\gamma(t-\tau(t))) + r(Q'^R - Q^R)g^I(\gamma(t-\tau(t))) + I'^I(t) - rI^I(t).
\end{aligned}
$$

It is important to show that $H(t)$ and $H^{'}(t)$ are bounded before getting to the main results. Using Assumptions 4 and 6 with ω-norm, we get

$$\begin{aligned}
\|H(t)\|_\omega \leq & |r|\|R' - R\|_\omega\|\upsilon(t)\|_\omega + |r|\|P'^R - P^R\|_\omega\|f^R(\gamma(t))\|_\omega \\
& + |r|\|P'^I - P^I\|_\omega\|f^I(\gamma(t))\|_\omega + |r|\|Q'^R - Q^R\|_\omega\|g^R(\gamma(t-\tau(t)))\|_\omega \\
& + |r|\|Q'^I - Q^I\|_\omega\|g^I(\gamma(t-\tau(t)))\|_\omega + \|I'^R(t) - rI^R(t)\|_\omega \\
\leq & |r|\|R' - R\|_\omega h + |r|\|P'^R - P^R\|_\omega hL_f + |r|\|P'^I - P^I\|_\omega hL_{f'} \\
& + |r|\|Q'^R - Q^R\|_\omega hL_g + |r|\|Q'^I - Q^I\|_\omega hL_{g'}, \\
\|H(t)\|_\omega \leq & k_1, \qquad (4.1.11) \\
\|H'(t)\|_\omega \leq & |r|\|R' - R\|_\omega\|\varrho(t)\|_\omega + |r|\|P'^I - P^I\|_\omega\|f^R(\gamma(t))\|_\omega \\
& + |r|\|P'^R - P^R\|_\omega\|f^I(\gamma(t))\|_\omega + |r|\|Q'^I - Q^I\|_\omega\|g^R(\gamma(t-\tau(t)))\|_\omega \\
& + |r|\|Q'^R - Q^R\|_\omega\|g^I(\gamma(t-\tau(t)))\|_\omega + \|I'^I(t) - rI^I(t)\|_\omega \\
\leq & |r|\|R' - R\|_\omega h + |r|\|P'^I - P^I\|_\omega hL_f + |r|\|P'^R - P^R\|_\omega hL_{f'} \\
& + |r|\|Q'^I - Q^I\|_\omega hL_g + |r|\|Q'^R - Q^R\|_\omega hL_{g'}, \\
\|H(t)\|_\omega \leq & k_2. \qquad (4.1.12)
\end{aligned}$$

Let us consider

$z(t) = \Big((\theta^R(t))^T, (\theta^I(t))^T\Big)^T,$

$\widetilde{F}^R(z(t)) = ((f^R(\theta(t)))^T, (f^R(\theta(t)))^T)^T,$

$\widetilde{F}^I(z(t)) = ((f^I(\theta(t)))^T, (f^I(\theta(t)))^T)^T,$

$\widetilde{G}^R(z(t-\tau(t))) = ((g^R(\theta(t-\tau(t))))^T, (g^R(\theta(t-\tau(t))))^T)^T,$

$\widetilde{G}^I(z(t-\tau(t))) = ((g^I(\theta(t-\tau(t))))^T, (g^I(\theta(t-\tau(t))))^T)^T,$

$S(t) = (H(t), H'(t))^T,$

with $R_1 = \begin{pmatrix} R' & 0 \\ 0 & R' \end{pmatrix}$, $P_1 = \begin{pmatrix} P'^R & 0 \\ 0 & P'^I \end{pmatrix}$, $P_2 = \begin{pmatrix} -P'^I & 0 \\ 0 & P'^R \end{pmatrix}$,

$Q_1 = \begin{pmatrix} Q'^R & 0 \\ 0 & Q'^I \end{pmatrix}$, $Q_2 = \begin{pmatrix} -Q'^I & 0 \\ 0 & Q'^R \end{pmatrix}$,

Therefore, from equation (4.1.10), we can get

$$\begin{aligned}\dot{z}(t) = &- R_1 z(t) + P_1 \widetilde{F}^R(z(t)) + P_2 \widetilde{F}^I(z(t)) + Q_1 \widetilde{G}^R(z(t-\tau(t))) \\ &+ Q_2 \widetilde{G}^I(z(t-\tau(t))) + S(t) + \Omega z(t). \end{aligned} \tag{4.1.13}$$

4.2 Main results

For the convenience of the notations, let us consider

$P_{31} = \begin{pmatrix} \hat{P}'^R & 0 \\ 0 & \hat{P}'^I \end{pmatrix}$, $P_{32} = \begin{pmatrix} \check{P}'^R & 0 \\ 0 & \check{P}'^I \end{pmatrix}$, $P_{41} = \begin{pmatrix} -\hat{P}'^I & 0 \\ 0 & \hat{P}'^R \end{pmatrix}$,

$P_{42} = \begin{pmatrix} -\check{P}'^I & 0 \\ 0 & \check{P}'^R \end{pmatrix}$, $Q_{31} = \begin{pmatrix} \hat{Q}'^R & 0 \\ 0 & \hat{Q}'^I \end{pmatrix}$, $Q_{32} = \begin{pmatrix} \check{Q}'^R & 0 \\ 0 & \check{Q}'^I \end{pmatrix}$,

$$Q_{41}=\begin{pmatrix}-\hat{Q}'^{I} & 0\\ 0 & \hat{Q}'^{R}\end{pmatrix},\quad Q_{42}=\begin{pmatrix}-\check{Q}'^{I} & 0\\ 0 & \check{Q}'^{R}\end{pmatrix},$$

$\hat{P}'^{R}=(\hat{p}'^{R}_{lm})_{n\times n}$, $\check{P}'^{R}=(\check{p}'^{R}_{lm})_{n\times n}$, $\hat{P}'^{I}=(\hat{p}'^{I}_{lm})_{n\times n}$, $\check{P}'^{I}=(\check{p}'^{I}_{lm})_{n\times n}$, $\hat{Q}'^{R}=(\hat{q}'^{R}_{lm})_{n\times n}$, $\check{Q}'^{R}=(\check{q}'^{R}_{lm})_{n\times n}$, $\hat{Q}'^{I}=(\hat{q}'^{I}_{lm})_{n\times n}$, $\check{Q}'^{I}=(\check{q}'^{I}_{lm})_{n\times n}$,

so that, we are able to establish the following main result.

Theorem 4.1. *Under Assumption 4, if the controller gain matrix Ω satisfies the following inequality*

$$\begin{aligned}&0<max\Big\{(m_k+n_k)(\|Q_{31}\|_\omega+\|Q_{41}\|_\omega),(m_k+n_k)(\|Q_{32}\|_\omega+\|Q_{42}\|_\omega)\Big\}<\\&-max\Big\{\mu_\omega(-R_1+\Omega)+(r_k+s_k)(\|P_{31}\|_\omega+\|P_{41}\|_\omega),\mu_\omega(-R_1+\Omega)\\&+(r_k+s_k)(\|P_{32}\|_\omega+\|P_{42}\|_\omega)\Big\},\end{aligned}\tag{4.2.1}$$

then the drive-response systems (4.1.1) and (4.1.5) will be quasi-projective synchronized under the controllers (4.1.9) with norm $\omega=1,2,\infty$.

Proof. Let us construct the Lyapunov function as

$$V_1(z(t))=\|z(t)\|_\omega.$$

In view of the definition of upper-right dini-derivative of $V_1(z(t))$ and Taylor's formula, we get

$$\begin{aligned}D^+V_1(z(t))=&\overline{lim}_{\epsilon\to0^+}\frac{\|z(t+\epsilon)\|_\omega-\|z(t)\|_\omega}{\epsilon}\\=&\overline{lim}_{\epsilon\to0^+}\Big[\frac{\|z(t)+\epsilon\dot{z}(t)+\mathcal{O}(\epsilon)\|_\omega-\|z(t)\|_\omega}{\epsilon}\Big].\end{aligned}\tag{4.2.2}$$

Case 1. If $\dot{\theta}_1^R(t) \geq \dot{\theta}_2^R(t)$, $\dot{\theta}_1^I(t) \geq \dot{\theta}_2^I(t)$, then

$$\begin{aligned}
&\|z(t)+\epsilon\dot{z}(t)+\mathcal{O}(\epsilon)\|_\omega - \|z(t)\|_\omega \\
&\quad =\Big\|z(t)+\epsilon\Big(-R_1 z(t)+P_{31}\widetilde{F}^R(z(t))+P_{41}\widetilde{F}^I(z(t))+Q_{31}\widetilde{G}^R(z(t-\tau(t))) \\
&\qquad +Q_{41}\widetilde{G}^I(z(t-\tau(t)))+S(t)+\Omega z(t)\Big)+\mathcal{O}(\epsilon)\Big\|_\omega - \|z(t)\|_\omega \\
&\quad \leq \|I+\epsilon(-R_1+\Omega)\|_\omega\|z(t)\|_\omega+\epsilon\|P_{31}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{41}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega \\
&\qquad +\epsilon\|Q_{31}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{41}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega-\|z(t)\|_\omega \\
&\quad \leq\Big\{\|I+\epsilon(-R+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{31}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{41}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega \\
&\qquad +\epsilon\|Q_{31}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{41}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega.
\end{aligned} \tag{4.2.3}$$

From the Assumption 4, we have

$$\begin{aligned}
\|\widetilde{F}^R(z(t))\|_\omega &\leq r_k\|z(t)\|_\omega, \\
\|\widetilde{F}^I(z(t))\|_\omega &\leq s_k\|z(t)\|_\omega.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega &\leq m_k\|z(t-\tau(t))\|_\omega, \\
\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega &\leq n_k\|z(t-\tau(t))\|_\omega.
\end{aligned} \tag{4.2.4}$$

Therefore, from equations (4.2.3) and (4.2.4), we get

$$\begin{aligned}
&\|z(t)+\epsilon\dot{z}(t)+\mathcal{O}(\epsilon)\|_\omega - \|z(t)\|_\omega \\
&\quad \leq\Big\{\|I+\epsilon(-R_1+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{31}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{41}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega \\
&\qquad +\epsilon\|Q_{31}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{41}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon K
\end{aligned}$$

$$\leq\Big\{\|I+\epsilon(-R_1+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{31}\|_\omega r_k\|z(t)\|_p+\epsilon\|P_{41}\|_\omega s_k\|z(t)\|_\omega$$
$$+\epsilon\|Q_{31}\|_\omega m_k\|z(t-\tau(t))\|_\omega+\epsilon\|Q_{41}\|_\omega n_k\|z(t-\tau(t))\|_\omega+\epsilon K. \qquad (4.2.5)$$

Using the inequality (4.2.5) in the equation (4.2.2), we get

$$D^+V_1(z(t))\leq\overline{lim}_{\epsilon\to0^+}\frac{\|I+\epsilon(-R_1+\Omega)\|_\omega-1}{\epsilon}\|z(t)\|_\omega+\|P_{31}\|_p r_k\|z(t)\|_\omega$$
$$+\|P_{41}\|_\omega s_k\|z(t)\|_\omega+\|Q_{31}\|_\omega m_k\|z(t-\tau(t))\|_\omega$$
$$+\|Q_{41}\|_\omega n_k\|z(t-\tau(t))\|_\omega+K. \qquad (4.2.6)$$

Based on Definition 1.4.4 and the equation (4.2.6), we obtain

$$D^+V_1(z(t))\leq\mu_\omega(-R_1+\Omega)\|z(t)\|_\omega+(r_k\|P_{31}\|_\omega+s_k\|P_{41}\|_\omega)$$
$$\|z(t)\|_\omega+(m_k\|Q_{31}\|_\omega+n_k\|Q_{41}\|_\omega)\|z(t-\tau(t))\|_\omega+K$$
$$\leq\Big\{\mu_\omega(-R_1+\Omega)+(r_k\|P_{31}\|_\omega+s_k\|P_{41}\|_\omega)\Big\}$$
$$\|z(t)\|_\omega+(m_k\|Q_{31}\|_\omega+n_k\|Q_{41}\|_\omega)\|z(t-\tau(t))\|_\omega+K$$
$$\leq\Big\{\mu_\omega(-R_1+\Omega)+(r_k\|P_{31}\|_\omega+s_k\|P_{41}\|_\omega)\Big\}$$
$$V_1(z(t))+(m_k\|Q_{31}\|_\omega+n_k\|Q_{41}\|_p)sup_{-\tau\leq s\leq0}V_1(z(s))+K. \qquad (4.2.7)$$

Case 2. If $\dot\theta_1^R(t)\geq\dot\theta_2^R(t),\dot\theta_1^I(t)<\dot\theta_2^I(t)$, then

$$\|z(t)+\epsilon\dot z(t)+\mathcal{O}(\epsilon)\|_\omega-\|z(t)\|_\omega$$
$$=\|z(t)+\epsilon\bigg(-R_1z(t)+P_{31}\widetilde F^R(z(t))+P_{41}\widetilde F^I(z(t))+Q_{32}\widetilde G^R(z(t-\tau(t)))$$
$$+Q_{42}\widetilde G^I(z(t-\tau(t)))+S(t)+\Omega z(t)\bigg)+\mathcal{O}(\epsilon)\|_\omega-\|z(t)\|_\omega$$
$$\leq\|I+\epsilon(-R_1+\Omega)\|_\omega\|z(t)\|_\omega+\epsilon\|P_{31}\|_\omega\|\widetilde F^R(z(t))\|_\omega+\epsilon\|P_{41}\|_\omega\|\widetilde F^I(z(t))\|_\omega$$
$$+\epsilon\|Q_{32}\|_\omega\|\widetilde G^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{42}\|_\omega\|\widetilde G^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega-\|z(t)\|_\omega$$

$$
\begin{aligned}
&\leq\Big\{\|I+\epsilon(-R+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{31}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{41}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega\\
&\quad+\epsilon\|Q_{32}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{42}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega\\
&\leq\Big\{\|I+\epsilon(-R_1+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{31}\|_\omega r_k\|z(t)\|_\omega+\epsilon\|P_{41}\|_\omega s_k\|z(t)\|_\omega\\
&\quad+\epsilon\|Q_{32}\|_\omega m_k\|z(t-\tau(t))\|_\omega+\epsilon\|Q_{42}\|_\omega n_k\|z(t-\tau(t))\|_\omega+\epsilon K.
\end{aligned}
$$

From Case 1, we get

$$
\begin{aligned}
D^+V_1(z(t))\leq&\Big\{\mu_p(-R_1+\Omega)+(r_k\|P_{31}\|_\omega+s_k\|P_{41}\|_\omega)\Big\}V_1(z(t))\\
&+(m_k\|Q_{32}\|_\omega+n_k\|Q_{42}\|_\omega)sup_{-\tau\leq s\leq 0}V_1(e(s))+K. \qquad (4.2.8)
\end{aligned}
$$

Case 3. If $\dot{\theta}_1^R(t)<\dot{\theta}_2^R(t),\dot{\theta}_1^I(t)\geq\dot{\theta}_2^I(t)$, then

$$
\begin{aligned}
&\|z(t)+\epsilon\dot{z}(t)+\mathcal{O}(\epsilon)\|_\omega-\|z(t)\|_\omega\\
&=\|z(t)+\epsilon\bigg(-R_1z(t)+P_{32}\widetilde{F}^R(z(t))+P_{42}\widetilde{F}^I(z(t))+Q_{31}\widetilde{G}^R(z(t-\tau(t)))\\
&\quad+Q_{41}\widetilde{G}^I(z(t-\tau(t)))+S(t)+\Omega z(t)\bigg)+\mathcal{O}(\epsilon)\|_\omega-\|z(t)\|_\omega\\
&\leq\|I+\epsilon(-R_1+\Omega)\|_\omega\|z(t)\|_\omega+\epsilon\|P_{32}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{42}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega\\
&\quad+\epsilon\|Q_{31}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{41}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega-\|z(t)\|_\omega\\
&\leq\Big\{\|I+\epsilon(-R+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{32}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{42}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega\\
&\quad+\epsilon\|Q_{31}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{41}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega\\
&\leq\Big\{\|I+\epsilon(-R_1+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{32}\|_\omega r_k\|z(t)\|_\omega+\epsilon\|P_{42}\|_\omega s_k\|z(t)\|_\omega\\
&\quad+\epsilon\|Q_{31}\|_\omega m_k\|z(t-\tau(t))\|_\omega+\epsilon\|Q_{41}\|_\omega n_k\|z(t-\tau(t))\|_\omega+\epsilon K.
\end{aligned}
$$

From Case 1, we get

$$D^+V_1(z(t)) \leq \Big\{\mu_\omega(-R_1+\Omega)+(r_k\|P_{32}\|_\omega+s_k\|P_{42}\|_\omega)\Big\}V_1(z(t)) \\ +(m_k\|Q_{31}\|_\omega+n_k\|Q_{41}\|_\omega)sup_{-\tau\leq s\leq 0}V_1(e(s))+K. \tag{4.2.9}$$

Case 4. If $\dot{\theta}_1^R(t) < \dot{\theta}_2^R(t), \dot{\theta}_1^I(t) < \dot{\theta}_2^I(t)$, then

$$\begin{aligned}
&\|z(t)+\epsilon\dot{z}(t)+\mathcal{O}(\epsilon)\|_\omega-\|z(t)\|_\omega \\
&=\|z(t)+\epsilon\bigg(-R_1z(t)+P_{32}\widetilde{F}^R(z(t))+P_{42}\widetilde{F}^I(z(t))+Q_{32}\widetilde{G}^R(z(t-\tau(t))) \\
&\quad+Q_{42}\widetilde{G}^I(z(t-\tau(t)))+S(t)+\Omega z(t)\bigg)+\mathcal{O}(\epsilon)\|_\omega-\|z(t)\|_\omega \\
&\leq\|I+\epsilon(-R_1+\Omega)\|_\omega\|z(t)\|_\omega+\epsilon\|P_{32}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{42}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega \\
&\quad+\epsilon\|Q_{32}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{42}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega-\|z(t)\|_\omega \\
&\leq\Big\{\|I+\epsilon(-R+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{32}\|_\omega\|\widetilde{F}^R(z(t))\|_\omega+\epsilon\|P_{42}\|_\omega\|\widetilde{F}^I(z(t))\|_\omega \\
&\quad+\epsilon\|Q_{32}\|_\omega\|\widetilde{G}^R(z(t-\tau(t)))\|_\omega+\epsilon\|Q_{42}\|_\omega\|\widetilde{G}^I(z(t-\tau(t)))\|_\omega+\epsilon\|S(t)\|_\omega \\
&\leq\Big\{\|I+\epsilon(-R_1+\Omega)\|_\omega-1\Big\}\|z(t)\|_\omega+\epsilon\|P_{32}\|_\omega r_k\|z(t)\|_\omega+\epsilon\|P_{42}\|_\omega s_k\|z(t)\|_\omega \\
&\quad+\epsilon\|Q_{32}\|_\omega m_k\|z(t-\tau(t))\|_\omega+\epsilon\|Q_{42}\|_\omega n_k\|z(t-\tau(t))\|_\omega+\epsilon K.
\end{aligned}$$

Therefore,

$$D^+V_1(z(t)) \leq \Big\{\mu_\omega(-R_1+\Omega)+(r_k\|P_{32}\|_\omega+s_k\|P_{42}\|_\omega)\Big\}V_1(z(t)) \\ +(m_k\|Q_{32}\|_\omega+n_k\|Q_{42}\|_\omega)sup_{-\tau\leq s\leq 0}V_1(e(s))+K. \tag{4.2.10}$$

Form the above analyses of the equations (4.2.7)-(4.2.10), and denoting

$k_1 = -max\Big\{\mu_\omega(-R_1+\Omega)+(r_k\|P_{31}\|_\omega+s_k\|P_{41}\|_\omega), \mu_\omega(-R_1+\Omega)+(r_k\|P_{32}\|_\omega + s_k\|P_{42}\|_\omega)\Big\}$

and $\quad k_2 = max\Big\{(m_k\|Q_{31}\|_\omega+n_k\|Q_{41}\|_\omega), (m_k\|Q_{32}\|_\omega+n_k\|Q_{42}\|_\omega)\Big\}$, we obtain

$$D^+V_1(z(t)) \leq k_1V_1(z(t)) + k_2 sup_{t-\tau\leq s\leq t}V_1(z(s)) + 4K. \tag{4.2.11}$$

Then from the condition (4.2.11), we have $0 < k_2 < k_1$ and using Lemma 3.1, we get

$$V_1(z(t)) \leq \ sup_{-\tau\leq s\leq 0}V_1(z(s))e^{-\delta t} + \frac{4K}{\delta}. \tag{4.2.12}$$

As z(t) converges exponentially towards a desirable region $\gamma = \{z(t) : \|z(t)\| \leq \frac{k_3}{\delta}\}$, therefore from Definition 1.5.2, we can conclude that CVRNNs (4.1.1) and (4.1.5) exhibit quasi-projective synchronization. Thus proof is completed. □

Note. Now using Lemma 1.3, we get

$$\mu_\omega(-R_1 + \Omega) \leq \mu_\omega(-R_1) + \mu_\omega(\Omega), \tag{4.2.13}$$

which gives

$$\begin{aligned}
&-max\Big\{\mu_\omega(-R_1+\Omega) + (r_k\|P_{31}\|_\omega + s_k\|P_{41}\|_\omega), \mu_\omega(-R_1+\Omega) + (r_k\|P_{32}\|_\omega + s_k\|P_{42}\|_\omega)\Big\} \\
&\quad < -max\Big\{\mu_\omega(-R_1) + \mu_\omega(\Omega) + (r_k\|P_{31}\|_\omega + s_k\|P_{41}\|_\omega), \mu_\omega(-R_1+\Omega) \\
&\quad + (r_k\|P_{32}\|_\omega + s_k\|P_{42}\|_\omega)\Big\}.
\end{aligned}$$

Therefore from equation (4.2.13), we have

$$\begin{aligned}
&0 < max\Big\{(m_k\|Q_{31}\|_\omega + n_k\|Q_{41}\|_\omega), (m_k\|Q_{32}\|_\omega + n_k\|Q_{42}\|_\omega)\Big\} < \\
&\ - max\Big\{\mu_\omega(-R_1) + \mu(\Omega) + (r_k\|P_{31}\|_\omega + s_k\|P_{41}\|_\omega), \mu_\omega(-R_1) + \mu(\Omega) \\
&\ + (r_k\|P_{32}\|_\omega + s_k\|P_{42}\|_\omega)\Big\}.
\end{aligned} \tag{4.2.14}$$

Theorem 4.2. *Let us assume the Assumptions 3,4 and 5 hold, then the systems (4.1.1) and (4.1.5) will be quasi- projective synchronized , if ∃ diagonal matrices R,S,M,K s.t.,*

$$\lambda_1 = min\Big\{\lambda_{min}(P_{31}^TP_{31} + P_{41}^TP_{41} + Q_{31}^TQ_{31} + Q_{41}^TQ_{41} + R'^TR' + S^TS),$$
$$\lambda_{min}(P_{31}^TP_{31} + P_{41}^TP_{41} + Q_{32}^TQ_{32} + Q_{42}^TQ_{42} + R'^TR' + S^TS),$$
$$\lambda_{min}(P_{32}^TP_{32} + P_{42}^TP_{42} + Q_{31}^TQ_{31} + Q_{41}^TQ_{41} + R'^TR' + S^TS),$$
$$\lambda_{min}(P_{32}^TP_{32} + P_{42}^TP_{42} + Q_{32}^TQ_{32} + Q_{42}^TQ_{42} + R'^TR' + S^TS)\Big\},$$
$$with \quad \lambda_1 > \lambda_2 = max\{M^TM + K_1^TK_1\} > 0, \tag{4.2.15}$$

where $R' = diag\{r_1'^1, r_2'^1, ..., r_n'^1\}, S = diag\{s_1^1, s_2^1, ..., s_n^1\}, M = diag\{m_1^1, m_2^1, ..., m_n^1\}$ *and* $K_1 = diag\{k_1^1, k_2^1, ..., k_n^1\}$.

Proof. Let us assume Lyapunov-Krasovskii functional as

$$V(t) = \frac{1}{2}z^T(t)z(t)$$

Case 1. If $\dot{\theta}_1^R(t) \geq \dot{\theta}_2^R(t)$, $\dot{\theta}_1^I(t) \geq \dot{\theta}_2^I(t)$, the system (4.1.13) can be rewritten as

$$\dot{z}(t) = -R_1z(t) + P_{31}\widetilde{F}^R(z(t)) + P_{41}\widetilde{F}^I(z(t)) + Q_{31}\widetilde{G}^R(z(t-\tau(t))) + Q_{41}\widetilde{G}^I(z(t-\tau(t)))$$
$$+ S(t) + \Omega z(t). \tag{4.2.16}$$

By taking the upper-right Dini derivative with respect to t and from equation (4.2.16), we get

$$D^+V(t) = z^T(t)[-R_1z(t) + P_{31}\widetilde{F}^R(z(t)) + P_{41}\widetilde{F}^I(z(t)) + Q_{31}\widetilde{G}^R(z(t-\tau(t)))$$
$$+ Q_{41}\widetilde{G}^I(z(t-\tau(t))) + S(t) + \Omega z(t)]. \tag{4.2.17}$$

From Assumption 5, we have

$$\begin{aligned} z^T(t)P_{31}\widetilde{F}^R(z(t)) &\leq \frac{1}{2}z^T(t)P_{31}^TP_{31}z(t) + \frac{1}{2}z^T(t)R'^TR'z(t), \\ z^T(t)P_{41}\widetilde{F}^I(z(t)) &\leq \frac{1}{2}z^T(t)P_{41}^TP_{41}z(t) + \frac{1}{2}z^T(t)L^TLz(t), \\ z^T(t)Q_{31}\widetilde{G}^R(z(t-\tau(t))) &\leq \frac{1}{2}z^T(t)Q_{31}^TQ_{31}z(t) + \frac{1}{2}z^T(t-\tau(t))M^TMz(t-\tau(t)), \\ z^T(t)Q_{41}\widetilde{G}^R(z(t-\tau(t))) &\leq \frac{1}{2}z^T(t)Q_{41}^TQ_{41}z(t) + \frac{1}{2}z^T(t-\tau(t))K_1^TK_1z(t-\tau(t)), \\ z^TS(t) &\leq \frac{1}{2\mu}z^T(t)z(t) + \frac{1}{2}\mu S(t)^TS(t), \\ z^TS(t) &\leq \frac{1}{2\mu}z^T(t)z(t) + \frac{1}{2}\mu K^2. \end{aligned} \tag{4.2.18}$$

Substituting inequality (4.2.18) in (4.2.17), we obtain

$$\begin{aligned} D^+V(t) \leq\, & z^T(t)[-D_1 + \frac{1}{2\mu} + \frac{1}{2}(P_{31}^TP_{31} + P_{41}^TP_{41} + Q_{31}^TQ_{31} + Q_{41}^TQ_{41} + R'^TR' \\ & + L^TL + 2\Omega)]z(t) + z^T(t-\tau(t))[\frac{1}{2}(M^TM + K^TK)]z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\ \leq & -\lambda_{min}(P_{31}^TP_{31} + P_{41}^TP_{41} + Q_{31}^TQ_{31} + Q_{41}^TQ_{41} + R'^TR' + L^TL + 2\Omega)z^T(t)z(t) \\ & + \lambda_{max}(M^TM + K_1^TK_1)z^T(t-\tau(t))z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\ \leq & -\lambda_1V(t) + \lambda_2V(t-\tau(t)) + \frac{1}{2}\mu K^2. \end{aligned} \tag{4.2.19}$$

Case 2. If $\dot{\theta}_1^R(t) \geq \dot{\theta}_2^R(t)$, $\dot{\theta}_1^I(t) < \dot{\theta}_2^I(t)$, the system (4.1.13) can be rewritten as

$$\begin{aligned} \dot{z}(t) = & -R_1z(t) + P_{31}\widetilde{F}^R(z(t)) + P_{41}\widetilde{F}^I(z(t)) + Q_{32}\widetilde{G}^R(z(t-\tau(t))) + Q_{42}\widetilde{G}^I(z(t-\tau(t))) \\ & + S(t) + \Omega z(t). \end{aligned}$$

Similar to Case 1, we have

$$\begin{aligned}
D^+V(t) \leq & z^T(t)[-R_1 + \frac{1}{2}(P_{31}^T P_{31} + P_{41}^T P_{41} + Q_{32}^T Q_{32} + Q_{42}^T Q_{42} + R'^T R' + L^T L + 2\Omega)]z(t) \\
& + z^T(t-\tau(t))[\frac{1}{2}(M^T M + K_1^T K_1)]z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\
\leq & -\lambda_{min}(P_{31}^T P_{31} + P_{41}^T P_{41} + Q_{32}^T Q_{32} + Q_{42}^T Q_{42} + R'^T R' + L^T L + 2\Omega) z^T(t) z(t) \\
& + \lambda_{max}(M^T M + K_1^T K_1) z^T(t-\tau(t)) z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\
\leq & -\lambda_1 V(t) + \lambda_2 V(t-\tau(t)) + \frac{1}{2}\mu K^2. \qquad (4.2.20)
\end{aligned}$$

Case 3. If $\dot{\theta}_1^R(t) < \dot{\theta}_2^R(t)$, $\dot{\theta}_1^I(t) \leq \dot{\theta}_2^I(t)$, the system (4.1.13) can be rewritten as

$$\begin{aligned}
\dot{z}(t) = & -R_1 z(t) + P_{32}\widetilde{F}^R(z(t)) + P_{42}\widetilde{F}^I(z(t)) + Q_{31}\widetilde{G}^R(z(t-\tau(t))) + Q_{41}\widetilde{G}^I(z(t-\tau(t))) \\
& + S(t) + \Omega z(t).
\end{aligned}$$

Similar to Case 1, we have

$$\begin{aligned}
D^+V(t) \leq & z^T(t)[-R_1 + \frac{1}{2}(P_{32}^T P_{32} + P_{42}^T P_{42} + Q_{31}^T Q_{31} + Q_{41}^T Q_{41} + R'^T R' + L^T L + 2\Omega)]z(t) \\
& + z^T(t-\tau(t))[\frac{1}{2}(M^T M + K_1^T K_1)]z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\
\leq & -\lambda_{min}(P_{32}^T P_{32} + P_{42}^T P_{42} + Q_{31}^T Q_{31} + Q_{41}^T Q_{41} + R'^T R' + L^T L + 2\Omega) z^T(t) z(t) \\
& + \lambda_{max}(M^T M + K_1^T K_1) z^T(t-\tau(t)) z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\
\leq & -\lambda_1 V(t) + \lambda_2 V(t-\tau(t)) + \frac{1}{2}\mu K^2. \qquad (4.2.21)
\end{aligned}$$

Case 4. If $\dot{\theta}_1^R(t) < \dot{\theta}_2^R(t)$, $\dot{\theta}_1^I(t) < \dot{\theta}_2^I(t)$, the system (4.1.13) can be rewritten as

$$\begin{aligned}
\dot{z}(t) = & -R_1 z(t) + P_{32}\widetilde{F}^R(z(t)) + P_{42}\widetilde{F}^I(z(t)) + Q_{32}\widetilde{G}^R(z(t-\tau(t))) + Q_{42}\widetilde{G}^I(z(t-\tau(t))) \\
& + S(t) + \Omega z(t).
\end{aligned}$$

Similar to Case 1, we have

$$\begin{aligned}
D^+V(t) \leq\, & z^T(t)[-R_1 + \frac{1}{2}(P_{32}^T P_{32} + P_{42}^T P_{42} + Q_{32}^T Q_{32} + Q_{42}^T Q_{42} + R^T R + L^T L + 2\Omega)]z(t) \\
& + z^T(t-\tau(t))[\frac{1}{2}(M^T M + K_1^T K_1)]z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\
\leq & -\lambda_{min}(P_{32}^T P_{32} + P_{42}^T P_{42} + Q_{32}^T Q_{32} + Q_{42}^T Q_{42} + R^T R + L^T L + 2\Omega)z^T(t)z(t) \\
& + \lambda_{max}(M^T M + K_1^T K_1)z^T(t-\tau(t))z(t-\tau(t)) + \frac{1}{2}\mu K^2 \\
\leq & -\lambda_1 V(t) + \lambda_2 V(t-\tau(t)) + \frac{1}{2}\mu K^2. \qquad (4.2.22)
\end{aligned}$$

From the above analyses of the equations (4.2.19)-(4.2.22), and by using Lemma 3.1, we get

$$V_1(\theta(t)) \leq \; sup_{-\tau\leq s\leq 0} V_1(\theta(s))e^{-\delta t} + \frac{\mu K^2}{2\delta},$$

where r is the unique solution of the equation

$$\delta = \lambda_1 - \lambda_2 e^{\delta\tau}.$$

As $z(t)$ converges exponentially towards a desirable region $\gamma = \{z(t) : \|z(t)\| \leq \frac{\mu K^2}{2\delta}\}$, then from Definition 1.5.2, we can conclude that CVRNNs (4.1.1) and (4.1.5) exhibit quasi-projective synchronization. The proof is completed. □

Remark 4.3. The errors are taken as $\theta^R(t) = \widetilde{\alpha}(t) - r\alpha(t), \theta^I(t) = \widetilde{\beta}(t) - r\beta(t)$. In particular, if $r = 1$, the problem can be extended to the quasi-completely synchronization of MCVRNNs with time-varying delay and mismatched parameters. Also, if r = -1, it will be quasi anti-synchronization of MCVRNNs.

Remark 4.4. In the articles [107, 108], the quasi-synchronization of real-valued MBNNs has been discussed, while in [109, 110], the projective synchronization of real-valued MBNNs is shown and these works are only based on Lyapunov functional method. However, this chapter has focused on quasi-projective synchronization of MCVRNNs with parameters mismatched by matrix measure method and Lyapunov functional method. The conditions in Theorem 3.1 are rendered in the form of matrix measure, which make the conditions more general and broad. Because of our proposed systems are on the complex valued domain, which is the generalization of real valued domain and has ability to deal with multi-dimensional data. The existing results [107, 108, 109, 110] are on the real valued domain. By comparing with above existing results, we can find that those results are the particular cases of our present results. Thus, the obtained results of this chapter are less conservative and more general than previous results. In addition, as compared with the articles [92, 111, 112, 113], the control gain fluctuation is displayed in the controller, which makes the obtained results more significant.

Remark 4.5. In the synchronization results [114, 115, 106], the drive and response systems are identical. However, in practical application, there are always mismatched between drive and response systems which can destroy the synchrony state of drive and response systems. Therefore, the research on quasi-projective synchronization of MCVRNNs of mismatched parameters has importance and significance from the application point of view, because it is impossible to construct two absolutely identical systems in practical synchronization implementation. Here, Theorems 4.1 and 4.2 provide sufficient conditions to guarantee the achievements of quasi-projective synchronization of MCVRNNs associated with a relatively small error bound for the first time. It is believed that this results are more useful and effective for the purposes of design and applications of MCVRNNs.

Remark 4.6. Theorem 4.1 is proved by using matrix measure method to ensure the quasi-projective synchronization between the systems (4.1.1) and (4.1.4). Based on Lemmas 1.3 and 3.1, Assumptions 3, 4 and 5, Definitions 1.4.4 and 1.5.2, the Theorem 4.1 is derived. During derivation of the Theorem 3.2, the appropriate

Lyapunov functional is chosen through which some sufficient conditions have been obtained by using Lemmas 1.3 and 3.1, and Assumption 5 and Definition 1.5.2. So, the Theorems 4.1 and 4.2 are complement to each other.

4.3 Numerical Simulation

Here, two numerical simulation results are given to demonstrate the effectiveness and usefulness of the proposed quasi-projective synchronization scheme.

Example 4.3.1. Let us consider memristor-based CVRNNs in presence of time-varying delay as the drive system as

$$\dot{\gamma}(t) = -R\gamma(t) + P(\gamma(t))f(\gamma(t)) + Q(\gamma(t))f(\gamma(t-\tau(t))) + I(t), \tag{4.3.1}$$

with the following parameters

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} -0.2-0.2i & -0.1-0.1i \\ 0.4+0.1i & 0.2-0.3i \end{pmatrix}, Q = \begin{pmatrix} -0.1-0.3i & -0.1-0.2i \\ -0.2+0.1i & 0.2+0.4i \end{pmatrix},$$

$$I(t) = \begin{pmatrix} 0.5sin(t) + i0.2cos(t) \\ 0.6cos(t) + i0.5sin(t) \end{pmatrix}, \tau(t) = 0.6exp(t), \text{ and}$$

$$p_{11}^R(t) = \begin{cases} 1.5, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.2, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{a\to s^-} p_{11}^R(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases} ; p_{12}^R(t) = \begin{cases} 0.4, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.5, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{a\to s^-} p_{12}^R(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases}$$

$$p_{21}^R(t)=\begin{cases}0.6, & \dot{\upsilon}_1(t)>\dot{\upsilon}_2(t)\\ 0.8, & \dot{\upsilon}_1(t)<\dot{\upsilon}_2(t)\\ lim_{a\to s^-}p_{21}^R(s), & \dot{\upsilon}_1(t)=\dot{\upsilon}_2(t)\end{cases} ;\, p_{22}^R(t)=\begin{cases}0.9, & \dot{\upsilon}_1(t)>\dot{\upsilon}_2(t)\\ 0.2, & \dot{\upsilon}_1(t)<\dot{\upsilon}_2(t)\\ lim_{a\to s^-}p_{22}^R(s), & \dot{\upsilon}_1(t)=\dot{\upsilon}_2(t)\end{cases}$$

$$p_{11}^I(t)=\begin{cases}0.3, & \dot{\upsilon}_1(t)>\dot{\upsilon}_2(t)\\ 1.6, & \dot{\upsilon}_1(t)<\dot{\upsilon}_2(t)\\ lim_{a\to s^-}p_{11}^I(s), & \dot{\upsilon}_1(t)=\dot{\upsilon}_2(t)\end{cases} ;\, p_{12}^I(t)=\begin{cases}1.4, & \dot{\upsilon}_1(t)>\dot{\upsilon}_2(t)\\ 0.8, & \dot{\upsilon}_1(t)<\dot{\upsilon}_2(t)\\ lim_{a\to s^-}p_{12}^I(s), & \dot{\upsilon}_1(t)=\dot{\upsilon}_2(t)\end{cases}$$

$$p_{21}^I(t)=\begin{cases}0.5, & \dot{\upsilon}_1(t)>\dot{\upsilon}_2(t)\\ 1.2, & \dot{\upsilon}_1(t)<\dot{\upsilon}_2(t)\\ lim_{a\to s^-}p_{21}^I(s), & \dot{\upsilon}_1(t)=\dot{\upsilon}_2(t)\end{cases} ;\, p_{22}^I(t)=\begin{cases}0.4, & \dot{\upsilon}_1(t)>\dot{\upsilon}_2(t)\\ 0.1, & \dot{\upsilon}_1(t)<\dot{\upsilon}_2(t)\\ lim_{a\to s^-}p_{22}^I(s), & \dot{\upsilon}_1(t)=\dot{\upsilon}_2(t)\end{cases}$$

$$q_{11}^R(t)=\begin{cases}0.6, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 1.2, & \dot{\varrho}_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{11}^R(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t)\end{cases} ;\, q_{12}^R(t)=\begin{cases}0.3, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 0.8, & \dot{\varrho}_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{12}^R(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t)\end{cases}$$

$$q_{21}^R(t)=\begin{cases}0.8, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 0.9, & \dot{\varrho}_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{21}^R(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t)\end{cases} ;\, q_{22}^R(t)=\begin{cases}0.2, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 0.5, & \dot{\varrho}_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{22}^R(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t)\end{cases}$$

$$q_{11}^I(t)=\begin{cases}1.3, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 0.5, & \dot{\varrho}_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{11}^I(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t)\end{cases} ;\quad q_{12}^I(t)=\begin{cases}0.5, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 1.7, & \dot{\varrho}_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{12}^I(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t)\end{cases}$$

$$q_{21}^{I}(t)=\begin{cases}0.3, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 0.2, & \dot{\varrho}_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{21}^{I}(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t)\end{cases} \quad ; \quad q_{22}^{I}(t)=\begin{cases}1.6, & \dot{\varrho}_1(t)>\dot{\varrho}_2(t)\\ 0.7, & \varrho_1(t)<\dot{\varrho}_2(t)\\ lim_{a\to s^-}q_{22}^{I}(s), & \dot{\varrho}_1(t)=\dot{\varrho}_2(t).\end{cases}$$

The activation functions are taken as

$$\begin{aligned} f_m(\gamma_m(t)) =& \frac{|\upsilon_m+1|-|\upsilon_m-1|}{2}+i\frac{|\varrho_m+1|-|\varrho_m-1|}{2},\\ g_m(\gamma_m(t)) =& \frac{1}{1+e^{(-\upsilon_m+2\varrho_m)}}+i\frac{1-e^{(-2\upsilon_m-\varrho_m)}}{1+e^{(-2\upsilon_m-\varrho_m)}}, (m=1,2). \end{aligned}$$

Figure 4.1 depicts the plots of the trajectories $\upsilon_1(t), \varrho_1(t)$ and $\upsilon_2(t), \varrho_2(t)$ of the system (4.3.1) in three-dimensional at time t in the presence of mismatched parameters and time-varying delay $\tau(t) = 0.6exp(t)$ which clearly show the chaotic behavior of the real and imaginary parts of trajectories with respect to time t.

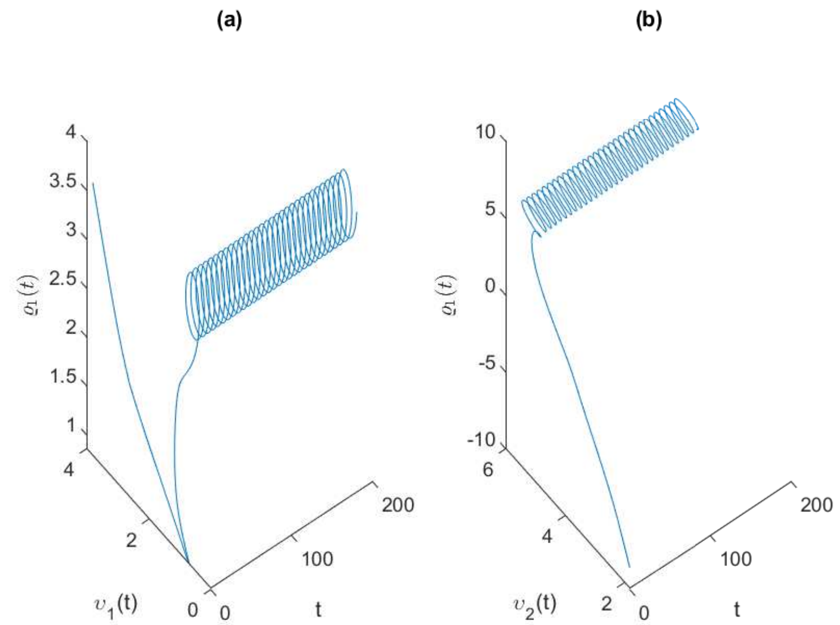

Figure 4.1: The state trajectories of the system (4.3.1) in three-dimensional space for **(a)** $\upsilon_1(t)$, $\varrho_1(t)$, and **(b)** $\upsilon_2(t)$ and $\varrho_2(t)$.

The corresponding response system is considered as

$$\dot{\tilde{\gamma}}(t) = -R'\tilde{\gamma}(t) + P'(\tilde{\gamma}(t))f(\tilde{\gamma}(t)) + Q'(\tilde{\gamma}(t))g(\tilde{\gamma}(t-\tau(t))) + I'(t) + \Xi(t), \quad (4.3.2)$$

with the following parameters

$$R' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P' = \begin{pmatrix} -0.3+0.1i & -0.1+0.1i \\ 0.2-0.1i & 0.3+0.2i \end{pmatrix}, Q' = \begin{pmatrix} -0.3-0.1i & -0.4-0.2i \\ -0.3-0.2i & 0.3-0.2i \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -64 & 10 \\ 15 & -60 \end{pmatrix}, I'(t) = \begin{pmatrix} 0.2cos(t) + i0.3sin(t) \\ 0.5cos(t) + i0.5sin(t) \end{pmatrix} \text{ and also}$$

$$p'^{R}_{11}(t) = \begin{cases} 0.4, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 1.3 & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{a\to s^-} p'^{R}_{11}(s), \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases} ; \quad p'^{R}_{12}(t) = \begin{cases} 0.8, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 1.3, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{a\to s^-} p'^{R}_{12}(s), \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases}$$

$$p'^{R}_{21}(t) = \begin{cases} 0.9, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.2, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{a\to s^-} p'^{R}_{21}(s), \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases} ; \quad p'^{R}_{22}(t) = \begin{cases} 0.4, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 1.8, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{a\to s^-} p'^{R}_{22}(s), \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases}$$

$$p'^{I}_{11}(t) = \begin{cases} 0.8, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.3, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{a\to s^-} p'^{I}_{11}(s), \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases} ; p'^{I}_{12}(t) = \begin{cases} 0.9, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.1, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{a\to s^-} p'^{I}_{12}(s), \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases}$$

$$p_{21}'^{I}(t)=\begin{cases}0.5, & \dot{\tilde{\upsilon}}_1(t)>\dot{\tilde{\upsilon}}_2(t)\\ 1.3, & \dot{\tilde{\upsilon}}_1(t)<\dot{\tilde{\upsilon}}_2(t)\\ lim_{a\to s^-}p_{21}'^{I}(s), & \dot{\tilde{\upsilon}}_1(t)=\dot{\tilde{\upsilon}}_2(t)\end{cases};\ p_{22}'^{I}(t)=\begin{cases}0.8, & \dot{\tilde{\upsilon}}_1(t)>\dot{\tilde{\upsilon}}_2(t)\\ 1.2, & \dot{\tilde{\upsilon}}_1(t)<\dot{\tilde{\upsilon}}_2(t)\\ lim_{a\to s^-}p_{22}'^{I}(s), & \dot{\tilde{\upsilon}}_1(t)=\dot{\tilde{\upsilon}}_2(t)\end{cases}$$

$$q_{11}'^{R}(t)=\begin{cases}0.8, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.3, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{11}'^{R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{12}'^{R}(t)=\begin{cases}0.8, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.3, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{12}'^{R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases}$$

$$q_{21}'^{R}(t)=\begin{cases}0.6, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.8, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{21}'^{R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{22}'^{R}(t)=\begin{cases}0.9, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.3, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{22}'^{R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases}$$

$$q_{11}'^{I}(t)=\begin{cases}0.2, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.4, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{11}'^{I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{12}'^{I}(t)=\begin{cases}0.5, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.3, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{12}'^{I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases}$$

$$q_{21}'^{I}(t)=\begin{cases}0.8, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.3, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{21}'^{I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{22}'^{I}(t)=\begin{cases}0.8, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.4, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{a\to s^-}q_{22}'^{I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t).\end{cases}$$

Consider $\omega=1$, then the Assumption 4 of the system holds for the parameters given as $r_k=1, s_k=1, m_k=\sqrt{2}/2$ and $n_k=\sqrt{2}$. The initial conditions of drive-response systems (4.3.1) and (4.3.2) are taken as

$\gamma_1(s)=3.8+3.65i, \gamma_2(s)=1.8-8.57i,$

$\tilde{\gamma}_1(s)=1.5-6.5i,\ \tilde{\gamma}_2(s)=-4.5-3.9i.$

Then, we can verify that

$k_1 = -max\Big\{\mu_\omega(-R_1 + \Omega) + (r_k + s_k)(\|P_{31}\|_\omega + \|P_{41}\|_\omega), \mu_\omega(-R_1 + \Omega) + (r_k + s_k)(\|P_{32}\|_\omega + \|P_{42}\|_\omega)\Big\} = 11.3201$

$k_2 = \Big\{(m_k + n_k)(\|Q_{31}\|_\omega + \|Q_{41}\|_\omega), (m_k + n_k)(\|Q_{32}\|_\omega + \|Q_{42}\|_\omega)\Big\} = 8.2426,$

i.e., $k_1 > k_2$. Hence, according to Theorem 4.1, the systems (4.3.1) and (4.3.2) will be quasi-projective synchronized.

Figures 4.2 and 4.3 represent the state evaluation curves with respect to time t of the systems (4.3.1) and (4.3.2) under the controllers with mismatched parameters. It is seen that the state variables of drive and response systems cannot achieve an identical behavior with time flowing and always there will be a little difference i.e., systems cannot achieve complete synchronization. Figure 4.4(a) shows the evaluation curves of the error system (4.1.10) without controllers, indicating that the quasi-synchronization is not achieved, while Fig. 4.4(b) shows the quasi-synchronization with time varying delay and mismatched parameters of the concerned error system having error bound = 1.5131, in the presence of controllers.

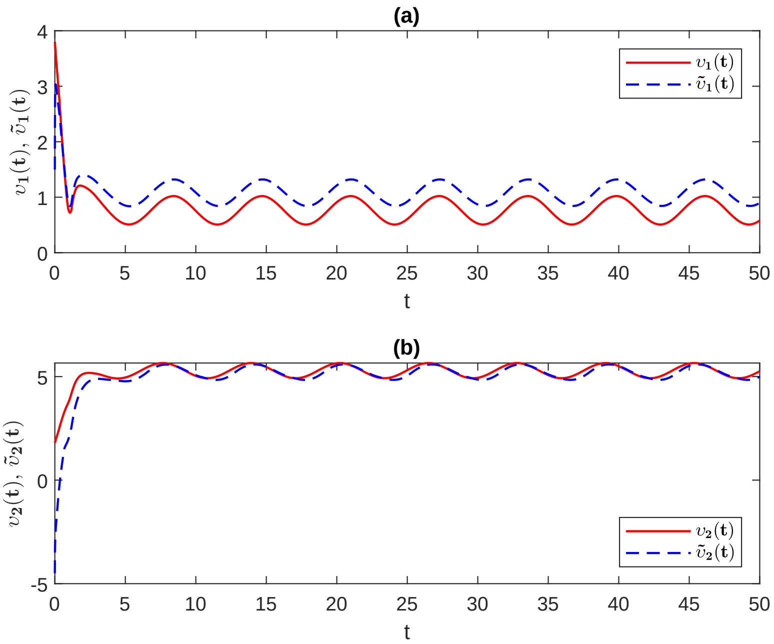

Figure 4.2: The state trajectories of the systems (4.3.1) and (4.3.2) for **(a)** $\upsilon_1(t)$, $\widetilde{\upsilon}_1(t)$, and **(b)** $\upsilon_2(t)$, $\widetilde{\upsilon}_2(t)$ with time t for the Example 4.3.1

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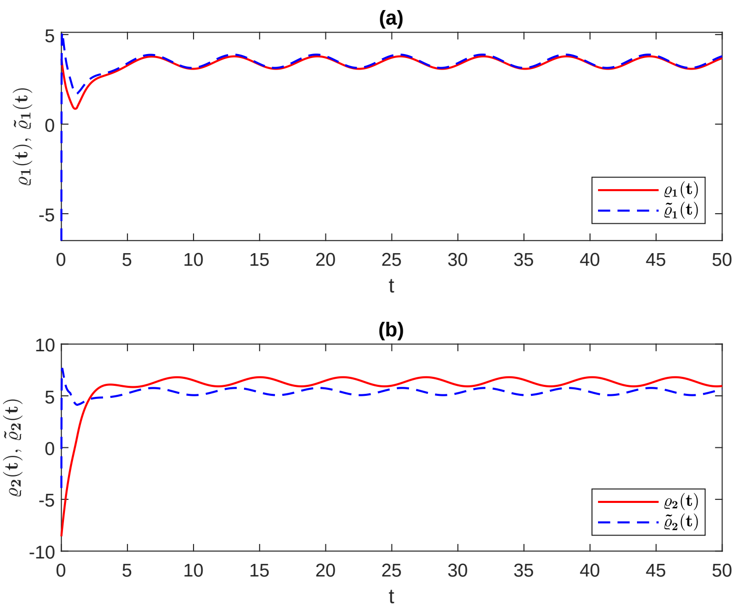

Figure 4.3: The state trajectories of the systems (4.3.1) and (4.3.2) for **(a)** $\varrho_1(t)$, $\widetilde{\varrho}_1(t)$, and **(b)** $\varrho_2(t)$, $\widetilde{\varrho}_2(t)$ with time t for the Example 4.3.1

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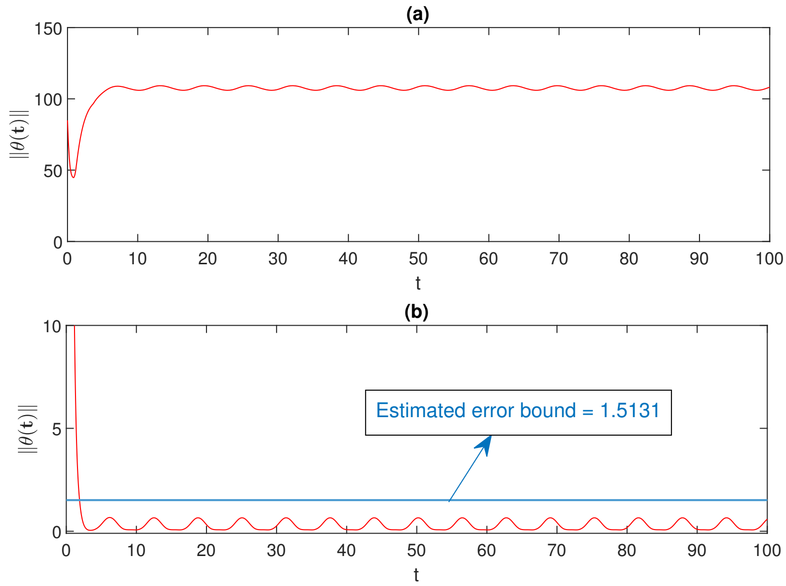

Figure 4.4: Plots of the system (4.3.1) for **(a)** without controllers and **(b)** with controllers for the Example 4.3.1

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Example 4.3.2. Let us consider following parameters of the drive system (4.3.1) as

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} -0.3-0.4i & -0.2-0.1i \\ 0.5+0.1i & 0.3-0.1i \end{pmatrix}, Q = \begin{pmatrix} -0.2-0.4i & -0.2-0.1i \\ -0.3+0.2i & 0.3+0.2i \end{pmatrix},$$

$$L(t) = \begin{pmatrix} 0.3sin(t) + i0.4cos(t) \\ 0.5cos(t) + i0.4sin(t) \end{pmatrix}, \tau(t) = sin(t) + 2,$$

and

$$p_{11}^R(t) = \begin{cases} 0.6, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.8, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{11}^R(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases} ; p_{12}^R(t) = \begin{cases} 0.4, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.7, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{12}^R(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases}$$

$$p_{21}^R(t) = \begin{cases} 0.3, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.5, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{21}^R(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases} ; p_{22}^R(t) = \begin{cases} 0.4, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.3, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{22}^R(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases}$$

$$p_{11}^I(t) = \begin{cases} 0.8, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.6, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{11}^I(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases} ; p_{12}^I(t) = \begin{cases} 0.9, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.5, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{12}^I(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases}$$

$$p_{21}^I(t) = \begin{cases} 0.5, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 1.7, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{21}^I(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases} ; p_{22}^I(t) = \begin{cases} 1.2, & \dot{\upsilon}_1(t) > \dot{\upsilon}_2(t) \\ 0.5, & \dot{\upsilon}_1(t) < \dot{\upsilon}_2(t) \\ lim_{p\to s^-} p_{22}^I(s), & \dot{\upsilon}_1(t) = \dot{\upsilon}_2(t) \end{cases}$$

$$q_{11}^R(t) = \begin{cases} 1.3, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 0.5, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q\to s^-} q_{11}^R(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t) \end{cases} ; q_{12}^R(t) = \begin{cases} 0.4, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 1.7, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q\to s^-} q_{12}^R(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t) \end{cases}$$

$$q_{21}^R(t) = \begin{cases} 1.3, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 0.5, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q\to s^-} q_{21}^R(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t) \end{cases} ; q_{22}^R(t) = \begin{cases} 0.2, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 1.1, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q\to s^-} q_{22}^R(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t) \end{cases}$$

$$q_{11}^{I}(t) = \begin{cases} 0.4, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 1.2, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q \to s^-} q_{11}^{I}(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t) \end{cases} ; q_{12}^{I}(t) = \begin{cases} 0.7, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 0.3, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q \to s^-} q_{12}^{I}(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t) \end{cases}$$

$$q_{21}^{I}(t) = \begin{cases} 0.5, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 1.2, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q \to s^-} q_{21}^{I}(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t) \end{cases} ; q_{22}^{I}(t) = \begin{cases} 0.8, & \dot{\varrho}_1(t) > \dot{\varrho}_2(t) \\ 1.2, & \dot{\varrho}_1(t) < \dot{\varrho}_2(t) \\ lim_{q \to s^-} q_{22}^{I}(s), & \dot{\varrho}_1(t) = \dot{\varrho}_2(t). \end{cases}$$

The activation functions are taken as

$$f_m(\gamma_m(t)) = g_m(\gamma_m(t)) = tanh(\upsilon_m(t)) + itanh(\varrho_m(t)), \quad m = 1, 2.$$

Figure 4.5 shows three-dimensional plots of the trajectories $\upsilon_1(t), \varrho_1(t)$ and $\upsilon_2(t), \varrho_2(t)$ of the drive system (4.3.1) in presence of mismatched parameters and time-varying delay $\tau(t) = sin(t) + 2$.

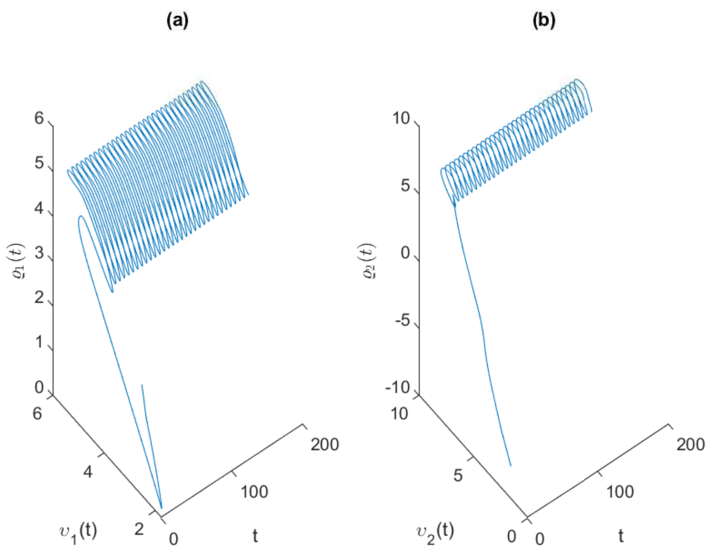

Figure 4.5: The state trajectories of the system (4.3.1) in three-dimensional space for **(a)** $\upsilon_1(t)$, $\varrho_1(t)$, and **(b)** $\upsilon_2(t)$ and $\varrho_2(t)$.

The parameters of corresponding response system is considered as

$$R' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P' = \begin{pmatrix} -0.2-0.2i & -0.1-0.1i \\ 0.4+0.1i & 0.2-0.3i \end{pmatrix}, Q' = \begin{pmatrix} -0.1-0.3i & -0.1-0.2i \\ -0.2+0.1i & 0.2+0.4i \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -64 & 10 \\ 15 & -60 \end{pmatrix}, L'(t) = \begin{pmatrix} 0.5sin(t)+i0.2cos(t) \\ 0.6cos(t)+i0.5sin(t) \end{pmatrix}$$

and

$$p'^{R}_{11}(t) = \begin{cases} 1.4, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.2, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{R}_{11}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases} ; p'^{R}_{12}(t) = \begin{cases} 0.7, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.5, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{R}_{12}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases}$$

$$p'^{R}_{21}(t) = \begin{cases} 0.4, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 1.9, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{R}_{21}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases} ; p'^{R}_{22}(t) = \begin{cases} 0.3, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 1.4, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{R}_{22}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases}$$

$$p'^{I}_{11}(t) = \begin{cases} 0.3, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.9, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{I}_{11}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases} ; p'^{I}_{12}(t) = \begin{cases} 0.3, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 1.6, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{I}_{12}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases}$$

$$p'^{I}_{21}(t) = \begin{cases} 1.3, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.4, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{I}_{21}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases} ; p'^{I}_{22}(t) = \begin{cases} 0.8, & \dot{\tilde{v}}_1(t) > \dot{\tilde{v}}_2(t) \\ 0.4, & \dot{\tilde{v}}_1(t) < \dot{\tilde{v}}_2(t) \\ lim_{p\to s^-} p'^{I}_{22}(s), & \dot{\tilde{v}}_1(t) = \dot{\tilde{v}}_2(t) \end{cases}$$

$$q_{11}^{\prime R}(t)=\begin{cases}0.2, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.8, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{11}^{\prime R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{12}^{\prime R}(t)=\begin{cases}0.9, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.4, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{12}^{\prime R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases}$$

$$q_{21}^{\prime R}(t)=\begin{cases}1.5, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.3, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{21}^{\prime R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{22}^{\prime R}(t)=\begin{cases}0.6, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.7, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{22}^{\prime R}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases}$$

$$q_{11}^{\prime I}(t)=\begin{cases}0.8, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.4, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{11}^{\prime I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{12}^{\prime I}(t)=\begin{cases}0.6, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.4, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{12}^{\prime I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases}$$

$$q_{21}^{\prime I}(t)=\begin{cases}1.5, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 0.7, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{21}^{\prime I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t)\end{cases};\ q_{22}^{\prime I}(t)=\begin{cases}0.4, & \dot{\tilde{\varrho}}_1(t)>\dot{\tilde{\varrho}}_2(t)\\ 1.6, & \dot{\tilde{\varrho}}_1(t)<\dot{\tilde{\varrho}}_2(t)\\ lim_{q\to s^-}q_{22}^{\prime I}(s), & \dot{\tilde{\varrho}}_1(t)=\dot{\tilde{\varrho}}_2(t).\end{cases}$$

Taking $\omega=1$, and the parameters of the systems are as $m_k=\sqrt{2}/2$, $n_k=\sqrt{2}, r_k=1$, and $s_k=1$. The initial conditions of drive-response systems (4.3.1) and (4.3.2) are taken as

$\gamma_1(s)=-1.14+2.46i, \gamma_2(s)=2.15+1.34i,$

$\tilde{\gamma}_1(s)=-1.35-1.67i,\ \tilde{\gamma}_2(s)=1.36+1.45i.$

Then, we can verify that

λ_1=10.4201,

λ_2=10.2426,

i.e., $\lambda_1>\lambda_2$. Hence, according to the Theorem 4.2, the systems (4.3.1) and (4.3.2) are quasi-projective synchronized.

Figures 4.6 and 4.7 present the state evaluation curves with respect to time t of the systems (4.3.1) and (4.3.2) under the controllers with mismatched parameters. Here the state variables of the drive and response systems cannot achieve an identical behavior with time flowing and always having a little difference. In other words, the systems cannot achieve complete synchronization. The evolution curves of the synchronization error in absence of controllers are depicted through Fig. 4.8(a), which indicates that quasi-synchronization is not achieved. Fig. 4.8(b) shows the said synchronization of the error system (4.1.10) under controllers with error bound = 1.9621.

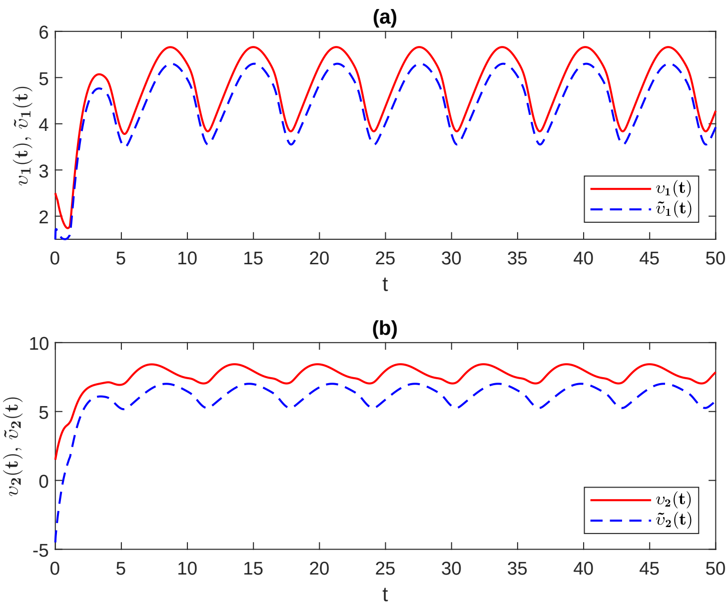

Figure 4.6: The state trajectories of the systems (4.3.1) and (4.3.2) for **(a)** $v_1(t)$, $\widetilde{v}_1(t)$, and **(b)** $v_2(t)$, $\widetilde{v}_2(t)$ with time t for the Example 4.3.2

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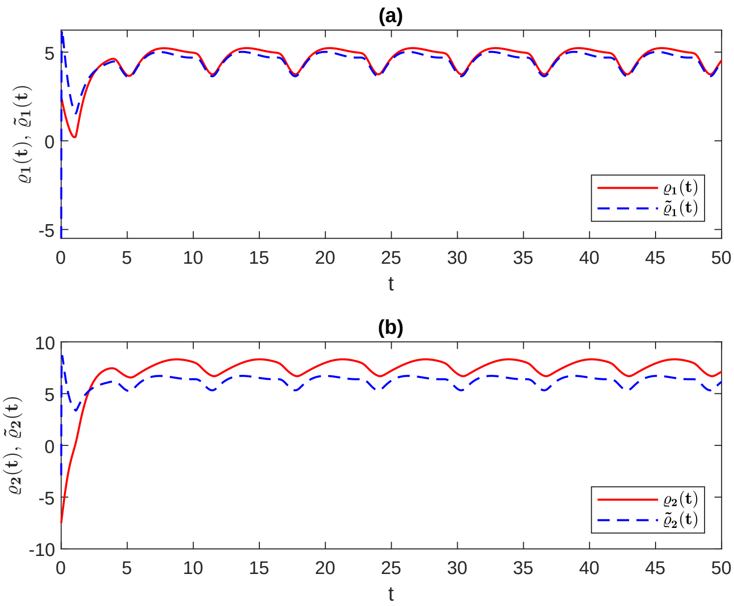

Figure 4.7: The state trajectories of the systems (4.3.1) and (4.3.2) for **(a)** $\varrho_1(t)$, $\widetilde{\varrho}_1(t)$, and **(b)** $\varrho_2(t)$, $\widetilde{\varrho}_2(t)$ with time t for the Example 4.3.2

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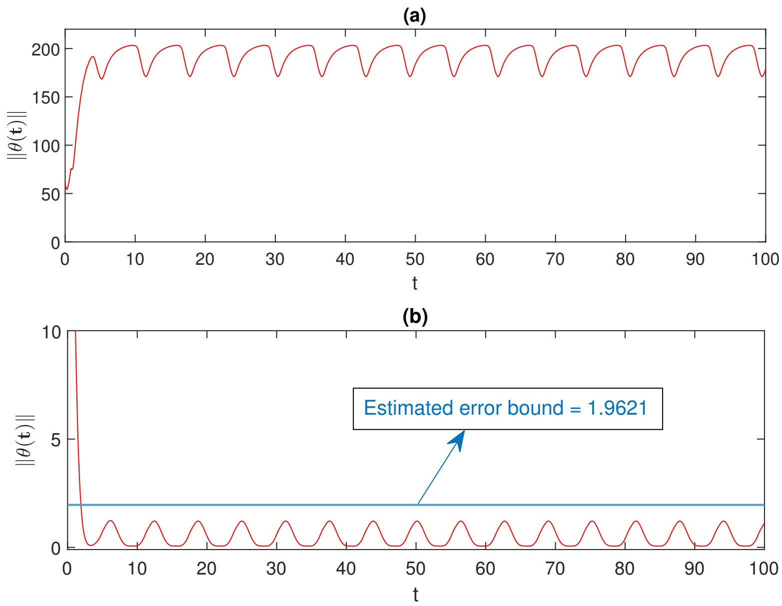

Figure 4.8: Plots of the system (4.3.1) for **(a)** without controllers and **(b)** with controllers for the Example 4.3.2

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4.4 Conclusion

In this chapter, the quasi-projective synchronization problem of two non-identical memristor-based CVRNNs with time-varying delays and mismatched parameters has been presented. To do this, the quasi-projective synchronization criteria have been introduced by matrix measure method as well as Lyapunov stability approach. Next, by constructing suitable controllers, the quasi-projective synchronization criteria of memristor-based CVRNNs with time-varying delays and mismatched parameters have been derived through the proper description of the said methods. The upper bound of synchronization error has been calculated. Finally, the effectiveness of the proposed method through two numerical simulation results have been demonstrated.

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