Chapter 3

Differentiator Design under Non-Differentiability

This chapter discusses about a discrete-time fractional-order differentiator utilizing the super-twisting algorithm based sliding mode control for second-order systems. The differentiator achieves higher performance with respect to the classical ones of integer order in terms of convergence time and robustness. It relaxes the classical boundedness condition required to be satisfied by the second-order derivatives of the signals in conventional differentiators based on sliding mode control. The numerical integration is performed by an implicit Euler discretization method based on the Fractional Adams-Moulton method. This discrete-time implementation technique significantly suppresses the chattering. The significance of the proposed differentiator is demonstrated through a simulation example, comparing with the classical ones.

3.1 Introduction

The problem of differentiator design has been one of the most interesting and important problems in control theory. There are numerous approaches in the literature to design differentiators [136] - [146]. Sliding mode based techniques are one of the most popular schemes used in the design of differentiators. Out of the various sliding mode based techniques, the Super-Twisting Algorithm (STA) is often used by the researchers. Being one of the most popular Higher-Order Sliding Mode (HOSM) algorithm [136] [191], it achieves the behaviour of second-order sliding mode while only requiring the position information for systems of relative degree one with respect to the sliding variable. It is widely used in the field of robust nonlinear control and observation.

This chapter discusses the design of a fractional-order differentiator using STA.

The STA based design generates smoother response as compared to the first-order sliding mode control techniques. However, its conventional explicit Euler implementation suffers from chattering which is an undesirable phenomenon. The magnitude of the chattering increases as the gain values of STA or the sizes of sampling time increases. Various methods have been proposed in the literature to minimise the effects of chattering. The use of sigmoid function is made in place of the signum function for this purpose. This results in a smoother control function. One of the other possible solutions is based on the implicit discretization of the continuous-time system [148] [149] [150]. Full Euler discretization schemes of the STA are proposed in [148], which achieve only standard first-order accuracy of Sliding Mode Control (SMC).

The existing sliding mode based techniques assume a bound on the second derivative of the signal for determining the first-order derivative. However, not all the practical signals can be second-order differentiable which may be the case with the signals in commonly used circuits and systems. This imposes a limitation on the class of signals that can be differentiated by using sliding mode control. Fractional-order operators can be used to overcome this restriction [10] [28] [206] [209] [153] [177]. In this work, the signal is assumed to have only Hölder continuous first-order derivatives. So, this technique addresses a large class of signals which can be differentiated. In this work, Fractional Adams-Moulton (FAM) method has been mainly used as the numerical scheme which is quite effective in dealing with chattering in the case of fractional-order differential equations with discontinuous right hand sides [155] [156]. The Riemann-Liouville definition of fractional-order derivatives has been used throughout the chapter [7]. According to this definition, the signal does not have to be integer-order differentiated for its fractional-order derivative to be defined.

Discretization is the key step in the implementation of control laws [205] [208] [208] [192] [132] [178] [152]. A detailed discussion of the significance of discrete-time fractional-order differentiators in physical systems is presented in [142]. There are various methods for discretizing fractional-order operators [147]. They are based on several methods including continued fraction expansion, radial basis function method, Tustin method, Taylor series, Newton series, least squares method [147] [144] [146]. Differentiator design for signals with error is considered in [143]. Here, an implicit Euler discretization based technique is proposed. The scheme is based on STA and utilizes FAM method for numerical simulation.

3.2 Motivation

Classical control design principles often require the information of the derivative of one function or the other. The function can be the reference function which is to be tracked or of any other signal which is to be differentiated. Generally, integer-order derivative is used in these problems. However, this operator requires the function to be differentiable for its application. Physical systems possess a wide variety of signals. Not all of the associated functions are necessarily differentiable. In such a scenario, there is a strong need for a more generalized operator which relaxes the condition of differentiability and considers a large class of functions. Fractional-order derivatives can be opted to serve the purpose.

3.3 Notations

Let us define the projection function as follows: for a set $\mathcal{A} = [A, B]$ where $A \leq B$,

$$\mathrm{proj}_{[A,B]}(x) = \begin{cases} B & \text{if } x > B \\ x & \text{if } x \in [A, B] \\ A & \text{if } x < A. \end{cases} \tag{3.1}$$

This chapter defines the signum function as:

$$\mathrm{sgn}(x) \triangleq \begin{cases} [-1, 1] & \text{if } x = 0 \\ \{x/|x|\} & \text{if } x \neq 0. \end{cases} \tag{3.2}$$

This type of set-valued definition has been employed in many previous papers [148, 159]. To distinguish the difference, $\mathrm{sign}(\cdot)$ is defined as the single-valued function: $\mathrm{sign}(x) := \mathrm{sgn}(x)$ for $x \neq 0$ and $\mathrm{sign}(x) := 0$ for $x = 0$. For a non-negative scalar $F \geq 0$,

$$x \in F\mathrm{sgn}(y - x) \iff x = \mathrm{proj}_{[-F,F]}(y), \tag{3.3}$$

of which the proofs are given in [159].

In continuous-time, the Riemann-Liouville definition of the fractional-order derivative of order α is given by [6]:

$${}_{t_0}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{(\alpha-m+1)}} d\tau \tag{3.4}$$

where $m \in \mathbb{N}$ such that $m \geq \lceil \alpha \rceil$, $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α with $0 < \alpha < 1$.

3.4 Problem Formulation

Let the signal $x_1 = f(t)$ be second-order differentiable, i.e., $f(t) \in C^2([0, +\infty), \mathbb{R})$, and it can be written in the state-space form as follows:

$$\dot{x}_1 = x_2; \tag{3.5}$$

$$\dot{x}_2 = \ddot{f}(t) \tag{3.6}$$

where $x := [x_1, x_2] \in \mathbb{R}^2$ is the state. The purpose is to design a differentiator based on STA to estimate the state x_2 with the available measurement output x_1 within a finite time. Such an estimator is given as follows [137] [138]:

$$\dot{\hat{x}}_1 = \kappa_1\sqrt{|x_1 - \hat{x}_1|}\text{sgn}(x_1 - \hat{x}_1) + \hat{x}_2 \tag{3.7a}$$

$$\dot{\hat{x}}_2 \in \kappa_2\text{sgn}(x_1 - \hat{x}_1) \tag{3.7b}$$

where $\hat{x} := [\hat{x}_1, \hat{x}_2]$ is the estimation of $x = [x_1, x_2]$, κ_1 and κ_2 are positive constant gains. Defining $e_1 := x_1 - \hat{x}_1$ and $e_2 := x_2 - \hat{x}_2$, the error dynamics of (3.5) and (3.7) has exactly the same form as the STA:

$$\dot{e}_1 = -\kappa_1\sqrt{|e_1|}\text{sgn}(e_1) + e_2 \tag{3.8a}$$

$$\dot{e}_2 \in -\kappa_2\text{sgn}(e_1) + \ddot{f}(t) \tag{3.8b}$$

where $\ddot{f}(t)$ represents the unknown disturbance assumed to be bounded by some constant $\mathcal{F}$, i.e., $|\ddot{f}(t)| \leq \mathcal{F}$, e_2 is unknown as it contains the unknown x_2. It has been shown in [137] that, with properly selected gains κ_1 and κ_2 based on the knowledge of $\mathcal{F}$, e_1 and e_2 converge to the origin in finite time. The problem of the differentiator (3.7) is two folds. First, it imposes a restriction on the signal $x_1 = f(t)$ that it should be second-order differentiable. Second, $\mathcal{F}$ has to be known prior to design the observer (3.7) with proper gains κ_1 and κ_2 which are usually difficult to know in practice.

3.5 Fractional-Order Differentiator

The inherent restrictions of the differentiator (3.7) can be removed by employing the following fractional-order differentiator:

$$\dot{\hat{x}}_1 = \kappa_1\sqrt{|x_1 - \hat{x}_1|}\text{sgn}(x_1 - \hat{x}_1) + \hat{x}_2 \tag{3.9a}$$

$$D^\alpha \hat{x}_2 \in \kappa_2\text{sgn}(x_1 - \hat{x}_1) \tag{3.9b}$$

where $0 < \alpha < 1$ is constant, D^{α} represents the differential operator of fractional order. Here, Riemann-Liouville operator has been used. Defining $e_1 := x_1 - \hat{x}_1$ and $e_2 := \dot{f}(t) - \hat{x}_2$, the error dynamics becomes [46]:

$$\dot{e}_1 = -\kappa_1 \sqrt{|e_1|}\mathrm{sgn}(e_1) + e_2 \tag{3.10a}$$

$$D^{\alpha} e_2 \in -\kappa_2 \mathrm{sgn}(e_1) + D^{1+\alpha} f(t) \tag{3.10b}$$

where $D^{\alpha} e_2 = D^{\alpha} \dot{f}(t) - D^{\alpha} \hat{x}_2 = D^{1+\alpha} f(t) - D^{\alpha} \hat{x}_2$. Integrating (3.9) numerically is a challenging work. On one side, the conventional discrete-time methods are explicit Euler methods, which result in numerical chattering and destroy the estimation accuracy. On the other side, the conventional numerical integration methods for the continuous fractional-order differential equations (FDEs) may not be applicable to FDEs with discontinuous right hand sides as in (3.9). Here, we propose a new discrete-time realization of (3.9) with an implicit Euler method that significantly suppresses the chattering.

3.6 Discretization of Fractional-Order Differential Equations

Sliding mode control techniques have been recently investigated for fractional-order systems [195] [23] [83] [84] [85] [165] [232]. In such techniques, the control law is not a continuous function of the states but it switches from one structure to another according to the position information of the system in the state space delimited by the sliding surface [71] [88] [128] [129] [130] [131] [132] [133] [134]. This switching appears as a discontinuity in the control law which creates significant challenges in such systems [76] [77] [78] [79] [80]. The overall closed-loop system becomes a fractional-order differential equation with discontinuity. This requires to explore the numerical techniques of such differential equations for their effective computation. Some of the reported results in this direction are [157] [71] [72] [73] [74] [75] [233] [234] [177] [178].

Implicit Euler discretization of non-smooth dynamical systems of integer order has been performed in [158]. It is observed that this technique allows to achieve smooth stabilization of the states on the sliding surface unlike the explicit Euler method which generates the undesired chattering phenomenon. In the framework of Filippov's differential inclusion, the implicit Euler method has been validated as a viable method for chattering suppression for integer-order systems [135]. For fractional-order systems with discontinuity, the counterparts of implicit

Euler discretization method have been studied in [155].

Different approaches can be opted for the generalization of implicit Euler method for fractional-order differential equations. According to the approach, there are different numerical methods available in the literature. However, not all the methods have the feature of chattering suppression as found in the integer-order systems. Product-integration rule is a commonly used numerical method in the literature [188]. In this method, the integrand function is replaced by a suitable piecewise interpolant polynomial of a certain degree k in each subinterval of integration. The method converges with order $k + 1$ as the step-size $h \to 0$ [189]. When generalized for fractional-order differential equations, the actual order of convergence of this method does not exceed $\alpha + 1$ as proved in [155]. It has been observed that only the method belonging to the class of fractional Adams method possesses the property of chattering suppression [155].

3.7 Implicit Euler Discretization

Let us redefine $e_2 := v + \varphi$ and rewrite (3.10):

$$\dot{e}_1 = -\kappa_1 |e_1|^{\frac{1}{2}} \mathrm{sgn}(e_1) + v + \varphi \tag{3.11a}$$

$$D^\alpha e_2 = D^\alpha v + D^\alpha \varphi \tag{3.11b}$$

$$D^\alpha v = -\kappa_2 \mathrm{sgn}(e_1); \quad D^\alpha \varphi = \Delta(t) \tag{3.11c}$$

where the unknown e_2 is decomposed into v and φ, v is a known intermediate variable and φ represents the unknown disturbance of which the fractional-order derivative is $D^\alpha \varphi := \Delta(t) := D^{1+\alpha} f(t)$. Let us first consider the full implicit Euler discretization of (3.11a) as follows:

$$\frac{e_{1,k} - e_{1,k-1}}{h} = e_{2,k} - \kappa_2 |e_{1,k}|^{\frac{1}{2}} \mathrm{sgn}(e_{1,k}) \tag{3.12}$$

where $e_{i,k} := e_i(kh), i \in \{1, 2\}$, $k \in \mathbb{N}$, $h > 0$ is the time-stepping size. The discretization of (3.11c) is based on the Fractional Adams-Moulton method [155]:

$$E_{2,k} = \omega_k^{(\alpha-1)} e_{2,0} - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)} e_{2,j} \tag{3.13a}$$

$$e_{2,k} - E_{2,k} = -h^\alpha \kappa_2 \mathrm{sgn}(e_{1,k}) + h^\alpha \Delta_k \tag{3.13b}$$

where $e_{2,k} = v_k + \varphi_k$, $E_{2,k}$ represents the lag-term, $\omega_j^{(\alpha)}$ are the coefficients in the power series expansion of $(1 - \xi)^\alpha$, i.e.,

$$(1 - \xi)^\alpha = \Sigma_{j=0}^{\infty} \omega_j^{(\alpha)} \xi^j$$

and

$$\omega_k^{(\alpha-1)} = \Sigma_{j=0}^{k}\omega_j^{(\alpha)}$$

Substituting (3.13) to (3.12) leads to,

$$e_{1,k} = e_{1,k-1} + hE_{2,k} + hu_{1,k} + h^{1+\alpha}\Delta_k \tag{3.14a}$$

$$u_{1,k} \in -(\kappa_1|e_{1,k}|^{\frac{1}{2}} + h^{\alpha}\kappa_2)\mathrm{sgn}(e_{1,k}) \tag{3.14b}$$

where $E_{2,k}$ is unknown to us because it is the sum of $e_{2,j}, j = 1, \cdots, k-1$, containing the unknown φ_k shown in (3.11). Considering the unknown $E_{2,k}$ and Δ_k as disturbance leads to the following nominal version of (3.14):

$$\tilde{e}_{1,k} = e_{1,k-1} + h\tilde{u}_{1,k} \tag{3.15a}$$

$$\tilde{u}_{1,k} \in -(\kappa_1|\tilde{e}_{1,k}|^{\frac{1}{2}} + h^{\alpha}\kappa_2)\mathrm{sgn}(\tilde{e}_{1,k}) \tag{3.15b}$$

where $\tilde{e}_{1,k}$ is the nominal state of $e_{1,k}$. From (3.15), for $\tilde{e}_{1,k} \neq 0$

$$|\tilde{e}_{1,k}| + h\kappa_1|\tilde{e}_{1,k}|^{\frac{1}{2}} + h^{1+\alpha}\kappa_2 = |e_{1,k-1}| \tag{3.16}$$

which means $|\tilde{e}_{1,k}| < |e_{1,k-1}|$. Due to the fact that larger gains lead to a shorter time of approaching the sliding surface, the gain,

$$(\kappa_1|\tilde{e}_{1,k}|^{\frac{1}{2}} + h^{\alpha}\kappa_2)$$

in (3.15) is replaced by,

$$(\kappa_1|e_{1,k-1}|^{\frac{1}{2}} + h^{\alpha}\kappa_2)$$

without explicitly solving (3.16). This results into the following equations:

$$\tilde{e}_{1,k} = e_{1,k-1} + h\tilde{u}_{1,k} \tag{3.17a}$$

$$\tilde{u}_{1,k} \in -(\kappa_1|e_{1,k-1}|^{\frac{1}{2}} + h^{\alpha}\kappa_2)\mathrm{sgn}(\tilde{e}_{1,k}) \tag{3.17b}$$

Using (3.3), one can obtain the equivalence of (3.17):

$$\tilde{e}_{1,k} = e_{1,k-1} + h\tilde{u}_{1,k} \tag{3.18}$$

$$\tilde{u}_{1,k} = -\frac{1}{h}\mathrm{proj}_{C_{1,k}}(e_{1,k-1}) \tag{3.19}$$

where proj$(\cdot)$ is defined as (3.1) and $C_{1,k}$ is the set defined as:

$$C_{1,k} := [-D_{1,k}, D_{1,k}] \tag{3.20a}$$

$$D_{1,k} := h\kappa_1|e_{1,k-1}|^{\frac{1}{2}} + h^{1+\alpha}\kappa_2 \tag{3.20b}$$

Let us define the sliding surface for (3.18):

Definition 3.1 *The discrete-time sliding surface of* (3.17) *or its equivalence* (3.18) *is defined as,*

$$\Sigma_d = \{\tilde{e}_{1,k} \in \mathbb{R} | \tilde{e}_{1,k} = 0\}$$

After a finite number of steps, (3.18) is on the sliding surface Σ_d and this is guaranteed by the following lemma:

Lemma 3.2 *Consider the discrete-time system* (3.17) *or its equivalence* (3.18) *with* $e_{1,0} := e_1(0)$. *After a finite number of steps,*

$$k_0 := \lceil |e_{1,0}|/(h^{1+\alpha}k_2) \rceil$$

(3.17) *or its equivalence* (3.18) *slides on the surface* Σ_d*, i.e.,* $\tilde{e}_{1,k} = 0$ *for* $k \geq k_0 + 1$.

Proof According to (3.18) and the definition of proj(·) function (3.1), for $|e_{1,k}| > D_{1,k} > 0$, one has, $\tilde{e}_{1,k} = e_{1,k-1} - D_{1,k}$ if $e_{1,k-1} > D_{1,k}$ and $\tilde{e}_{1,k} = e_{1,k-1} + D_{1,k}$ if $e_{1,k-1} < -D_{1,k}$. While for $|e_{1,k-1}| \leq D_{1,k}$, one has, $\tilde{e}_{1,k} = e_{1,k-1} - e_{1,k-1} = 0$. Let us begin with the case $|e_{1,0}| > D_{1,1}$. From the above three cases, after at most,

$$k_0 := \lceil |e_{1,0}|/(h^{1+\alpha}k_2) \rceil \geq 0$$

where $\lceil \cdot \rceil$ is the ceiling function, $|e_{1,k}| \leq D_{1,k}$ is satisfied and one has $\tilde{e}_{1,k} = 0$ at $k \geq k_0 + 1$. This completes the proof.

From Lemma 3.2, when (3.18) is on the sliding surface Σ_d, one has $\tilde{e}_{1,k} = 0$, $|e_{1,k-1}| \leq D_{1,k}$ and $\tilde{u}_{1,k} = -e_{1,k-1}/h$ for $k \geq k_0 + 1$ from definition (3.1). So, from above analysis, (3.18) changes into:

$$\tilde{e}_{1,k} = e_{1,k-1} + h\tilde{u}_{1,k}, \tag{3.21a}$$

$$\tilde{u}_{1,k} = -\frac{1}{h}\text{proj}_{C_{1,k}}(e_{1,k-1}) \tag{3.21b}$$

$$C_{1,k} := \begin{cases} [-D_{1,k}, D_{1,k}] & \text{if } |e_{1,k-1}| > D_{1,k} \\ [-D_{2,k}, D_{2,k}] & \text{else} \end{cases} \tag{3.21c}$$

where $D_{2,k} := h^{1+\alpha}k_2$. Consider that $u_{1,k}$ in (3.14) is approximated by $\tilde{u}_{1,k}$. Then, one has the following expression:

$$e_{1,k} = e_{1,k-1} + h\tilde{u}_{1,k} + hE_{2,k} + h^{1+\alpha}\Delta_k \tag{3.22a}$$

$$\tilde{u}_{1,k} = -\frac{1}{h}\text{proj}_{C_{1,k}}(e_{1,k-1}) \tag{3.22b}$$

By comparing (3.22) to (3.18), from Lemma 3.2, it is easy to derive the following conclusion:

Lemma 3.3 *Consider the discrete-time system* (3.22) *with the initial state* $e_{1,0} := e_1(t_0)$. *Assume that*

$$\sup_k |h^\alpha \Delta_k + E_{2,k}| < h^\alpha \kappa_2$$

Then, at most after a finite number of steps $k_0 := \lceil |e_{1,0}|/(h^{\alpha+1}\kappa_2)\rceil$, *the state of* (3.22) *stays in the following subset:*

$$\mathcal{R}_1 = \{e_{1,k} \in \mathbb{R} | e_{1,k} = h(E_k + h^\alpha \Delta_k)\}$$

for $k \geq k_0$.

Proof By employing the relation (3.3), the discrete-time system (3.22) can be equivalently rewritten as,

$$e_{1,k} \in e_{1,k-1} - D_{1,k}\mathrm{sgn}(\tilde{e}_{1,k}) + hE_{2,k} + h^{1+\alpha}\Delta_k$$

with $\tilde{e}_{1,k}$ defined in (3.17) and $D_{1,k}$ is given in (3.20). Then, by applying Proposition 1 in [159], one can prove the above conclusion.
This completes the proof.

Remark 3.4 *The scheme* (3.22) *is based on a semi-implicit Euler method because the gain,*

$$(\kappa_1 |\tilde{e}_{1,k}|^{\frac{1}{2}} + h^\alpha \kappa_2)$$

in (3.15) *is replaced by,*

$$(\kappa_1 |e_{1,k-1}|^{\frac{1}{2}} + h^\alpha \kappa_2)$$

as in [159, Section III.B.2]. In [148], a fully implicit Euler method was employed and $\tilde{e}_{1,k}$ *is analytically obtained. From* (3.21)*, one can see that this semi-implicit Euler method only works in the approaching phase, i.e.,* $\tilde{e}_{1,k} \neq 0$. *During the sliding phase,* (3.21) *is a scheme based on the fully implicit Euler method because it is equivalent to* (3.15). *The advantage of such a strategy is that it does not need to solve the equation* (3.15) *and the approximation only happens during the approaching phase with a quicker convergence manner.*

By inserting the correction term (3.22b), the estimation accuracy is determined by the disturbance $E_{2,k} + h^\alpha \Delta_k$ attenuated by the factor h. As $e_{2,k} = (v_k + \varphi_k)$ from (3.11), the estimation accuracy can be improved by using v_k. Assume that the discrete-time state of (3.22) belongs to $\mathcal{R}_1$ at some $k \geq k_0$. Thus, one has

$$e_{1,k} = hE_{2,k} + h^{1+\alpha}\Delta_k \tag{3.23a}$$

$$E_{2,k} = \omega_k^{(\alpha-1)}(v_0 + \varphi_0) - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)}(v_j + \varphi_j) \tag{3.23b}$$

where, v_k and φ_k are given as:

$$v_k = V_k + h^\alpha u_{2,k} \tag{3.24a}$$

$$u_{2,k} \in -\kappa_2 \mathrm{sgn}(e_{1,k}) \tag{3.24b}$$

$$\varphi_k = \Phi_k + h^\alpha \Delta_k; \tag{3.24c}$$

$$\Phi_k := \omega_k^{(\alpha-1)} \varphi_0 - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)} \varphi_j \tag{3.24d}$$

$$V_k := \omega_k^{(\alpha-1)} v_0 - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)} v_j \tag{3.24e}$$

according to the equations (3.11). From above equations, one has

$$E_{2,k} = V_k + \Phi_k$$

and

$$e_{1,k} = hV_k + h\varphi_k$$

where V_k is a known input that can be managed. To further improve the estimation accuracy, let us replace the input V_k in $e_{1,k} = hV_k + h\varphi_k$ by v_k. So, the correction term $u_{2,k}$ can be employed to compensate the matched disturbance φ_k:

$$e_{1,k} = hv_k + h\varphi_k \tag{3.25a}$$

$$v_k = V_k + h^\alpha u_{2,k}; \tag{3.25b}$$

$$u_{2,k} \in -k_2 \mathrm{sgn}(e_{1,k}) \tag{3.25c}$$

As $e_{1,k} = hv_k + h\varphi_k$ is satisfied only for $k \geq k_0$, let us split $E_{2,k}$ into two parts when $k = k_0 + i$, i.e. $E_{2,k} = \bar{E}_{2,k} + R_{1,k}$ where, $i \geq 1$ is an integer. According to (3.25),

$$\begin{aligned} e_{1,k} = hv_k + h\varphi_k &= hV_k + h^{1+\alpha} u_{2,k} + h\Phi_k + h^{1+\alpha}\Delta_k \\ &= h\bar{E}_{2,k} + hR_{1,k} + h^{1+\alpha}(u_{2,k} + \Delta_k) \end{aligned} \tag{3.26a}$$

$$v_k = V_k + h^\alpha u_{2,k}; \tag{3.26b}$$

$$u_{2,k} \in -\kappa_2 \mathrm{sgn}(e_{1,k}) \tag{3.26c}$$

where,

$$\bar{E}_{2,k} := \omega_k^{(\alpha-1)} e_{2,0} - \sum_{j=0}^{k_0-1} \omega_{k-j}^{(\alpha)} e_{2,j}; \tag{3.27a}$$

$$\begin{aligned} R_{1,k} &:= -\sum_{j=k_0}^{k_0+i-1} \omega_{k-j}^{(\alpha)} v_j - \sum_{j=k_0}^{k_0+i-1} \omega_{k-j}^{(\alpha)} \varphi_j \\ &= -\frac{1}{h} \sum_{j=k_0}^{k_0+i-1} \omega_{k-j}^{(\alpha)} e_{1,j} \end{aligned} \tag{3.27b}$$

For $k > k_0$, $\bar{E}_{1,k}$ and $R_{1,k}$ can be calculated separately. Particularly, because $e_{1,j}$ are known to us for $k > k_0$, $e_{2,j}$ in (3.27a) $j = 1, \cdots, k_0 - 1$ are calculated from (3.12) and $e_{2,0}$ can be estimated from (3.11a) using the explicit Euler method. Then, consider the following nominal form of (3.26):

$$\tilde{e}_{1,k} = h(\bar{E}_{2,k} + R_{1,k}) + h^{1+\alpha} \tilde{u}_{2,k} \tag{3.28a}$$

$$v_k = V_k + h^\alpha \tilde{u}_{2,k}; \; \tilde{u}_{2,k} = -\kappa_2 \mathrm{sgn}(\tilde{e}_{1,k}) \tag{3.28b}$$

Comparing (3.26c) to (3.15b), from (3.22b), it is easy to explicitly calculate $\tilde{u}_{2,k}$:

$$\tilde{e}_{1,k} = h(\bar{E}_{2,k} + R_{1,k} + h^\alpha \tilde{u}_{2,k}); v_k = V_k + h^\alpha \tilde{u}_{2,k} \tag{3.29a}$$

$$\tilde{u}_{2,k} = -\frac{1}{h^{1+\alpha}} \mathrm{proj}_{C_{2,k}} \left(h\bar{E}_{2,k} + hR_{1,k} \right) \tag{3.29b}$$

where, $C_{2,k} := [-D_{2,k}, D_{2,k}]$ with $D_{2,k}$ defined in (3.21). Using $\tilde{u}_{2,k}$ from the nominal dynamics (3.29) in the perturbed dynamics (3.25):

$$e_{1,k} = hv_k + h\varphi_k; \quad v_k = V_k + h^\alpha \tilde{u}_{2,k}; \tag{3.30a}$$

$$\tilde{u}_{2,k} = -\frac{1}{h^{1+\alpha}} \mathrm{proj}_{C_{2,k}} \left(h\bar{E}_{2,k} + hR_{1,k} \right), \tag{3.30b}$$

which has the following property:

Theorem 3.5 *Consider* (3.30) *with the initial state* $e_{1,0} := e_1(0)$. *Assume that* $e_{1,k}$ *is within the subset* $\mathcal{R}_1$ *for some* $k \geq k_0$ *and* $\sup_k |\Delta_k| < \mu$ *with some* $\mu > 0$. *Then,* $e_{1,k}$ *stays within the following subset:*

$$\mathcal{R}_2 = \{e_k \in \mathbb{R} | e_{1,k} = he_{2,k}, |e_{2,k}| < \mu h^\alpha\}$$

for $k \geq k_0 + 1$.

Proof Because $e_{1,k}$ is staying within the subset $\mathcal{R}_1$ for some $k \geq k_0$, from (3.29a), one has $e_{1,k} = hv_k + h\varphi_k, k = k_0, k_0+1, k_0+2, \cdots$. From the definition (3.23b)-(3.24e), one can define $\bar{V}_k$ and $\bar{\Phi}_k$ because of $\varphi_k = \Phi_k + h^\alpha \Delta_k$. As $e_{1,k-1} = hv_{k-1} + h\varphi_{k-1}$, the equation becomes,

$$e_{1,k} = e_{1,k-1} + h^{1+\alpha}\tilde{u}_{2,k} + h^{1+\alpha}\Delta_k \tag{3.31}$$

where, $\tilde{u}_{2,k}$ is given as in (3.29b). From Lemma 3.2 and Lemma 3.3, the fact that $e_{1,k}$ is within $\mathcal{R}_1$ implies that $\tilde{e}_{1,k}$ is on Σ_d. Then, $\tilde{e}_{1,k-1} = 0$, $e_{1,k-1} = \text{proj}_{C_k}(e_{1,k-1})$ and $\tilde{u}_{2,k} = -\frac{1}{h^{1+\alpha}}\text{proj}_{C_k}(e_{1,k-1}) = -e_{1,k-1}/h^{1+\alpha}$. Substituting $\tilde{u}_{2,k} = -e_{1,k-1}/h^{1+\alpha}$ into (3.31) leads to $e_{1,k} = h^{1+\alpha}\Delta_k$. Due to $e_{1,k} = h(v_k + \varphi_k) = he_{2,k}$ for $k \geq k_0$, one has $|e_{2,k}| = h^\alpha|\Delta_k| < \mu h^\alpha$ for $k \geq k_0$. Therefore, $e_{1,k}$ is within the subset $\mathcal{R}_2$ for $k \geq k_0 + 1$.
This completes the proof.
From Lemma 3.3 to Theorem 3.5, all of them are from the time step $k \geq k_0$, including v_k and $\tilde{u}_{2,k}$ (3.30). However, v_k has to be calculated from $k = 0$ to $k_0 - 1$, which depends on $\tilde{u}_{2,k}$ in (3.30) and is not defined. To be consistent with Lemma 3.3, let us define $\tilde{u}_{2,k}$ and v_k for $k > k_0$ and $k \leq k_0$:

$$e_{1,k} = e_{1,k-1} + hu_k + h\varphi_k \tag{3.32a}$$

$$u_k = \tilde{u}_{1,k} + v_k; \quad \tilde{u}_{1,k} = -\frac{1}{h}\text{proj}_{C_{1,k}}(e_{1,k-1}) \tag{3.32b}$$

$$v_k = V_k + h^\alpha \tilde{u}_{2,k}, \tilde{u}_{2,k} = -\frac{1}{h^{1+\alpha}}\text{proj}_{C_{1,k}}(E_{1,k}) \tag{3.32c}$$

where, $E_{1,k} := h(\bar{E}_{2,k} + R_{1,k})$. Applying the proposed integration method of the fractional-order STA (3.32) to the fractional-order differentiator (3.9):

$$\begin{aligned}
&\hat{x}_{1,k} = \hat{x}_{1,k-1} - h\tilde{u}_{1,k} + h\hat{x}_{2,k}; \hat{x}_{2,k} = \hat{X}_{2,k} - h^\alpha \tilde{u}_{2,k};\\
&\tilde{u}_{1,k} = -\frac{1}{h}\text{proj}_{C_{1,k}}(e_{1,k-1}); \tilde{u}_{2,k} = -\frac{1}{h^{1+\alpha}}\text{proj}_{C_{1,k}}(E_{1,k})\\
&\hat{X}_{2,k} := \omega_k^{(\alpha-1)}\hat{x}_{2,0} - \sum_{j=0}^{k-1}\omega_{k-j}^{(\alpha)}\hat{x}_{2,j}.
\end{aligned} \tag{3.33}$$

The discrete-time differentiator (3.33) is obtained by comparing (3.11) with (3.9) and its implicit Euler discretization (3.32).

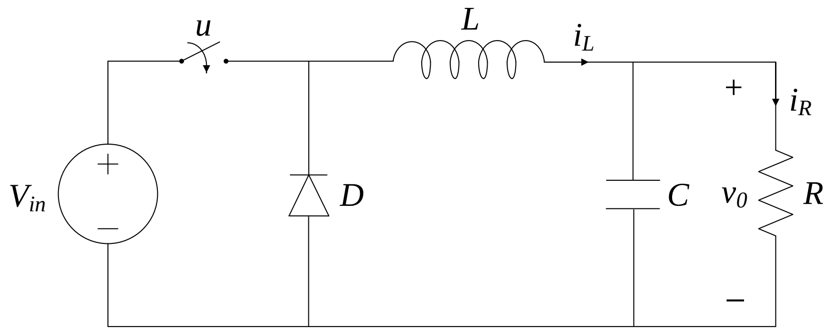

Figure 3.1: Basic circuit diagram of Buck Converter

3.8 Illustrative Example

Let us consider the example of buck converter circuit system [179] [180] [181] [182], as shown in Fig. 3.1:

$$\begin{aligned}\frac{\mathrm{d}v_0}{\mathrm{d}t} &= \frac{1}{C}\left(i_L - \frac{v_0}{R}\right)\\ \frac{\mathrm{d}i_L}{\mathrm{d}t} &= \frac{1}{L}\left(uV_{in} - v_0\right)\end{aligned} \tag{3.34}$$

where i_L is the inductor current, V_{in} is the voltage of DC, v_0 is the output voltage, L, C, and R are the values of inductor, capacitor and resistor, respectively, and u is the switch input.

To make the output v_0 track some desired trajectory in the presence of model uncertainties of parameters L, C, R, the model (3.34) can be rewritten as the canonical form [179, 180]:

$$\dot{x}_1 = x_2 \tag{3.35}$$

$$\dot{x}_2 = \phi(t, x) \tag{3.36}$$

where,

$$x_1 := v_0$$

$$x_2 := i_L/C - v_0/(RC)$$

$$\phi(t, x) = v_0/(R^2C^2) - v_0/(LC) - i_L/(RC^2) + V_{in0}/(LC)u$$

The state x_2 is required to be known for designing the control law u and it is usually estimated by the conventional integer-STA based differentiator (3.7). However, when x_1 is only first-order time differentiable and the derivative of x_2 does not exist at some time instants, the fractional-order differentiator (3.9) can be employed to estimate x_2.

Due to the focus on derivative estimation in this manuscript, for simplicity, let us suppose the output x_1 of the above circuit is measurable, for which the first-order derivative has to be

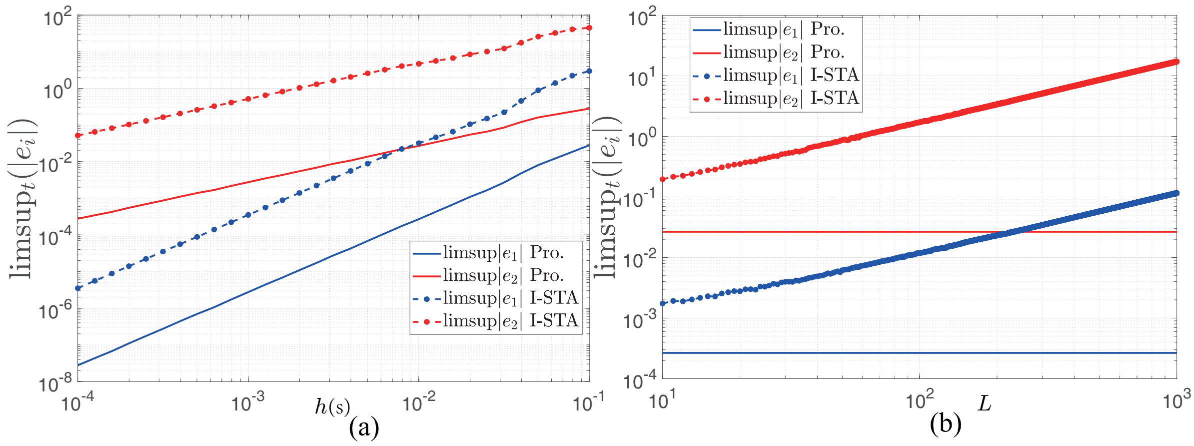

Figure 3.2: Estimation Precision of $|e_1|$ and $|e_2|$ of system (3.35) by using the differentiator (3.7) with conventional explicit Euler method denoted by "I-STA" vs the proposed fractional-order differentiator (3.9) with the scheme (3.33) denoted by "Pro." (a) $|e_1|$ and $|e_2|$ w.r.t. the time stepping size h and the parameters are set as $\alpha = 0.8, \kappa_1 = 1.5\sqrt{L}, \kappa_2 = 1.1L, L = 300$. (b) $|e_1|$ and $|e_2|$ w.r.t. the gain parameter L and the parameters are set as $\alpha = 0.8, \kappa_1 = 1.5\sqrt{L}, \kappa_2 = 1.1L, h = 0.01$s.

determined in the presence of an unknown model uncertainty,

$$\phi(t, x) = 2|z(t)|\ddot{z}(t) + 2\dot{z}^2(t)\text{sgn}(z(t))$$

with $z(t) = \sin(t)$. The estimation $\hat{x}_2$ of the integer-order differentiator (3.7) and proposed fractional-order differentiator (3.9) realized by (3.33) are shown Fig. 3.3. One can see that $e_2 = x_2 - \hat{x}_2$ of (3.7) is much larger than that of the proposed one. Some magnitude of chattering exists both in $\hat{x}_1$ and $\hat{x}_2$ of the integer-order differentiator (3.7).

The estimation precision of the fractional-order differentiator (3.9) and that of the differentiator (3.7) are compared w.r.t. the sampling time h and the gain parameter L, as shown in Fig. 3.2. It shows that with fixed $L = 300$ and any values of h between 10^{-4}s and 10^{-1}s as well as with fixed $h = 10^{-2}$s and any values of L between 10 and 10^3, the proposed fractional-order differentiator achieves a higher asymptotical estimation accuracy, i.e., smaller magnitudes of $\limsup |e_i|$ than these of the conventional integer-order differentiator (3.7). The estimation accuracy of the proposed scheme is independent from the overestimation of gains κ_1 and κ_2. These results are illustrated by the fact that the curves of $\limsup |e_i|$ of the proposed one (3.9) are under the respective ones of the integer-STA (3.7). The robustness of the proposed technique is independent of the discretization step h [155]. However, the accuracy decreases with decreasing values of the fractional order α.

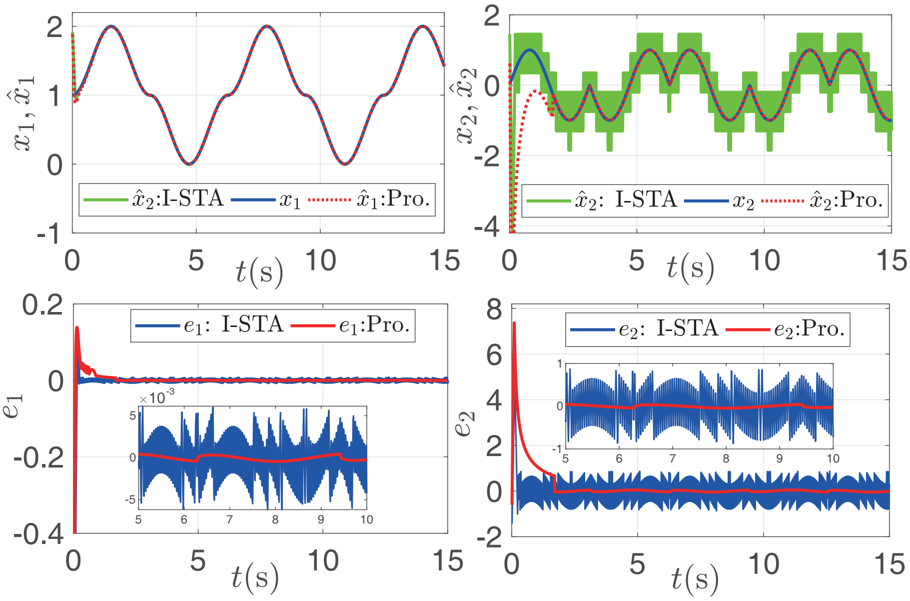

Figure 3.3: Estimated states $\hat{x}_1$ and $\hat{x}_2$ of system (3.35) in the presence of model uncertainty $\phi(t, x) = -2|\sin(t)|\sin(t) + 2\cos^2(t)\text{sgn}(\sin(t))$ with the proposed fractional-order differentiator (3.9) with the scheme (3.33) denoted by "Pro." and the conventional integer-STA based differentiator (3.7) denoted by "I-STA". The parameters $\alpha = 0.8, \kappa_1 = 1.5\sqrt{L}, \kappa_2 = 1.1L, L = 50, h = 0.01$s.

3.9 Summary

The proposed scheme keeps the precision consistency of the STA, i.e, $|x_1| \leq h^{1+\alpha}\mu_1$ and $|x_2| \leq h^{\alpha}\mu_2$ with the fractional-order parameter $0 < \alpha < 1$ and constants $\mu_1 > 0$, $\mu_2 > 0$. Further, it is observed that the results with the proposed fractional-order differentiator using Fractional Adams-Moulton method is more accurate than the conventional integer-order differentiator with explicit Euler method. Also, it estimates the value of the derivative at the non-differentiable points exactly where the integer-order differentiator fails. Numerical examples are taken to confirm these results. Thus, the technique can be useful for obtaining the required derivatives of output signals in circuit systems with high accuracy. More strict Lyapunov approaches remain to be explored.