Chapter 2

On pair correlation of Hermite coefficients of functions from the Hardy class

2.1 Statement of the result

Theorem 2.1. *Let $t > 0$. If $f \in E(\tanh 2t, \tanh 2t)$ then*

$$\langle f, \varphi_n \rangle + \left(\frac{n(n+2)}{(n+1)(n+3)} \right)^{1/2} e^{4t} \langle f, \varphi_{n+4} \rangle = O\left(n^{-3/4} e^{-nt} \right),$$

for $n = 1, 2, \ldots$. If $t_0 > 0$ then this estimate is uniform for $t \in [t_0, \infty)$.

Remark 2.2. In [13], Vemuri used (1.2) to prove a uniform Gaussian bound for the solution of the harmonic oscillator Schrödinger equation with initial value $\varphi_0 \in H(\tanh 2t)$, and conjectured the sharp bound. Recently, substantial progress towards the conjecture was made in [8]. We hope that Theorem 2.1 could be useful in this direction.

2.2 Proof of the result

Define $\gamma_n(t) = \left(\frac{4n(n+2)}{\mu}\right)^{1/4} e^{it}$ for $0 \le t \le 2\pi$. By the Cauchy integral formula for derivatives, we have $Bf(w) = \sum_{n=0}^{\infty} c_n w^n$ where

$$c_n = \frac{1}{2\pi i}\int_{\gamma_n} \frac{Bf(w)}{w^{n+1}}\, dw.$$

Therefore

$$\left| c_n + \frac{4n(n+2)}{\mu} c_{n+4} \right| = \frac{1}{2\pi}\left| \int_{\gamma_n} \left(w^4 + \frac{4n(n+2)}{\mu} \right) \frac{Bf(w)}{w^{n+5}} dw \right|.$$

Thus

$$\begin{aligned} \left| c_n + \frac{4n(n+2)}{\mu} c_{n+4} \right| &\le \frac{1}{2\pi}\left(\frac{\mu}{4n(n+2)}\right)^{n/4} \int_0^{2\pi} \left| e^{4it} + 1 \right| \left| Bf\left(\left(\frac{4n(n+2)}{\mu}\right)^{1/4} e^{it} \right) \right| dt \\ &= \frac{1}{\pi}\left(\frac{\mu}{4n(n+2)}\right)^{n/4} \sum_{k=1}^{4} \int_{\frac{(k-1)\pi}{2}}^{\frac{k\pi}{2}} |\cos 2t| \left| Bf\left(\left(\frac{4n(n+2)}{\mu}\right)^{1/4} e^{it} \right) \right| dt. \end{aligned} \tag{2.1}$$

By inequalities (1.8), (1.9), and (1.10) we have

$$\int_0^{\pi/2} |\cos 2t| \left| Bf\left(\left(\frac{4n(n+2)}{\mu}\right)^{1/4} e^{it} \right) \right| dt \le C\sqrt{\frac{2}{1+a}}(I_n + J_n + K_n)$$

where

$$\begin{aligned} I_n &= \int_0^{\theta_0} |\cos 2t| \exp\left(\frac{\sqrt{n(n+2)}(\mu + (1-\mu)\sin^2 t)}{2\sqrt{\mu}} \right) dt, \\ J_n &= \int_{\theta_0}^{\frac{\pi}{2}-\theta_0} |\cos 2t| \exp\left(\frac{\sqrt{n(n+2)}}{2} \sin 2t \right) dt, \quad \text{and} \\ K_n &= \int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}} |\cos 2t| \exp\left(\frac{\sqrt{n(n+2)}(\mu + (1-\mu)\cos^2 t)}{2\sqrt{\mu}} \right) dt. \end{aligned}$$

For $n \in \mathbb{N}$, let $l_n : (\theta_0, \frac{\pi}{4}) \to \mathbb{R}$ be defined by

$$l_n(t) = \sin 2t + \frac{2}{\sqrt{n(n+2)}} \log|\cos 2t|\,.$$

Clearly, $l_n(t) = l_n(\frac{\pi}{2} - t)$ *for all* $t \in (\frac{\pi}{4}, \frac{\pi}{2} - \theta_0)$. It follows that

$$J_n = \int_{\theta_0}^{\frac{\pi}{2}-\theta_0} \exp\left(\frac{\sqrt{n(n+2)}}{2} l_n(t)\right) dt = 2\int_{\theta_0}^{\frac{\pi}{4}} \exp\left(\frac{\sqrt{n(n+2)}}{2} l_n(t)\right) dt. \tag{2.2}$$

Since

$$\left(\sqrt{\frac{n}{n+2}}\right)^2 + \left(\sqrt{\frac{2}{n+2}}\right)^2 = 1,$$

there exists a unique $t_0 \in (0, \frac{\pi}{4})$ such that

$$\sin 2t_0 = \sqrt{\frac{n}{n+2}} \quad \text{and} \quad \cos 2t_0 = \sqrt{\frac{2}{n+2}}.$$

Thus, we have

$$l_n'(t_0) = 2\cos 2t_0 - \frac{4\tan 2t_0}{\sqrt{n(n+2)}} = 0, \tag{2.3}$$

and

$$l_n''(t_0) = -4\sin 2t_0 - \frac{4\sec^2 2t_0}{\sqrt{n(n+2)}} = \frac{-8(n+1)}{\sqrt{n(n+2)}} < 0. \tag{2.4}$$

Also, observe that, for large enough $n \in \mathbb{N}$, we have $\theta_0 \leq t_0 < \frac{\pi}{4}$. Therefore, we may estimate J_n by the use of Laplace's method (see theorem 1.12). In our case (see equation (2.2)), $G(t) = 1$, $x = \frac{\sqrt{n(n+2)}}{2}$, and $H(t) = l_n(t)$. Therefore

$$\left[\frac{-2\pi}{xH''(t_0)}\right]^{1/2} = \left[\frac{\pi}{2(n+1)}\right]^{1/2}, \tag{2.5}$$

and

$$e^{xH(t_0)} = e^{\frac{\sqrt{n(n+2)}}{2}\left(\sqrt{\frac{n}{n+2}} + \frac{2}{\sqrt{n(n+2)}}\log\sqrt{\frac{2}{n+2}}\right)} = \sqrt{\frac{2}{n+2}}e^{n/2}. \tag{2.6}$$

From equations (2.2), (2.3), (2.4), (2.5), (2.6), and Theorem 1.12, we get

$$J_n \sim 2\sqrt{\pi}\, n^{-1} e^{n/2}.$$

Observe that

$$\mu + (1-\mu)\sin^2 t \leq \mu + (1-\mu)\sin^2\theta_0 \quad \textit{for all} \quad t \in [0, \theta_0],$$

and

$$\mu + (1-\mu)\sin^2\theta_0 \leq \sqrt{\mu}\sin 2t \quad \textit{for all} \quad t \in \left[\theta_0, \frac{\pi}{4}\right].$$

Therefore

$$I_n \leq \sin 2\theta_0 \exp\left(\frac{\sqrt{n(n+2)}(\mu + (1-\mu)\sin^2\theta_0)}{2\sqrt{\mu}}\right),$$

and

$$J_n \geq 2(1 - \sin 2\theta_0) \exp\left(\frac{\sqrt{n(n+2)}(\mu + (1-\mu)\sin^2\theta_0)}{2\sqrt{\mu}}\right).$$

Thus

$$I_n = O\left(J_n\right).$$

Note that the implied constant depends on μ. Therefore, the estimate in the theorem is not uniform in t. However, if $\mu_0 \in (0,1)$ then the constant may be chosen independently of μ for $\mu \in (0, \mu_0)$. Clearly

$$K_n = I_n = O\left(J_n\right).$$

Hence, we get

$$\int_0^{\pi/2} |\cos 2t| \left| Bf\left(\left(\frac{4n(n+2)}{\mu}\right)^{1/4} e^{it}\right)\right| dt = O\left(n^{-1}e^{n/2}\right).$$

The other three integrals in (2.1) are also $O\left(n^{-1}e^{n/2}\right)$ by equation (1.10), and the fact that the right hand sides of inequalities (1.8) and (1.9) do not change when we replace θ by $\pi-\theta$ or $2\pi-\theta$. We conclude from equation (2.1) that

$$c_n + \frac{4n(n+2)}{\mu}c_{n+4} = O\left[n^{-1}\left(\frac{\sqrt{\mu}e}{2n}\right)^{n/2}\right]. \tag{2.7}$$

On the other hand, we obtain

$$\langle f, \varphi_n\rangle + \frac{1}{\mu}\left(\frac{n(n+2)}{(n+1)(n+3)}\right)^{1/2}\langle f, \varphi_{n+4}\rangle = \sqrt{2^n n! \pi^{1/2}}\left[c_n + \frac{4n(n+2)}{\mu}c_{n+4}\right].$$

It follows from equation (2.7) and Stirling's formula that

$$\langle f, \varphi_n\rangle + \frac{1}{\mu}\left(\frac{n(n+2)}{(n+1)(n+3)}\right)^{1/2}\langle f, \varphi_{n+4}\rangle = O\left(n^{-3/4}\ \mu^{n/4}\right).$$

Taking $\mu = e^{-4t}$ gives the result stated in Theorem 2.1. □