Chapter 3

Finite Direct Projective Modules

In this chapter, we introduce the notion of finite direct projective modules, which is an extension of the concept of direct projective modules [34] and dual to the concept of finite direct injective modules [30]. To illustrate this extension, a counterexample is given. Actually, the class of finite direct projective modules lies between the class of direct projective modules and the class of simple direct projective modules[18]. We study the properties of finite direct projective modules with respect to their summands. Also, we provide the characterizations of von Neumann regular rings in terms of the endomorphism ring of finite direct projective modules. Moreover, we investigate the relationships among various types of modules, like Rickart modules, D_3 modules, direct projective modules, finite direct projective modules, and endoregular modules. Finally, we provide the characterization of semi-hereditary rings, S-rings, and semi-simple Artinian rings in terms of finite direct projective modules.

3.1 Finite Direct Projective Modules

Definition 3.1.1. *An R-module M is called finite direct projective if for any submodule T of M and any direct summand K of M that is finitely generated, if M/T is isomorphic to K, then T is also a direct summand of M.*

Lemma 3.1.2. *An R-module M is finite direct projective if every epimorphism f from M to a finitely-generated summand K of M, splits, i.e., there exists an endomorphism g such that $fog = p_K$, where p_K is the projection map on K.*

In the following example, we provide a module that is finite direct projective, but not direct projective.

Example 3.1.3. *Let M be any injective module over a semi-hereditary ring R, then M is a finite direct projective module but need not be a direct-projective module. For this, consider semi-hereditary ring $\mathbb{Z}$ and injective module $\mathbb{Q} \bigoplus \mathbb{Q}/\mathbb{Z}$ then $\mathbb{Z}$-module $\mathbb{Q} \bigoplus \mathbb{Q}/\mathbb{Z}$ is a finite direct projective module but it is not a direct projective module.*

Proposition 3.1.4. *The direct summand of a finite direct projective module is a finite direct projective.*

Proof:- Let M be a finite direct projective module, and N be a direct summand of M. If N has no finitely generated summand, then N is a finite direct projective module trivially. Otherwise, we can find a finitely generated summand P of N. Since P is a direct summand of N, it is also a direct summand of M. By the definition of finite direct projectivity, there exists a submodule X of M such that M/X is isomorphic to $P \leq^{\oplus} M$. Also $P \oplus X \cong M$ and $(P \oplus X) \cap N \cong M \cap N$ by Modular law $P \oplus (X \cap N) \cong N$ implies $(X \cap N) \cong K \leq^{\oplus} N$ and $N/(X \cap N) \cong P \leq^{\oplus} N$. Hence N is a finite direct projective module.

Every submodule of a finite direct projective module need not be finite direct projective. As an example, we consider a module $\mathbb{Q} \bigoplus \mathbb{Z}_2$ over ring $\mathbb{Z}$, which is a finite direct projective module. But its submodule $Z \bigoplus \mathbb{Z}_2$ is not finite direct projective. But, over a semi-simple ring, every submodule of a finite direct projective module is a finite direct projective module.

The next proposition gives a sufficient condition for a finite direct projective module to be direct-projective.

Proposition 3.1.5. *A finitely generated R-module M is a finite direct projective if and only if it is a direct projective module. In particular, a ring R is the finite direct projective if and only if it is direct-projective.*

Proof:- Let M be a finite direct projective module, and N is any direct summand of M which satisfies $M/P \cong N \leq^{\bigoplus} M$ where P is a submodule of M. Since M is finitely generated, this implies N is finitely generated. Since N is a direct summand of M and M is finite direct projective, therefore P will be a direct summand of M. This shows that M is a direct projective module. The converse is obvious.

Proposition 3.1.6. *Let M be a finite direct projective module, and $M = M_1 \bigoplus M_2$ for some direct summands M_1 and M_2 with M_2 is finitely generated. If $f : M_1 \to M_2$ is a homomorphism and $Img(f) \leq^{\oplus} M_2$, then $Ker(f) \leq^{\bigoplus} M_1$.*

Proof:- Let $f : M_1 \to M_2$ be a module homomorphism. Consider canonical projection $\pi : M \to M_1$, then $f \circ \pi : M \to M_2$. Also $Img(f \circ \pi) = Img(f)$. Therefore $M/Ker(f \circ \pi) \cong Img(f \circ \pi) = Img(f)$. Since $Img(f) \leq^{\bigoplus} M_2$ and M_2 finitely generated, this implies $Img(f)$ is finitely generated. Hence by definition of finite direct projective modules $Ker(f \circ \pi) \leq^{\bigoplus} M$. And $Ker(f \circ \pi) = M_2 \oplus Ker(f)$ therefore, $Ker(f) \leq^{\bigoplus} M$. Since $Ker(f)$ is a submodule of M_1 implies $Ker(f) \leq^{\bigoplus} M_1$.

Corollary 3.1.7. *Let M be a finite direct projective module, and $M = M_1 \bigoplus M_2$ for some direct summands M_1 and M_2 with M_2 is finitely generated. If $f : M_1 \to M_2$ is an epimorphism, then $Ker(f) \leq^{\oplus} M_1$.*

We have proved in Proposition 3.1.4 that any direct summand of the finite direct projective module is finite direct projective. But, the following example shows that the direct sum of two finite direct projective modules need not be a finite direct projective module.

Example 3.1.8. Consider $\mathbb{Z}$ module $M = \mathbb{Z}_3 \bigoplus \mathbb{Z}_9$. Here $\mathbb{Z}$ module $\mathbb{Z}_3$ and $\mathbb{Z}$ module $\mathbb{Z}_9$ are both finite direct projective modules trivially since both the modules have no proper direct summand. We claim that M is not a finite direct projective module. To prove this consider the direct summands of M those are $A =< (1,3) >$, $B =< (1,1) >$, $C = \mathbb{Z}_3 \bigoplus < 0 >$ and $D =< 0 > \bigoplus \mathbb{Z}_9$. Since A and B are direct summands of M that are finitely generated. Also $M = A \bigoplus B$. Let $g : B \to A$ be defined by $g(1,1) = (1,3)$, here g is well defined epimorphism map with $Ker(f)$=$<(0,3) > \not\leq^{\oplus} B$. Hence module M is not a finite direct projective module.

In the next proposition, we find a connection between D_3-modules and finite direct projective modules.

Proposition 3.1.9. *The following statements are true for an R-module M:*

1. *If M is a finite direct projective module, then for any two direct summands M_1 and M_2 of M with $M_1 + M_2 = M$ and M_2 is finitely generated, the intersection of M_1 and M_2 is a direct summand of M.*

2. *If $M \bigoplus M$ is a D_3-module, then M is a finite direct projective module.*

Proof:-

1. Suppose M is a finite direct projective module and $M_1 \leq^{\oplus} M$ and $M_2 \leq^{\oplus} M$ with $M_1 + M_2 = M$, and M_2 finitely generated. If M_2 is a subset of M_1, their intersection is direct summand trivially in the first case. In another case, since we have $M = M_1 + M_2$, we can write $M = M_1 \bigoplus M_1'$ and $M = M_2 \bigoplus M_2'$ for some direct summand M_1' and M_2'. Consider $f : M_2' \to M_2$ be an homomorphism, such that $Img(f)$ is direct summand of M_2. By Proposition 3.1.6 we know that $Ker(f)$ is direct summand of M_2'. Let $\pi_{M_2'} : M \to M_2'$ be the natural projection on M_2'. Consider $f \circ \pi_{M_2'} : M \to M_2$. Then $Ker(f \circ \pi_{M_2'}) = (M_1 \cap M_2) \bigoplus (M_1' \cap M_2) \bigoplus Ker(f)$. Since M is finite direct projective and $Img(f \circ \pi_{M_2'}) = Img(f)$, implies $Img(f \circ \pi_{M_2'})$ is direct summand of M_2'. Hence $Ker(f \circ \pi_{M_2'})$ is direct summand of M. Therefore $(M_1 \cap M_2)$ is direct summand of M.

2. For a D_3 module M, and $M = M_1 \bigoplus M_2$ where M_1 and M_2 direct summands of M. If $f : M_1 \to M_2$ is a homomorphism and $Img(f) \leq^{\oplus} M_2$ then $Ker(f) \leq^{\oplus} M_1$. Consider $M \bigoplus M$ as a D_3 module., for a finitely generated direct summand F, $M \bigoplus F$ is a direct summand of $M \bigoplus M$. This implies $M \bigoplus F$ is D_3 as D_3 modules are closed under direct summand. Therefore every epimorphism $g : M \to F$ splits and F is an arbitrary finitely generated summand. Hence M is a finite direct projective from Lemma 3.1.2.

Remark 3.1.10. *The second result of Proposition 3.1.9 is also true for the finite direct sum of copies of modules. That is if finite copies M is D_3 module, then M is a finite direct projective module.*

Proposition 3.1.11. *The following statements are equivalent for an R-module M:*

1. *M is a finite direct projective module;*

2. *If Q is a direct summand of M and P is finitely generated direct summand of M, then every epimorphism from Q to P splits;*

3. *If P is a finitely generated direct summand of M, then every epimorphism M to P splits;*

4. *If P is finitely generated direct summand of M and α,γ are epimorphisms from M onto P, then there exists an endomorphism β of M such that $\alpha\beta = \gamma$.*

Proof:- Follows from the definition of a finite direct projective module.

3.2 Characterization of rings using Finite Direct Projective Modules

We characterize finite direct projective modules with respect to their Endomorphism rings. Then we find their equivalence with several modules, such as Endoregular, SIP, Dual-Rickart, and Hopfian.

Lemma 3.2.1. *[27, Remark 2.8] Let M be an R-module then, $S = End_R(M)$ is a von Neumann regular ring if and only if $Ker(s)$ and $Img(s)$ are direct summands of M $\forall$ $s \in S$*

Proposition 3.2.2. *Let M be an R-module and $S = End_R(M)$. Then the following statements are equivalent:*

1. *M is a finite direct projective module;*

2. *For any $s \in S$, $Ker(s)$ is a direct summand of M if $Img(s)$ is finitely generated summand of M.*

Proof:- Follows from Proposition 3.1.6.

Every endoregular module is a finite direct projective module. Still, its converse need not be true as $\mathbb{Z}$-module $\mathbb{Z}_n$ is a finite direct projective module but not endoregular module. In the following Proposition, we find that when the class of finite direct projective modules is equivalent to the class of endoregular, SSP, and Dual Rickart modules.

Proposition 3.2.3. *Let M be a finitely generated module M and $S = End_R(M)$, then following statements are equivalent:*

1. *M is an endoregular module;*
2. *M is a finite direct projective module and $M \bigoplus M$ is an SSP module;*
3. *M is a finite direct projective and dual-Rickart module.*

Proof:- (1) $\Rightarrow$ (2) Let M be an endoregular module. This implies that M is a finite direct projective module. To show that $M \bigoplus M$ is a SSP module. We set $A = End_R(M \bigoplus M) \cong Mat_2(S)$. Using fact from [27, Theorem 2.4] $Mat_2(S)$ is von Neumann regular ring and $M \bigoplus M$ is a SSP module.

(2) $\Rightarrow$ (3) It is given that $M \bigoplus M$ is a SSP module. Therefore, $Img(s)$ is a direct summand of M for all $s \in S$. Hence, M is a dual-Rickart module.

(3) $\Rightarrow$ (1) Since M is a dual-Rickart module, it implies $Img(s)$ is direct summand of M for all $s \in S$. Also, given that M is finite direct projective, then by Proposition 3.1.6, $Ker(s)$ is the direct summand of M for all $s \in S$. Hence by Lemma 3.2.1, M is an endoregular module.

Corollary 3.2.4. *A module M is an endoregular module if and only if M is a dual-Rickart module and $M \bigoplus M$ is a D_3 module.*

Proof:- It follows from Proposition 3.1.9 and 3.2.3.

Proposition 3.2.5. *Let M be a finitely generated finite direct projective module. Then the following assertions hold:*

1. *$S = End_R(M)$ is SIP ring if M is a dual-Rickart module;*

2. *$Mat_2(S)$ is SIP ring if M is a dual-Rickart module.*

Proof:-

1. If M is a dual Rickart module, then $s(M)$ is a direct summand of M $\forall s \in S$. Since M is finite direct projective and $s(M)$ is a finitely generated direct summand of M $\forall s \in S$, then by Proposition 3.2.2, we get that $Ker(s)$ is a direct summand of M $\forall s \in S$. Hence, S is a von Neumann regular ring which implies S is a SIP ring.

2. As in above proof, S is a von Neumann regular ring, therefore $Mat_2(S)$ is a von Neumann regular ring. Hence $Mat_2(S)$ is SIP ring.

Proposition 3.2.6. *The following statements are equivalent for a finitely generated module M:*

1. *M is a finite direct projective module;*

2. *M is a direct-projective module;*

3. *M is a D_3 module;*

4. *M is a Rickart module;*

5. *M has SIP property;*

6. *Each submodule of a M is a direct summand.*

Proof:- (1) $\Rightarrow$ (2) From Proposition 3.1.5, a finitely generated finite direct projective module is direct projective.

(2) $\Rightarrow$ (3) As every direct-projective module is D_3 module.

(3) $\Rightarrow$ (4) Consider, finitely generated R-module M and $S = End_R(M)$. We are required to show that M is a Rickart module, i.e. for any $s \in S$, $Ker(s) \leq^{\bigoplus} M$. Since M is finitely generated, hence module $M \bigoplus s(M)$ is finitely generated. Therefore by (3), it is D_3-module. Hence, by property of D_3 module [48, Proposition 4], $Ker(s) \leq^{\bigoplus} M$. Therefore M is a Rickart module.

(4) $\Rightarrow$ (5) Since every Rickart module satisfies the SIP property [25, Proposition 2.16]. Therefore by (4), every finitely generated R-module has the SIP property.

(5) $\Rightarrow$ (6) Let N be any submodule of an R-module M that is finitely generated, then M/N is also finitely generated. Therefore $(M/N) \bigoplus M$ is also finitely generated, and by (5), it has the SIP property. Let $f : M \to M/N$ be a R-homomorphism and $Ker(f) \cong N$. Hence it is a direct summand of M.

(6) $\Rightarrow$ (1) Follows from the definition of direct projective modules.

Next, we characterize some well-known rings, such as Semi hereditary rings, S rings, and semisimple Artinian rings in terms of finite direct projective modules.

Lemma 3.2.7. *Let P be any projective module, and $P \bigoplus M$ is finite direct projective with M finitely generated module. If an epimorphism exists from P to M, then M is a projective module.*

Proof:- Since $P \bigoplus M$ is a finite direct projective module. Let f be an epimorphism from P to M from Corollary 3.1.7, we get $Ker(f) \leq^{\oplus} P$. Hence f splits and M is a projective module.

According to Puninski and Rothmaler [38], a ring R is called S-ring if every finitely generated flat R-module is projective. Now we characterize finite direct projective modules over S-ring.

Proposition 3.2.8. *The following statements are equivalent:*

1. *R is a S-ring;*

2. *Every finitely generated flat R-module is a finite direct projective module.*

Proof:- (1) $\Rightarrow$ (2) By definition of S-ring every finitely generated flat module is projective which implies it is finite direct projective.
(2) $\Rightarrow$ (1) Let N be a finitely generated flat module. We must prove that N is projective. Consider an R-module $M = F \bigoplus N$, where F is free R-module of finite rank and M is finitely generated flat R-module. As the direct sum of flat modules is flat. Also, a R-epimorphism $f : F \longrightarrow N$ exists. Since $M = F \bigoplus N$ is a finite direct projective module therefore homomorphism f splits. Therefore from Lemma 3.2.7 N is projective.

Next, we characterize finite direct projective modules with respect to P.P. rings. Any ring R is called P.P. if every principal ideal (generated by a single element) of R is a projective module.

Lemma 3.2.9. *The following statements are equivalent:*

1. *R is a P.P. ring.*

2. *Every principal ideal for $M_2(R)$ is finite direct projective module.*

Proof:- (1) $\Rightarrow$ (2). Since ring R is P.P., every principal ideal of $M_2(R)$ is projective. Hence every ideal of $M_2(R)$ is a finite direct projective ideal.

(2) $\Rightarrow$ (1). Let $T = M_2(R)$, $a \in R$, and S be a finite direct projective principal ideal of T generated by a diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$. If $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,then $Se_1 \cong aR \bigoplus R$. Also, $Se_1 \leq^{\oplus} S$ implies Se_1 is a finite direct projective ideal represented as $Se_1 \cong aR \bigoplus R$. So for any canonical epimorphism $\psi : R \to aR$ splits follow from Corollary 3.1.7. Hence aR is a projective ideal, from Lemma 3.2.7, and R is a P.P ring.

Now, we find an equivalent condition of semi-hereditary rings and finite direct projective module. A ring is called semi-hereditary if every finitely generated submodule of a projective module is a projective module.

Note 3.2.10. R is a semi-hereditary ring if for all $n \in \mathbb{N}$ every principal ideal of $M_n(R)$ is a projective module.

Proposition 3.2.11. *The following statement is equivalent for a ring R:*

1. *R is a semi-hereditary ring;*

2. *Each finitely generated submodule of a projective module is a finite direct projective module;*

3. *Each principal ideal of $P = End_R(F)$ is a finite direct projective, for any finite free R-module F.*

Proof:- (1) $\Rightarrow$ (2) Since over semi-hereditary ring every finitely generated submodule of a projective module is projective. And every projective module is finite direct projective. Hence we got our claim.

(2) $\Rightarrow$ (1) Let N be a finitely generated submodule of a projective R-module P. Let F be a free R-module of finite rank with a surjective map $f : F \to N$. Then $F \bigoplus N$ will be a submodule of $F \bigoplus P$. Since $F \bigoplus P$ is a projective module, this implies $F \bigoplus N$ is a finite direct projective module. Therefore N is a projective module from the Lemma 3.2.7.

(1) $\Rightarrow$ (3) From the above remark.

(3) $\Rightarrow$ (1) Let F be a free R-module which is of finite rank. And $P = End_R(F) \cong M_n(R)$, this implies $F \bigoplus F$ is an R-module which is also free and $End(F \bigoplus F) = M_2(P)$. Following from Lemma 3.2.9, every principal ideal of $M_2(P)$ is a finite direct projective module. Hence P is a P.P ring. Using the above Remark, R is a semi-hereditary ring.

In the next proposition we combine Proposition 3.2.8 and 3.2.11 and find the equivalence between the finite direct projective module and semi-hereditary S-ring.

Proposition 3.2.12. *The following statements are equivalent:*

1. *R is a semi-hereditary S-ring;*
2. *Every finitely generated submodule of a flat R-module is a finite direct projective module.*

Proof:- Follows from above Proposition 3.2.8 and 3.2.11.

In the next Proposition we use the fact that over every semi-simple Artinian ring every cyclic module is projective.

Proposition 3.2.13. *The following are equivalent for a ring R:*

1. *R is a semi-simple Artinian ring;*

2. *Every finitely generated R-module is finite direct projective;*

3. *Every 2-generated R-module is finite direct projective;*

4. *The direct sum of two finite direct projective R-modules is finite direct projective.*

Proof:- (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are straight-forward as R is semi-simple Artinian ring.

(3) $\Rightarrow$ (1) Let N be cyclic R-module. We need to prove N is projective. Consider $R \bigoplus N$ it is finite direct projective with N generated by a single element. Also, there exists R-epimorphism $f : R \longrightarrow N$. Then from Lemma 3.2.7 N is projective.

(4) $\Rightarrow$ (1) Let N be cyclic R-module, then $R \bigoplus N$ is finite direct projective. Hence N is projective.