Chapter 4

A Primal-dual Interior-point Technique to Solve Multiobjective Optimization Problems with an Application to Optimal Control Problem

4.1 Introduction

Over the past few years, the research on IPMs and their applications has been widely spread. Initially, IPM was developed to solve linear programming problems. Thereafter, new versions of IPM were developed for addressing nonlinear, convex and nonconvex optimization problems. The first IPM (logarithmic barrier method) was introduced by Frisk [110] in 1955. The modern era of IPM started with Karmarker's paper [55]. Affine-scaling methods, projective methods and primal-dual methods are three common types of interior-point methods. The primal-dual algorithm is known as one of the most efficient algorithms among the different types of IPMs.

4.2 Motivation

Most of the IPMs can be trapped in the areas where the constraints of the problem is inconsistent, for example, the constraints $x^2 - x \geq 0$, $x \geq 10^{-5}$ are inconsistent around $x = 0$. Therefore, if an initial point $x^0 = 0$, the algorithm stays in the neighbourhood of 0 and never converges to a feasible point. To overcome this deficiency, Griva et al. [111] proposed a globally convergent primal-dual interior-point method to solve nonlinear optimization problems. An important feature of this algorithm is that, in the worst case, it can stabilize a sequence of primal iterates and find a first-order optimality point. In this chapter, the author implement this variant of interior-point method to solve IPMs.

4.3 Contributions

This chapter introduces a primal-dual interior-point technique to determine the Pareto optimal solution for MOPs. A direction-based approach [47] is utilized to transform MOPs into a set of single-objective optimization subproblems. Then, the subproblems are solved by using a primal-dual interior-point method. A merit function is introduced to take the suitable step lengths along the search directions. To demonstrate the efficiency of the proposed method, we applied it to some constrained test problems. As an application, we use proposed algorithm to an optimal control problem.

Original contributions of this chapter are listed below:

(i) It is shown that the stationary point of the proposed merit function satisfies the KKT conditions of the barrier problem. It is also proven that the Newton direction is descent for the proposed merit function.

(ii) The summary of the proposed algorithm is explained. Also, the pseudo code of the proposed algorithm is provided.

(iii) The proposed algorithm is tested on some constrained multiobjective optimization problems. Also, the proposed algorithm is implemented to an optimal control problem of carbon dioxide emission from energy sector.

In this chapter, the following MOP is considered:

$$\left.\begin{aligned} &\text{minimize} \quad F(x) = (f_1(x), f_2(x), \ldots, f_{\mathcal{P}}(x))^\top, \ \mathcal{P} \geq 2 \\ &\text{subject to} \quad x \in \mathcal{X}, \end{aligned}\right\} \tag{4.1}$$

where $\mathcal{X} = \{x \in \mathbb{R}^n : h_j(x) \geq 0, \ j = 1, 2, \ldots, \mathcal{J}\}$, $f_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable functions for all $i = 1, 2, \ldots, \mathcal{P}$ and $j = 1, 2, \ldots, \mathcal{J}$.

4.4 Description of the algorithm

As described in [47], firstly, it determines the ideal point $F^* = (f_1^*, f_2^*, \ldots, f_{\mathcal{P}}^*)^\top$, where $f_i^* = \min\{f_i(x) : x \in \mathcal{X}\}$. Then, by solving the following minimization problem corresponding to a particular $\hat{\beta} \in \mathbb{S}_{\geqq}^{\mathcal{P}-1} = \mathbb{S}^{\mathcal{P}-1} \cap \mathbb{R}_{\geqq}^{\mathcal{P}}$ (where $\mathbb{S}^{\mathcal{P}-1}$ represents the unit sphere in $\mathbb{R}^{\mathcal{P}}$), the solution of the MOP (4.1) can be obtained:

$$\left.\begin{aligned} &\text{minimize} \quad t \\ &\text{subject to} \quad t\hat{\beta} \geqq F(x) - F^*, \\ &\qquad\qquad\quad h_i(x) \geq 0, \ i = 1, 2, \ldots, \mathcal{J}, \\ &\qquad\qquad\quad t \geq 0. \end{aligned}\right\} \tag{4.2}$$

Or equivalently,

$$\left.\begin{aligned} &\text{minimize} \quad c^\top \mathrm{x} \\ &\text{subject to} \quad \hat{\beta} c^\top \mathrm{x} - f(\mathrm{x}) - v = 0, \\ &\qquad\qquad\quad h(\mathrm{x}) - w = 0, \\ &\qquad\qquad\quad v \geqq 0, \ w \geqq 0, \end{aligned}\right\} \tag{4.3}$$

where $\mathrm{x} = (x_1, x_2, \ldots, x_n, t)^\top$, $c = (0, 0, \ldots, 0, 1)^\top$, $f(\mathrm{x}) = F(x) - F^*$, $h(\mathrm{x}) = \big(h_1(\mathrm{x}), h_2(\mathrm{x}), \ldots, h_{\mathcal{J}}(\mathrm{x}), h_{\mathcal{J}+1}(\mathrm{x})\big)^\top$ with $h_{\mathcal{J}+1}(\mathrm{x}) = c^\top \mathrm{x}$ and $v = (v_1, v_2, \ldots, v_{\mathcal{P}})^\top$ and $w = \big(w_1, w_2, \ldots, w_{\mathcal{J}}, w_{\mathcal{J}+1}\big)^\top$ being vectors of slack variables.

From the complete weakly nondominated set $\mathcal{Y}_{wN}$, the process to find the set of all nondominated points is detailed in [47].

We note that problem (4.2) is a single-objective parametric problem, with the parameter $\hat{\beta}$, corresponding to the MOP (4.1). To generate the complete nondominated set, one needs to solve the problem (4.2) for all $\hat{\beta}$'s in $\mathbb{S}^{\mathcal{P}-1}_{\geqq}$. For obtaining uniformly spread nondominated points, the work in [47] suggests to take $\hat{\beta}$ from (2.3).

4.5 Summary of the primal-dual interior-point algorithm

In this section, a primal-dual interior-point method (PD-IPM) is discussed to solve (4.3). In the sequel, (4.3) is formulated into barrier problem and then KKT conditions are derived. Thereafter, PD-IPM takes the advantage of Newton method to solve the system of KKT. An overview of the method is described below.

For a given $\mu > 0$, a barrier problem corresponding to problem (4.3) is

$$\left.\begin{array}{ll} \text{minimize} & b_\mu(\mathrm{x}, v, w) = c^\top \mathrm{x} - \mu \left(\displaystyle\sum_{i=1}^{\mathcal{P}} \log(v_i) + \sum_{j=1}^{\mathcal{J}+1} \log(w_j) \right) \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v = 0, \\ & h(\mathrm{x}) - w = 0. \end{array}\right\} \tag{4.4}$$

The Lagrangian for the problem (4.4) is given by

$$L_{\hat{\beta}}(\Omega, \mu) = b_\mu(\mathrm{x}, v, w) - y^\top \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v \right) - z^\top (h(\mathrm{x}) - w), \;\; y \in \mathbb{R}^{\mathcal{P}} \;\text{ and }\; z \in \mathbb{R}^{\mathcal{J}+1}.$$

The first-order KKT conditions for a minimum of (4.4) are

$$\begin{bmatrix} c - \nabla_{\mathrm{x}} \left(\hat{\beta} c^\top \mathrm{x} - f(\mathrm{x}) \right)^\top y - (\nabla_{\mathrm{x}} h(\mathrm{x}))^\top z \\ -\mu e + VYe \\ -\mu e + WZe \\ -F(\mathrm{x}) + \hat{\beta} c^\top \mathrm{x} - v \\ h(\mathrm{x}) - w \end{bmatrix} = 0, \quad (y, z) \geq 0 \tag{4.5}$$

In this chapter, we aim to solve (4.5) by decreasing the value of μ. Interior-point methods utilize the Newton method to solve the system (4.5). Before solving the system (4.5) by Newton method, we introduce the following notations:

$$\begin{aligned} & p = (\mathrm{x}, v, w), \quad d = (y, z), \quad \Omega = (p, d) = (\mathrm{x}, v, w, y, z), \\ & A_{\hat{\beta}}(\mathrm{x}) = \nabla_{\mathrm{x}} \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) \right), \quad B(\mathrm{x}) = \nabla_{\mathrm{x}} h(\mathrm{x}), \quad \gamma_1 = \mu V^{-1} e - y, \quad \gamma_2 = \mu W^{-1} e - z, \\ & \sigma_{\hat{\beta}}(\mathrm{x}, y, z) = c - (A_{\hat{\beta}}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z, \quad \gamma(v, w, y, z) = (\gamma_1, \ \gamma_2)^\top, \\ & \rho_{\hat{\beta}}(p) = (\rho_1, \ \rho_2)^\top, \quad \rho_1 = F(\mathrm{x}) + v - \hat{\beta} c^\top \mathrm{x}, \quad \rho_2 = w - h(\mathrm{x}), \\ & \Im(\Omega) = \left(\sigma_{\hat{\beta}}^\top, (VYe)^\top, (WZe)^\top, -\rho_{\hat{\beta}}^\top \right)^\top, \quad \Im_\mu(\Omega) = \left(\sigma_{\hat{\beta}}^\top, (VYe)^\top - \mu e^\top, (WZe)^\top - \mu e^\top, -\rho_{\hat{\beta}}^\top \right)^\top. \end{aligned}$$

Also, the following merit functions are used to control the convergence:

$$\begin{aligned} \nu(\Omega) &= \|\Im(\Omega)\| = \max\{\|\sigma_{\hat{\beta}}\|, \|\rho_{\hat{\beta}}\|, \|VYe\|, \|WZe\|\}, \\ \nu_\mu(\Omega) &= \|\Im_\mu(\Omega)\| = \max\{\|\sigma_{\hat{\beta}}\|, \|\rho_{\hat{\beta}}\|, \|V\gamma_1\|, \|W\gamma_2\|\} \text{ and} \\ \mathcal{M}_{\eta,\mu}(\Omega) &= b_\mu(p) + d^\top \rho_{\hat{\beta}} + \frac{\eta}{2} \rho_{\hat{\beta}}^\top \rho_{\hat{\beta}}. \end{aligned}$$

The function $\nu(\Omega)$ determines how close the current solution is to the KKT point of the problem (4.3). Also, for a given $\epsilon > 0$, if $\nu(\Omega^*) < \epsilon$, then Ω^* is an approximated KKT point. The function $\nu_\mu(\Omega)$ calculates the distance between the current point and a KKT point of the barrier problem (4.4). The merit function $\mathcal{M}_{\eta,\mu}$ has the following property.

Theorem 4.1 *Consider the barrier problem (4.4). Let the point* $\Omega^* = (\mathrm{x}^*, v^*, w^*, y^*, z^*)$

be such that $\Im_\mu(\Omega^*) = 0$ *for a given* $\mu > 0$*. Then, for any* $\eta > 0$*, the point* x^* *is stationary point of the merit function* $\mathcal{M}_{\eta,\mu}(\mathrm{x}, v^*, w^*, y^*, z^*)$*.*

Proof: The gradient of the merit function $\mathcal{M}_{\eta,\mu}$ at the point $(\mathrm{x}, v^*, w^*, y^*, z^*)$ is

$$\begin{aligned}\nabla_{\mathrm{x}}\mathcal{M}_{\eta,\mu}(\mathrm{x}, v^*, w^*, y^*, z^*) &= c + (\nabla_{\mathrm{x}}\rho_1(\mathrm{x}, v^*))^\top y^* + (\nabla_{\mathrm{x}}\rho_2(\mathrm{x}, w^*))^\top z^* + \eta\left(\rho_{\hat{\beta}}(\mathrm{x}, v^*, w^*)\right)^\top \nabla_{\mathrm{x}}\rho_{\hat{\beta}}(\mathrm{x}, v^*, w^*)\\ &= c - A_{\hat{\beta}}(\mathrm{x})^\top y^* - B(\mathrm{x})^\top z^* + \eta\left(\rho_{\hat{\beta}}(\mathrm{x}, v^*, w^*)\right)^\top \nabla_{\mathrm{x}}\rho_{\hat{\beta}}(\mathrm{x}, v^*, w^*).\end{aligned}$$

For any $\mu > 0$, at the point Ω^* the term $\rho_{\hat{\beta}}(\mathrm{x}^*, v^*)$ vanishes. Therefore, after replacing x by x^* in the last expression, we get

$$\nabla_{\mathrm{x}}\mathcal{M}_{\eta,\mu}(\mathrm{x}^*, v^*, w^*, y^*, z^*) = c - A_{\hat{\beta}}(\mathrm{x}^*)^\top y^* - B(\mathrm{x}^*)^\top z^* + \eta\left(\rho_{\hat{\beta}}(\mathrm{x}^*, v^*, w^*)\right)^\top \nabla_{\mathrm{x}}\rho_{\hat{\beta}}(\mathrm{x}^*, v^*, w^*) = 0.$$

The last equality holds because $\Im_\mu(\Omega^*) = 0$. □

4.6 Newton method

To obtain the solution of (4.5), PD-IPM utilizes the Newton method. For a given barrier parameter $\mu > 0$, the Newton direction $\Delta\Omega = (\Delta\mathrm{x}, \Delta v, \Delta w, \Delta y, \Delta z)$ at a point z is obtained by solving the following Newton system:

$$\mathcal{D}(\Omega)\Delta z = -\Im_\mu(\Omega), \tag{4.6}$$

where

$$\mathcal{D}(\Omega) = \begin{bmatrix} H(\mathrm{x}, y, z) & 0 & 0 & -(A_{\hat{\beta}}(\mathrm{x}))^\top & -(B(\mathrm{x}))^\top \\ 0 & Y & 0 & V & 0 \\ 0 & 0 & Z & 0 & W \\ A_{\hat{\beta}}(\mathrm{x}) & -I & 0 & 0 & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & 0 \end{bmatrix}$$

and

$$H(\mathrm{x}, y, z) = \sum_{i=1}^{\mathcal{P}} y_j \nabla^2 f_j(\mathrm{x}) - \sum_{j=1}^{\mathcal{J}+1} z_i \nabla^2 h_i(\mathrm{x}).$$

In this method, instead of solving (4.6), we aim to solve the following system for $\Delta\Omega = (\Delta \mathrm{x},\, \Delta v,\, \Delta w,\, \Delta y,\, \Delta z)$:

$$\begin{bmatrix} H(\mathrm{x},y,z) & 0 & 0 & -(A_{\hat\beta}(\mathrm{x}))^\top & -(B(\mathrm{x}))^\top \\ 0 & Y & 0 & V & 0 \\ 0 & 0 & Z & 0 & W \\ A_{\hat\beta}(\mathrm{x}) & -I & 0 & \lambda_d I & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & \lambda_d I \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} c - (A_{\hat\beta}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z \\ -\mu e + VYe \\ -\mu e + WZe \\ \hat\beta c^\top \mathrm{x} - F(\mathrm{x}) - v \\ h(\mathrm{x}) - w \end{bmatrix}, \tag{4.7}$$

where $\lambda_d > 0$ is regularizing parameter. Note that if $\lambda_d = 0$ the system (4.7) is same as (4.6). The parameter λ_d is chosen such that it becomes zero when the algorithm approaches the primal-dual solution of the barrier problem (4.4). In (4.7), we multiply the first equation by -1, the second equation by $-V^{-1}$ and the third equation by $-W^{-1}$. Accordingly, we get

$$\begin{bmatrix} -H(\mathrm{x},y,z) & 0 & 0 & (A_{\hat\beta}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ 0 & -V^{-1}Y & 0 & -I & 0 \\ 0 & 0 & -W^{-1}Z & 0 & -I \\ A_{\hat\beta}(\mathrm{x}) & -I & 0 & \lambda_d I & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & \lambda_d I \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sigma_{\hat\beta} \\ -\gamma_1 \\ -\gamma_2 \\ \varrho_{\hat\beta} \\ \rho \end{bmatrix}. \tag{4.8}$$

The explicit formulas for the solution to the primal-dual system (4.7) are

$$\left.\begin{aligned} \Delta\mathrm{x} &= N_{\hat\beta}^{-1}\left(-\sigma_{\hat\beta} + (A_{\hat\beta}(\mathrm{x}))^\top (VY^{-1}+\lambda_d I)(VY^{-1}\gamma_1 + \varrho_{\hat\beta}) + (B(\mathrm{x}))^\top (WZ^{-1}+\lambda_d I)(WZ^{-1}\gamma_2+\rho)\right), \\ \Delta v &= -\varrho_{\hat\beta} + A_{\hat\beta}(\mathrm{x})\Delta\mathrm{x} + \lambda_d \Delta y, \\ \Delta w &= -\rho + B(\mathrm{x})\Delta\mathrm{x} + \lambda_d\Delta z, \\ \Delta y &= (VY^{-1}+\lambda_d I)^{-1}(VY^{-1}\gamma_1 + \varrho_{\hat\beta} - A_{\hat\beta}(\mathrm{x})\Delta\mathrm{x}), \\ \Delta z &= (WZ^{-1}+\lambda_d I)^{-1}(WZ^{-1}\gamma_2 + \rho - B(\mathrm{x})\Delta\mathrm{x}), \end{aligned}\right\} \tag{4.9}$$

where $N_{\hat\beta} = H(\mathrm{x},y,z) + (A_{\hat\beta}(\mathrm{x}))^\top (V^{-1}Y + \lambda_d I)A_{\hat\beta}(\mathrm{x}) + (B(x))^\top (W^{-1}Z + \lambda_d I)B(\mathrm{x})$.

If the matrix $N_{\hat\beta}$ is not positive definite the algorithm replaces it with the regularized matrix

$$\tilde N_{\lambda_d}(\mathrm{x},y,z) = N_{\lambda_d}(\mathrm{x},y,z) + \lambda I, \qquad \lambda > 0, \tag{4.10}$$

where I is the identity matrix of order $(n+1)\times(n+1)$.

The following theorem ensures that the primal direction $\Delta p = (\Delta \mathrm{x}, \Delta v, \Delta w)$ is descent for the merit function $\mathcal{M}_{\eta,\mu}$.

Theorem 4.2 *Consider the barrier problem (4.4). Suppose that* $N_{\hat{\beta}} = H(\mathrm{x}, y, z) + (A_{\hat{\beta}}(\mathrm{x}))^\top (V^{-1}Y + \eta^{-1}I)A_{\hat{\beta}}(\mathrm{x}) + (B(x))^\top (W^{-1}Z + \eta^{-1}I)B(\mathrm{x})$ *is positive definite at* $\Omega = (\mathrm{x}, v, w, y, z)$*. Then, for any* $\mu > 0$ *and* $\eta > 0$*, the primal direction* $\Delta p = (\Delta \mathrm{x}, \Delta v, \Delta w)$ *obtained as the solution of the system (4.7) with regularization parameter* $\lambda_d = \frac{1}{\eta}$ *is a descent direction for* $\mathcal{M}_{\eta,\mu}(p, y, z)$*.*

Proof: For the regularization parameter $\lambda_d = \frac{1}{\eta}$, the primal-dual system (4.7) is as follows:

$$
\begin{bmatrix}
H(\mathrm{x},y,z) & 0 & 0 & -(A_{\hat{\beta}}(\mathrm{x}))^\top & -(B(\mathrm{x}))^\top \\
0 & Y & 0 & V & 0 \\
0 & 0 & Z & 0 & W \\
A_{\hat{\beta}}(\mathrm{x}) & -I & 0 & \frac{1}{\eta}I & 0 \\
B(\mathrm{x}) & 0 & -I & 0 & \frac{1}{\eta}I
\end{bmatrix}
\begin{bmatrix}
\Delta \mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta y \\ \Delta z
\end{bmatrix}
=
\begin{bmatrix}
-c + (A_{\hat{\beta}}(\mathrm{x}))^\top y + (B(\mathrm{x}))^\top z \\
\mu e - VYe \\
\mu e - WZe \\
-\hat{\beta} c^\top \mathrm{x} + F(\mathrm{x}) + v \\
-h(\mathrm{x}) + w
\end{bmatrix}. \tag{4.11}
$$

Eliminate Δy and Δz by solving last two equations of system (4.11), we obtain the following system :

$$
\begin{bmatrix}
H(\mathrm{x},y,z) + \eta\Omega & -\eta(A_{\hat{\beta}}(\mathrm{x}))^\top & -\eta(B(\mathrm{x}))^\top \\
-\eta A_{\hat{\beta}}(\mathrm{x}) & V^{-1}Y + \eta I & 0 \\
-\eta B(\mathrm{x}) & 0 & W^{-1}Z + \eta I
\end{bmatrix}
\begin{bmatrix}
\Delta \mathrm{x} \\ \Delta v \\ \Delta w
\end{bmatrix}
=
\begin{bmatrix}
-\sigma_{\hat{\beta}} + \eta(A_{\hat{\beta}}(\mathrm{x}))^\top \rho_1 + \eta B(\mathrm{x})^\top \rho_2 \\
\gamma_1 - \eta\rho_1 \\
\gamma_2 - \eta\rho_2
\end{bmatrix}, \tag{4.12}
$$

where $\Omega = (A_{\hat{\beta}}(\mathrm{x}))^\top A_{\hat{\beta}}(\mathrm{x}) + (B(\mathrm{x}))^\top B(\mathrm{x})$.

On the other hand, the gradient of $\mathcal{M}_{\eta,\mu}(\Omega)$ with respect to x, v and w is as follows:

$\nabla_{\mathrm{x}}\mathcal{M}_{\eta,\mu}(\Omega) = \sigma_{\hat{\beta}} - \eta(A_{\hat{\beta}}(\mathrm{x}))^\top \rho_1 - \eta B(\mathrm{x})^\top \rho_2$, $\nabla_v \mathcal{M}_{\eta,\mu}(\Omega) = -\gamma_1 + \eta\rho_1$, and $\nabla_w \mathcal{M}_{\eta,\mu}(\Omega) = -\gamma_2 + \eta\rho_2$.

Therefore,

$$\begin{bmatrix} \nabla_{\mathrm{x}}\mathcal{M}_{\eta,\mu}(\Omega) \\ \nabla_{v}\mathcal{M}_{\eta,\mu}(\Omega) \\ \nabla_{w}\mathcal{M}_{\eta,\mu}(\Omega) \end{bmatrix}^{\top} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} = \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix}^{\top} \begin{bmatrix} H(\mathrm{x},y,z)+\eta\Omega & -\eta(A_{\hat{\beta}}(\mathrm{x}))^{\top} & -\eta(B(\mathrm{x}))^{\top} \\ -\eta A_{\hat{\beta}}(\mathrm{x}) & V^{-1}Y+\eta I & 0 \\ -\eta B(\mathrm{x}) & 0 & W^{-1}Z+\eta I \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix}. \quad (4.13)$$

Since $N_{\hat{\beta}} = H(\mathrm{x},y,z) + (A_{\hat{\beta}}(\mathrm{x}))^{\top}(V^{-1}Y + \eta^{-1}I)A_{\hat{\beta}}(\mathrm{x}) + (B(x))^{\top}(W^{-1}Z + \eta^{-1}I)B(\mathrm{x})$ is positive definite, the matrix in the right side of (4.13) is positive definite (see Lemma 1 from the Appendix of [111]). Hence,

$$\begin{bmatrix} \nabla_{\mathrm{x}}\mathcal{M}_{\eta,\mu}(\Omega) \\ \nabla_{v}\mathcal{M}_{\eta,\mu}(\Omega) \\ \nabla_{w}\mathcal{M}_{\eta,\mu}(\Omega) \end{bmatrix}^{\top} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} < 0.$$

$\square$

4.6.1 Summary of the algorithm

For each successive iteration, the proposed procedure to identify a Pareto optimal point of the MOP (4.1) calculates the merit function ν and then the barrier parameter μ, withe the help of the following formula:

$$\mu = \min\{\wp\nu(\Omega), \nu(\Omega)^2\}, \quad (4.14)$$

where $0 < \wp < 1$. Thereafter, the algorithm determines the merit function ν_μ and the dual regularization parameter as follows:

$$\lambda_d = \min\{\tfrac{1}{\eta_0}, \nu_\mu(\Omega)\} \quad (4.15)$$

Afterwards, the algorithm calculates the primal-dual Newton directions Δz with the help of sparse Cholesky factorization [87]. While performing the factorization, it may be possible that the matrix $N(\mathrm{x}, y, z)$ is not positive definite. Then, the matrix $N(\mathrm{x}, y, z)$ is replaced by formula (4.10) and the factorization is repeated again. The algorithm keeps increasing the parameter λ in formula (4.10) until a positive definite factorization is completed. Then, the algorithm computes the primal and dual steplengths α_p and α_d by the following formulas:

$$\left.\begin{aligned} \alpha_p &= \min\{1, -\kappa\frac{v_i}{\Delta v_i}, -\kappa\frac{w_i}{\Delta w_i} : \Delta v_i < 0,\ \Delta w_i < 0\} \\ \alpha_d &= \min\{1, -\kappa\frac{y_i}{\Delta y_i}, -\kappa\frac{z_i}{\Delta z_i} : \Delta y_i < 0,\ \Delta z_i < 0\}, \end{aligned}\right\} \tag{4.16}$$

where the parameter κ is chosen by the following formula:

$$\kappa = \max\{0.95, 1 - \nu(\Omega)\}. \tag{4.17}$$

Thereafter, the algorithm updates the primal-dual point by the following rule:

$$\left.\begin{aligned} \hat{\mathrm{x}} &= \mathrm{x} + \alpha_p \Delta \mathrm{x} \\ \hat{v} &= v + \alpha_p \Delta v \\ \hat{w} &= w + \alpha_p \Delta w \\ \hat{y} &= y + \alpha_d \Delta y \\ \hat{z} &= z + \alpha_d \Delta z. \end{aligned}\right\} \tag{4.18}$$

Now, if the new point $\hat{\Omega} = (\hat{\mathrm{x}}, \hat{v}, \hat{w}, \hat{y}, \hat{z})$ is able to reduce the merit function $\nu(\Omega)$ and the matrix $N_{\lambda_d}(\hat{\Omega})$ is positive definite, then $\hat{\Omega}$ is accepted and the algorithm continues. If either the merit function $\nu(\Omega)$ does not reduce or the matrix $N_{\lambda_d}(\hat{\Omega})$ is not positive definite, then the new point $\hat{\Omega}$ is not accepted. Also, in both the cases the Lagrange multipliers y and z are not updated. The primal direction $\Delta p = (\Delta \mathrm{x}, \Delta v, \Delta w)$ is shown to be descent direction for the merit function $\mathcal{M}_{\eta,\mu}$ (see Theorem 4.2). In this case, the primal steplength α_p is backtracked by using the following Armijo rule:

$$\mathcal{M}_{\eta,\mu}(p+\alpha_p\Delta p)-\mathcal{M}_{\eta,\mu}(p)\leq \delta\alpha_p\left(\nabla\mathcal{M}_{\eta,\mu}\right)^\top\Delta p, \tag{4.19}$$

where $0<\delta<1$. When the algorithm achieves an approximation $\hat{p}=(\hat{\mathrm{x}},\hat{v},\hat{w})$ of the first order optimality point of the merit function $\mathcal{M}_{\eta,\nu}$, then the dual variables are updated by the following formula:

$$\left.\begin{aligned}\hat{y} &= y+\eta\rho_1(\hat{\mathrm{x}},\hat{v})\\ \hat{z} &= z+\eta\rho_2(\hat{\mathrm{x}},\hat{w}),\end{aligned}\right\} \tag{4.20}$$

where $\rho_1(\mathrm{x},v)=F(\mathrm{x})+v-\hat{\beta}c^\top\mathrm{x}$ and $\rho_2(\mathrm{x},w)=w-h(x)$. If the merit function $\nu_\mu(\hat{\Omega})$ reduces then the algorithm accepts $\hat{\Omega}$ as a new primal-dual approximation. Otherwise, the algorithm keeps the variables y and z unchanged and increases the penalty parameter $\eta=\frac{1}{\lambda_d}$. The algorithm is explained in detail in [111]. The formal description of the proposed algorithm is described in the following algorithm.

4.7 Numerical results and application

This subsection reports the outcomes of the application of the proposed primal-dual IPM on several types of test problems found in the literature. The performance of the Algorithm 5 is tested on constrained multiobjective multiobjective problems. The test have been carried out on a PC with Intel Core i7-4770U 3.40 GHz CPU and 4GB RAM in MATLAB 2020a. We take some widely used multiobjective test problems (BNH, SRN, TNK, CONSTR, Kita, SWG) to test the performance of Algorithm 5. The test problem Kita [97] is a maximization problem and remaining are the minimization problems. The details of these five constrained test problems are provided in Table 4.1. The Pareto points of these problems are shown in blue (see Figure 4.1, Figure 4.2 and Figure 4.3).

Algorithm 5 Ideal cone primal-dual interior-point method for MOPs

Inputs:
(a) Given MOP (4.1)
(b) Provide the number of subproblems to be solved, N

Initialization:
Set Pareto set $\leftarrow \emptyset$
Provide an initial point $\Omega^{(0)} = (\mathrm{x}^{(0)}, s^{(0)}, v^{(0)}, w^{(0)}, y^{(0)}, z^{(0)})$.
Provide the parameters $0 < \delta < 1$, $0 < \wp < 1$, $\tau > 0$, $0 < \xi < 1$
Give the accuracy precision $\epsilon > 0$
Set $\Omega = \Omega^{(0)}, r = \nu(\Omega^{(0)}), \mu = \min\{\wp r, r^2\}, r_\mu = \nu_\mu(\Omega^{(0)}), \eta = \eta_0 \geq 2(\mathcal{P} + \mathcal{J} + 1)\mu, \lambda_d = \min\{\nu(\Omega^{(0)}), \frac{1}{\eta_0}\}$.
Main Part:
for $i = 1 : 1 : N$ **do**
  Choose randomly a direction $\hat{\beta}$ using (2.3).
  while $r \geq \epsilon$ **do**
    Calculate the direction $\Delta\Omega$ to solve the system (4.8) by Cholesky factorization
    Set $\kappa = \max\{0.95, 1 - r\}$.
    Choose primal and dual steplength α_p and α_d by the formula (4.16)
    Set $\hat{p} = p + \alpha_p \Delta p$, $\hat{y} = y + \alpha_d \Delta y$ and $\hat{z} = z + \Delta z$
    if $\nu(\hat{\Omega}) \leq \nu(\Omega)$ and $N_{\lambda_d}(\hat{\Omega})$ is positive definite **then**
      Set $\Omega = \hat{\Omega}$, $r = \nu(\hat{\Omega})$, $\mu = \min\{\wp r, r^2\}$, $r_\mu = \nu_\mu(\hat{\Omega})$, $\lambda_d = \min\{\nu_\mu(\hat{\Omega}), \frac{1}{\eta}\}$
    else
      Set $\eta = \frac{1}{\lambda_d}$
      Find α_p such that the following Armijo condition is satisfied

$$\mathcal{M}_{\eta,\mu}(p + \alpha_p \Delta p, y, z) \leq \mathcal{M}_{\eta,\mu}(p, y, z) + \delta \alpha_p \left(\nabla_p \mathcal{M}_{\eta,\mu}\right)^\top \Delta p$$

      Set $\hat{p} = p + \alpha_p \Delta p$
      if $\|\nabla_p \mathcal{M}_{\eta,\mu}(\hat{p}, y, z)\| \leq \min\{\tau \|\rho(\hat{p})\|, \frac{\eta}{k}\}$, $y + \eta\rho_1(\hat{p}) \geq \xi\mu\hat{V}^{-1}e$, $z + \eta\rho_2(\hat{p}) \geq \xi\mu\hat{W}^{-1}e$ **then**
        Set $\hat{y} = y + \eta\rho_1(\hat{p})$, $\hat{z} = z + \eta\rho_2(\hat{p})$
        if $\nu_\mu(\hat{\gamma}) \leq r_\mu$, **then**
          Set $\Omega = \hat{\Omega}$, $r_\mu = \nu_\mu(\hat{\gamma})$
          if $\nu(\hat{\Omega}) \leq r$ **then**
            Set $r = \nu(\hat{\Omega})$, $\mu = \min\{\wp r, r^2\}$, $\lambda_d = \min\{\nu_\mu(\hat{\Omega}), \frac{1}{\eta}\}$, $r_\mu = \nu_\mu(\hat{\Omega})$
          end if
        else
          Set $p = \hat{p}$, $\eta = 2\eta$, $\lambda_d = \frac{1}{\eta}$
        end if
      else
        Set $p = \hat{p}$
      end if
    end if
  end while
  Calculate $F(p) = \hat{\beta} c^T \mathrm{x} - v$
  Update Pareto set $\leftarrow$ Pareto set $\bigcup\{F(p)\}$
end for
return Pareto set

Table 4.1: Constrained test problems used in this study

Problem	Source	n	$\mathcal{P}$	Number of subproblems	Accuracy
BNH	[94]	2	2	300	$1.0E-6$
SRN	[96]	2	2	75	$1.0E-6$
TNK	[95]	2	2	100	$1.0E-6$
CONSTR	[97]	2	3	400	$1.0E-6$
Kita	[97]	2	3	150	$1.0E-6$
SGW	[102]	2	2	300	$1.0E-6$

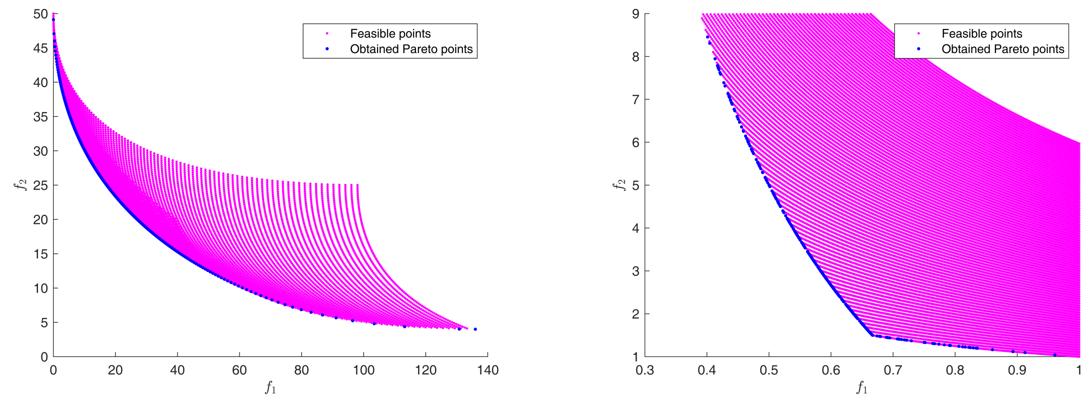

Figure 4.1: Obtained Pareto points of BNH and CONSTR problems by Algorithm 5

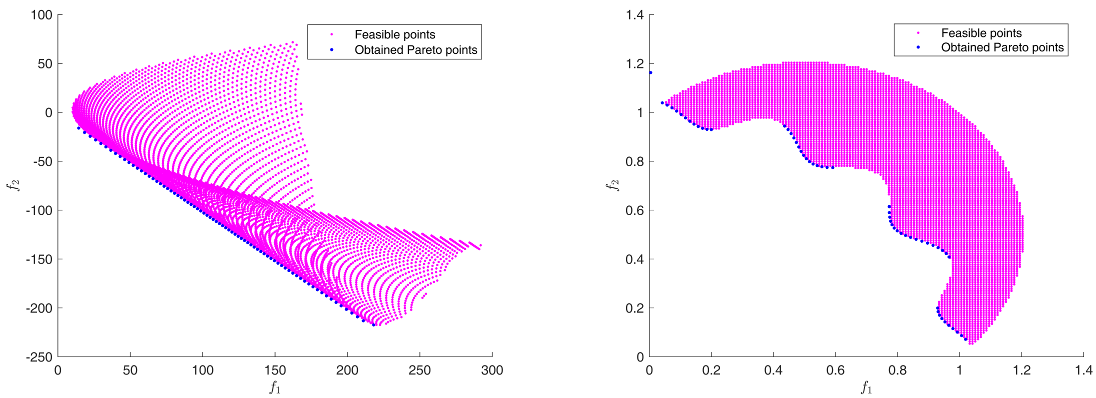

Figure 4.2: Obtained Pareto points of SRN and TNK problems by Algorithm 5

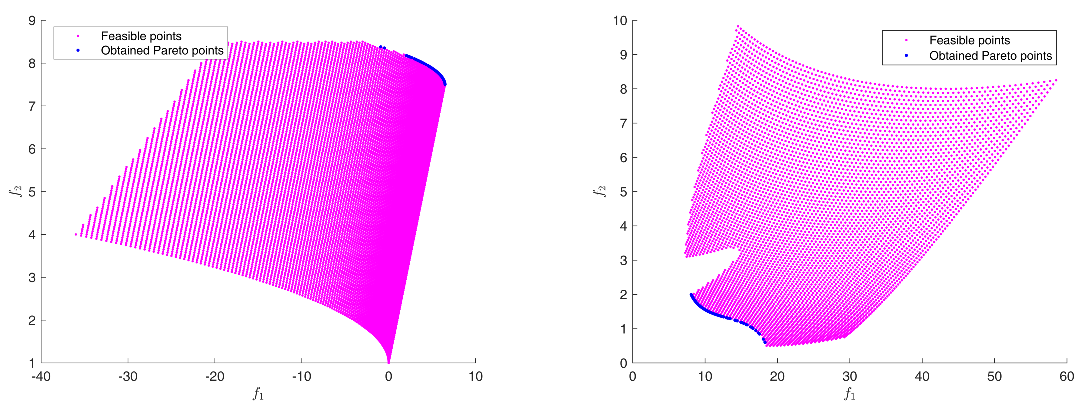

Figure 4.3: Obtained Pareto points of Kita and SGW problem by Algorithm 5

4.7.1 Carbon dioxide emission model with control

For the application, we have considered the optimal control mathematical model by Verma et al. [112]. Carbon dioxide levels in the atmosphere can be controlled by reducing the rate of CO_2 emission from energy generation and reducing the rate of energy consumption. The most desirable strategies are ones that can lower the CO_2 levels with the least amount of mitigation costs. The purpose is to find the optimal control strategies for available mitigation options, such that the their cost of implementation is minimum and at the same time it brings down the the concentration of CO_2 level in the atmosphere.

Based on the given model system (4.21), the population is categorized into three types:

- $C(t)$ - it indicates the concentration of CO_2 in atmosphere at any time t;
- $N(t)$ - it indicates the human population at any time t;
- $E(t)$ - it represents the energy usage at any time t.

The control variable of the model system are as follows:

- $u_1(t)$ - it measures the efficient use of mitigation options in order to decrease the energy consumption by enhancing energy efficiency and modifying people's behavior.
- $u_2(t)$ - it measures the effectiveness of mitigation options to reduce the CO_2 emission rate per unit of energy use. .

According to [112], the carbon dioxide emission control problem with $u_1(t)$ and $u_2(t)$ as their control variable is modeled by the following nonlinear time-varying state equations:

$$\left.\begin{aligned} \dot{C} &= -\alpha_r(C - C_0) + \lambda_1 N + \lambda_2(1 - u_2(t))E, \\ \dot{N} &= rN(1 - \frac{N}{L}) + \beta_1 NE + \beta_2 N^2 E - \theta_{\mathrm{M}}(C - C_0)N, \\ \dot{E} &= (1 - u_1(t))\frac{\gamma_e NE}{K + N} - \gamma_0 E^2, \end{aligned}\right\} \tag{4.21}$$

with the initial conditions

$$C(0) = 316.91 \text{ ppm}, \ N(0) = 3.032 \text{ billion}, \ E(0) = 40.5889 \text{ pWh (pentawatt-hours)}. \tag{4.22}$$

Following is the table that summarizes the value of the parameter considered within the control system (4.21). We obtain the values of parameters used in model system (4.21) from [112].

Symbols	Description	Value
C_0	Pre-industrial CO_2 concentration	280 ppm
λ_1	CO_2 emission rate coefficient from non-energy sectors	0.1025
α_r	Removal rate of atmospheric $C0_2$ by the sink of $C0_2$	0.01621
λ_2	CO_2 emission rate coefficient from energy sectors	0.02698
r_{In}	Intrinsic growth rate	0.0265
L	Carrying capacity of the population	11
β_1	Growth rate coefficient of the population	1.178×10^{-5}
β_2	Carrying capacity of population due to energy use	1.2×10^{-6}
θ_{M}	Mortality rate coefficient of the population due to the adverse impact posed by enhanced CO_2 levels	2.2183×10^{-7}
γ_e	Growth rate of energy use	0.08595
K	The population at which the growth rate of energy use is half of its maximum level	3.2
γ_0	Depletion rate of energy use	0.0002575

Table 4.2

4.8 Formulation of an optimal control problem

Our aim is to determine the optimal values u_1^* and u_2^* of the control u_1 and u_2, such that the corresponding state trajectories C^*, N^*, and E^* are the solution of the system (4.21) in the time interval $[0, T]$, with the initial conditions (4.22) and minimize an objective functional. Here, the objective functional considers the atmospheric CO_2 level and the implementation cost of the strategies associated with the controls $u_i, i = 1, 2$. The controls u_1 and u_2 are bounded above by $u_{1\,\max}$ and $u_{2\,\max}$, respectively, where

$u_{1\max} = 0.3$ and $u_{2\max} = 0.5$, the values of bounds for control variables are taken from [112]. We observe the optimal solution of the system (4.21) by taking $T = 83$ years.

Consider the model system of ordinary differential equations (4.21) and the set of permissible control functions given by a Lebesgue measurable set Φ, which is defined as $\Phi = \{(u_1, u_2) : 0 \leq u_1(t) \leq u_{1\max},\ 0 \leq u_2(t) \leq u_{2\max},\ \forall\ t \in [0, T]\}$. Hence, the proposed problem is to minimize the objective functional J:

$$J(u_1, u_2) = \int_0^T \big[P_0 C(t) + P_1 u_1^2(t) + P_2 u_2^2(t)\big]\, dt, \tag{4.23}$$

where the constants P_0, P_1 and P_2 are the positive weights parameters. The minimization of J has two main aspects: (i) decreasing the level of CO_2 in atmosphere and (ii) reducing the cost of implementation of mitigation options. We consider the optimal control problem of determining (C^*, N^*, E^*), associated to an permissible control pair $(u_1^*, u_2^*) \in \Phi$ on the time interval $[0, T]$, satisfying (4.21), with initial condition (4.22), and minimizing the cost functional (4.23), i.e.,

$$J(u_1^*, u_2^*) = \min_{\Phi}\ J(u_1, u_2). \tag{4.24}$$

4.9 Multiobjective approach to the optimal control problem (4.24)

The available approaches that use the idea of optimal control theory have a significant disadvantage that it can only yield a single optimal solution for the problem (4.24), which is defined from some decision-makers perspective for some choice of constants P_0, P_1 and P_2, for detail see [113, 114], and the references therein. However, the choice of parameters P_0, P_1 and P_2 require some prior knowledge about the problem and decision makers preferences, which may not always be available. A single optimal solution for the problem (4.24) does not provide all helpful insight into the optimal strategies and corresponding dynamics. Thus, many optimal alternatives remains unexplored by using the approaches based on optimal control theory.

This work proposes an approach that uses the idea of multiobjective optimization, which may provide the Pareto solutions instead of a single optimal solution. In the proposed approach, we decompose the cost functional shown in (4.23) into two components, each representing a certain aspects of model system (4.21) that must be taken into consideration.

The bi-objective formulations corresponding to (4.24) can be defined by the following:

$$\left.\begin{array}{ll} \text{minimize} & (\eta_1, \eta_2) \\ \text{subject to} & \dot{C} = -\alpha_r(C - C_0) + \lambda_1 N + \lambda_2(1 - u_2(t))E, \\ & \dot{N} = r_{\mathrm{In}} N(1 - \dfrac{N}{L}) + \beta_1 NE + \beta_2 N^2 E - \theta_{\mathrm{M}}(C - C_0)N, \\ & \dot{E} = (1 - u_1(t))\dfrac{\gamma_e NE}{K + N} - \gamma_0 E^2, \\ & C(0) = 316.91,\ N(0) = 3.032,\ E(0) = 40.5889, \\ & 0 \le u_1 \le u_{1\,\max}, \\ & 0 \le u_2 \le u_{2\,\max}, \end{array}\right\} \tag{4.25}$$

where

$$\eta_1(u_1, u_2) = \int_0^T C(t)dt \text{ and} \tag{4.26}$$

$$\eta_2(u_1, u_2) = \int_0^T \big(u_1^2(t) + u_2^2(t)\big)dt. \tag{4.27}$$

In the above formulation, the first objective function η_1 represents total amount of CO_2 which is concentrated in the atmosphere during the period $[0, T]$, and the second objective function η_2 represents the cost associated with mitigation during the period $[0, T]$. In the above formulation, the constants P_0, P_1 and P_2 are ignored. The underlying conflict highlights the difficulty of decision making, thus solving (4.25) is exciting and challenging.

4.10 Numerical results

In the following section, we present and analyze the numerical results obtained by applying Algorithm 5 to the model system (4.21) using multiobjective optimization approach. We study the optimal control strategies under different values of growth rate of energy use γ_e for model system (4.21).

In the following part, we describe the obtained optimal solutions to the problem (4.25), taking into account the variations of the parameter γ_e.

Figures 4.4 and 4.5 plot the feasible criterion region and Pareto optimal front of the problem (4.25) for $\gamma_e = 0.06$ and 0.1, respectively, obtained from Algorithm 5.

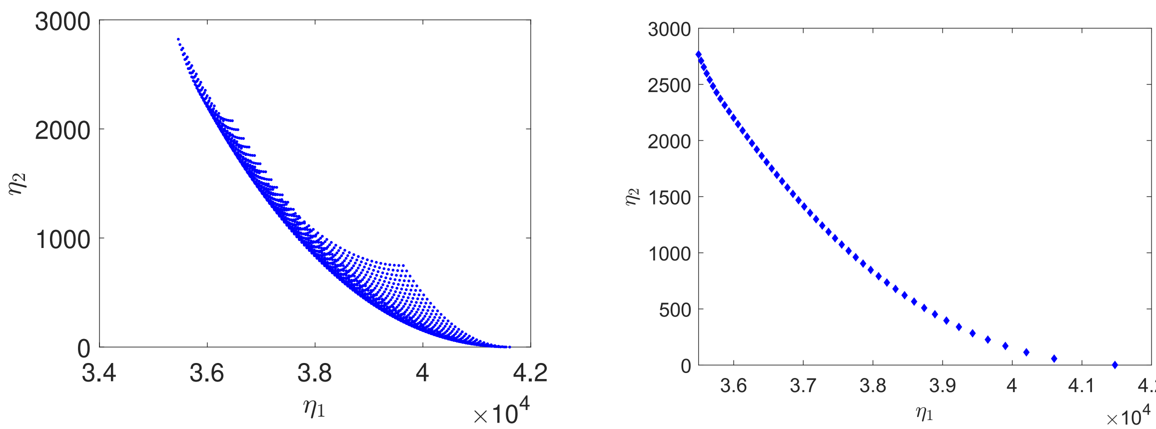

Figure 4.4: Feasible criterion region (left) and Pareto front for $\gamma_e = 0.06$ (right)

The left picture in Figure 4.6 plots the trade-off curve obtained for three different values of growth rate of energy use γ_e. From the figures, we observe that the difference between the best and worst values of η_1 increases when γ_e increases.

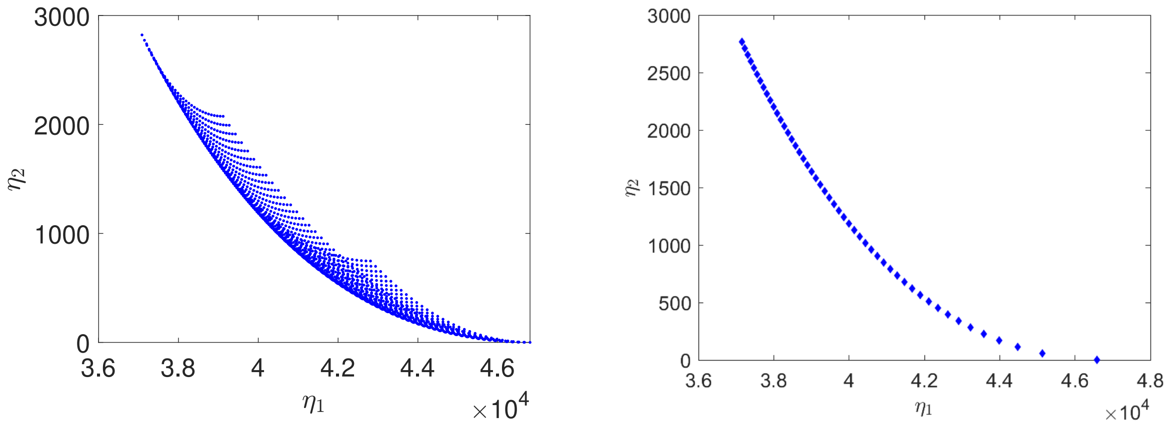

Figure 4.5: Feasible criterion region (left) and Pareto front for $\gamma_e = 0.1$ (right)

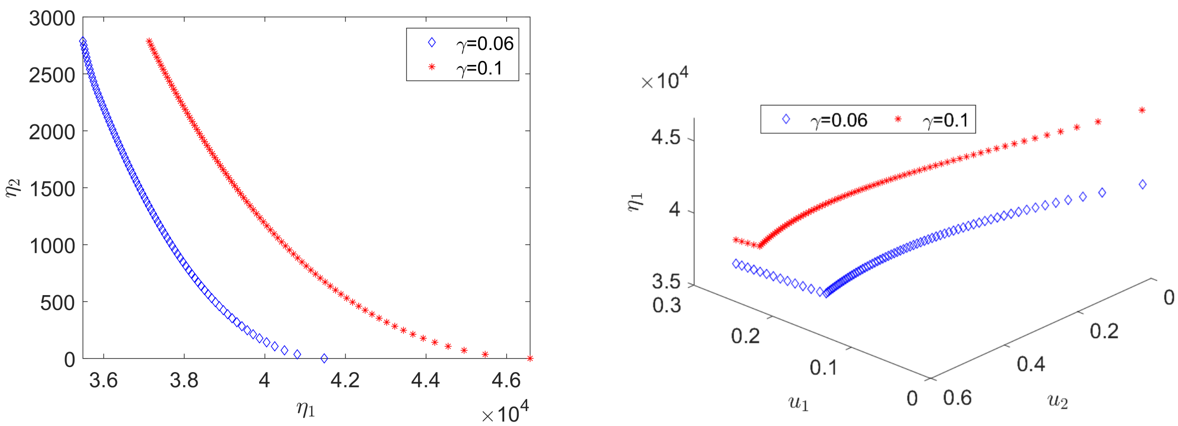

Figure 4.6: Trade-off curves for different values of γ_e (left) and representation of η_1 as a function of Pareto optimal strategies u_1 and u_2 for different values of γ_e (right)

The right picture in Figure 4.6 plots the values of η_1 (total amount of CO_2 which is concentrated the time span of 83 years) as a function of Pareto optimal control strategy u_1^*, u_2^* for two different values of γ_e. From the figures, we observe that the minimum and maximum values of η_1 increases as we increase the value of γ_e.

Figures 4.7 plots the amount of CO_2 concentration in the atmosphere, human population and energy usage, respectively, with and without control for two different values of γ_e, and for the time period of 83 years. For without control we assume a fixed value of $u_1 = 0.001$, i.e., $u_2 = 0$. To plot the amount of CO_2 concentration in the atmosphere, human population and energy usage with control, we have used $(u_1^*, u_2^*) = (0.2887, 0.5)$, a Pareto optimal solution corresponding to maximum control obtained from Algorithm 5 for $\gamma_e = 0.06$ and $\gamma_e = 0.1$. From the plots in Figure 4.7, we observe that the CO_2 concentration with control strategy is significantly lower than without control strategy by the end of 21 st century, regardless the values of γ_e (growth rate of energy use).

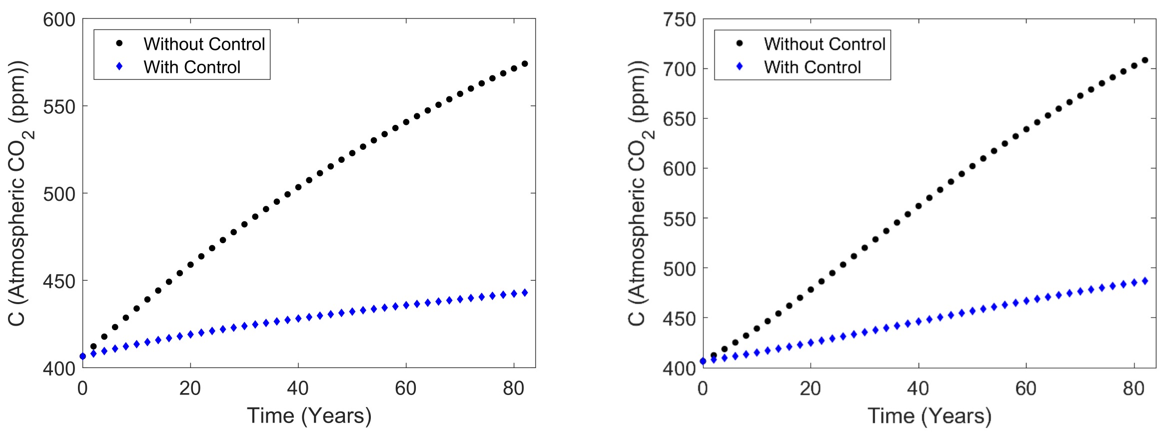

Figure 4.7: CO_2 concentration with and without control for $\gamma_e = 0.06$ (left) and CO_2 concentration with and without control for $\gamma_e = 0.1$ (right)

4.11 Conclusion

This chapter has introduced a primal-dual interior-point approach (PD-IPM), with the help of the cone method, to find a subset of nondominated points of an MOPs. In the proposed method, cone method has been used to transform an MOP into a collection of single objective optimization problems. Each single objective optimization problem of the collection has been solved by PD-IPM. To find the solution of each subproblem by PD-IPM, a barrier problem and its KKT conditions have been derived. In order to the solve the pertaining KKT systems, the iteration started with a given initial point and then calculated the direction by Newton method. Thereafter, step length has been chosen so that the nonnegative variables remain nonnegative. In the sequel, we have used a merit function to take a appropriate step length along the search direction. A sufficient condition for a point to be stationary of the used merit function has been also provided (Theorem 4.1). It has been found that the chosen primal directions in the movement of Algorithm 5 are descent directions of the used merit function (Theorem 4.2).

To show the diversity of the proposed method, we applied the proposed method to solve an optimal control problem of carbon dioxide emission from energy sector with the efficiencies of mitigation options to reduce energy consumption rate and the CO_2 emission rate as their control variables. For which we proposed a multiobjective optimization approach to find the optimal strategies to minimize both—the CO_2 level from energy sector and the implementation cost of the control strategies. The result obtained by applying Algorithm 5 to the problem (4.25) has been analyzed for two different values of growth rate of energy use, i.e., $\gamma = 0.06$ and 0.1. We have obtained the trade-off curves for two values γ which shows that the difference between the best and worst case scenario for η_1 increases when γ increases. We have also observed that the CO_2 concentration with control strategy is significantly lower than without control strategy by the end of 21st century, regardless the values of γ (growth rate of energy use).

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