Chapter 4

Skew Constacyclic Codes over a Class of Non-chain Rings

In this chapter, we explore skew constacyclic codes over a general class of non-chain rings. We begin by examining the key properties of this non-chain ring $\mathcal{T} := \mathbb{F}_q[u_1, u_2, \ldots, u_r]/\langle f_j(u_j), u_iu_j - u_ju_i\rangle$, and recall the definition of the Gray map along with its notable properties, such as distance preserving and Euclidean orthogonality preserving properties, as discussed in [14]. Additionally, we prove that the Gray map preserves Hermitian orthogonality. One of the key results presented is an explicit form for the generator matrix of the Gray image of a linear code over $\mathcal{T}$.

In Section 4.2, we delve into essential properties of automorphisms and units of $\mathcal{T}$, followed by an investigation into the structural properties of skew constacyclic codes over $\mathcal{T}$. Finally, we demonstrate that the Gray image of a skew constacyclic code over $\mathcal{T}$ corresponds to a class of linear codes that represents a specific case of skew multi-twisted codes (where all blocks have the same length), and serves as a generalization of skew quasi-twisted codes (allowing for potentially different twisting constants for each block).

4.1 The Ring $\mathcal{T}$ and Linear Codes over $\mathcal{T}$

In this section, we study some basic properties of a general non-chain ring $\mathcal{T}$ which represents a class of non-chain rings. Further, we study linear codes over $\mathcal{T}$ and their properties.

4.1.1 The Ring $\mathcal{T}$

Let $\mathbb{F}_q$ be a finite field of order $q = p^m$, where p is a prime and m is a positive integer. Consider $\mathcal{T} := \mathbb{F}_q[u_1, u_2, \ldots, u_r]/\langle f_j(u_j), u_iu_j - u_ju_i\rangle$, where $f_j(u_j)$ is a non-constant, monic polynomial which splits into distinct linear factors, for all $j = 1, 2, \ldots, r$. Then, $\mathcal{T}$ is a finite commutative ring with identity. Let $T_1 = \mathbb{F}_q[u_1]\langle f_1(u_1)\rangle$ and $T_{j+1} = T_j[u_{j+1}]\langle f_{j+1}(u_{j+1})\rangle$. Thus, $\mathcal{T} = T_r$.

Consider two ideals $I_1 = \langle u_1 - \alpha_{i_1}\rangle$ and $I_2 = \langle u_2 - \alpha_{i_2}\rangle$ of $\mathcal{T}$, where α_{i_j} is a zero of $f_j(u_j)$, $j = 1, 2$. Then, $u_2 - \alpha_{i_2} \notin I_1$ and $u_1 - \alpha_{i_1} \notin I_2$. Therefore, $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$ i.e. I_1 and I_2 are two incomparable ideals of $\mathcal{T}$. Hence, $\mathcal{T}$ is a **non-chain ring**.

Furthermore, let us assume that, $deg(f_j(u_j)) = l_j$ and $\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jl_j}$ are the zeros of $f_j(u_j)$, $j = 1, 2, \ldots, r$. Then,

$$f_j(u_j) = (u_j - \alpha_{j1})(u_j - \alpha_{j2}) \cdots (u_j - \alpha_{jl_j}).$$

Now, let us take

$$\kappa_{i_j} = \frac{(u_j - \alpha_{j1})(u_j - \alpha_{j2}) \cdots (u_j - \alpha_{j(i-1)})(u_j - \alpha_{j(i+1)}) \cdots (u_j - \alpha_{jl_j})}{(\alpha_{ji} - \alpha_{j1})(\alpha_{ji} - \alpha_{j2}) \cdots (\alpha_{ji} - \alpha_{j(i-1)})(\alpha_{ji} - \alpha_{j(i+1)}) \cdots (\alpha_{ji} - \alpha_{jl_j})},$$

and define $\eta_{i_1 i_2 \dots i_r} := \prod_{j=1}^{r} \kappa_{i_j},\ i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r.$

Lemma 4.1.1. [14, Lemma 2] The set $\{\eta_{i_1 i_2 \dots i_r} : i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r\}$ is set of primitive orthogonal idempotents in $\mathcal{T}$, i.e.

$$\begin{aligned} \eta^2_{i_1 i_2 \dots i_r} &= \eta_{i_1 i_2 \dots i_r}; \\ \eta_{i_1 i_2 \dots i_r} \eta_{k_1 k_2 \dots k_r} &= 0 \quad \text{for} \quad (i_1, i_2, \dots, i_r) \neq (k_1, k_2, \dots, k_r); \\ \sum_{i_1, i_2, \dots, i_r} \eta_{i_1 i_2 \dots i_r} &= 1. \end{aligned} \tag{4.1}$$

for all $i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r$, where $\sum_{i_1, i_2, \dots, i_r} := \sum_{i_1=1}^{l_1} \dots \sum_{i_r=1}^{l_r}$.

Thus, by a decomposition theorem of ring theory 1.1.48, we have

$$\mathcal{T} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{T} \cong \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathbb{F}_q.$$

Therefore, any element $\mathsf{v} \in \mathcal{T}$ can be expressed as

$$\mathsf{v} = \sum_{i_1, i_2, \dots, i_r} \eta_{i_1 i_2 \dots i_r} v_{i_1 i_2 \dots i_r}$$

in a unique way, where $v_{i_1 i_2 \dots i_r} \in \mathbb{F}_q$ and $i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r$.

Moreover, we can view the ring $\mathcal{T}$ as a finite $\mathbb{F}_q$-algebra with $\mathcal{B} = \{\eta_{i_1 i_2 \dots i_r} : i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r\}$ as a basis. We define a bilinear form $(,) : \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{F}_q$ as $(\mathsf{v}, \mathsf{w}) = \left(\sum_{i_1, i_2, \dots, i_r} \eta_{i_1 i_2 \dots i_r} v_{i_1 i_2 \dots i_r}, \sum_{i_1, i_2, \dots, i_r} \eta_{i_1 i_2 \dots i_r} w_{i_1 i_2 \dots i_r} \right) = \sum_{i_1, i_2, \dots, i_r} \eta_{i_1 i_2 \dots i_r} v_{i_1 i_2 \dots i_r} w_{i_1 i_2 \dots i_r}$. Then, $(\mathsf{v}, \mathsf{w}) = 0,\ \forall\ \mathsf{w} \in \mathcal{T} \implies \mathsf{v} = 0$. This can be easily proved by taking $\mathsf{w} = \eta_{i_1 i_2 \dots i_r},\ i_j \in \{1, 2, \dots, l_j\}, j = 1, 2, \dots, r$. This shows that $(,)$ is non-degenerate. Therefore, $\mathcal{T}$ is a commutative, finite-dimensional Frobenius algebra over a finite field and hence it is a **Frobenius ring**.

4.1.2 Gray Map

A Gray map $\phi : \mathcal{T} \to \mathbb{F}_q^{l_1 l_2 \dots l_r}$ is defined as

$$\begin{aligned}\phi(\mathsf{v}) =& (v_{i_1 i_2 \dots i_r})_{i_1, i_2, \dots, i_r} M \\ =& \big(v_{11\dots11}, v_{11\dots12} \dots, v_{11\dots1l_r}, v_{11\dots21}, v_{11\dots22} \dots, v_{11\dots2l_r}, \dots, v_{1l_2\dots l_{r-1}l_r}, \\ & v_{21\dots11}, v_{21\dots12} \dots, v_{21\dots1l_r}, v_{21\dots21}, v_{21\dots22} \dots, v_{21\dots2l_r}, \dots, v_{2l_2\dots l_{r-1}l_r}, \\ & \dots, v_{(l_1-1)1\dots11}, \dots, v_{(l_1-1)l_2\dots l_{r-1}l_r}, \dots, v_{l_1 l_2 \dots l_{r-1}1}, v_{l_1 l_2 \dots l_{r-1}2} \dots, v_{l_1 l_2 \dots l_{r-1} l_r}\big) M,\end{aligned}$$

for all $\mathsf{v} = \sum_{i_1, i_2, \dots, i_r} \eta_{i_1 i_2 \dots i_r} v_{i_1 i_2 \dots i_r} \in \mathcal{T}$, where $M \in GL_{l_1 l_2 \dots l_r}(\mathbb{F}_q)$ is such that $MM^T = \lambda I_{l_1 l_2 \dots l_r}$, for some $\lambda \in \mathbb{F}_q^*$.

The Lee-weight of an element $\mathsf{v} \in \mathcal{T}$ is defined as

$$w_L(\mathsf{v}) = w_H(\phi(\mathsf{v})),$$

where w_H denotes the Hamming weight.

We can extend ϕ to $\mathcal{T}^n$ as $\Phi : \mathcal{T}^n \mapsto \mathbb{F}_q^{l_1 l_2 \dots l_r n}$ as

$$\Phi(\mathbf{v}) = (\phi(\mathsf{v}^0), \phi(\mathsf{v}^1), \dots, \phi(\mathsf{v}^{n-1})),$$

for all $\mathbf{v} = (\mathsf{v}^0, \mathsf{v}^1, \ldots, \mathsf{v}^{n-1}) \in \mathcal{T}^n$. For any $\mathbf{v} = (\mathsf{v}^0, \mathsf{v}^1, \ldots, \mathsf{v}^{n-1}) \in \mathcal{T}^n$, we define its Lee-weight as

$$w_L(\mathbf{v}) = \sum_{k=0}^{n-1} w_L(\mathsf{v}^k).$$

For any two $\mathbf{v}, \mathbf{w} \in \mathcal{T}^n$, their Lee distance is defined as

$$d_L(\mathbf{v}, \mathbf{w}) = w_L(\mathbf{v} - \mathbf{w}).$$

Theorem 4.1.2. [14, Theorem 6] The Gray map Φ is a bijective, linear map and it preserves the distance between $(\mathcal{T}^n, d_L)$ and $(\mathbb{F}_q^{l_1 l_2 \ldots l_r n}, d_H)$.

Definition 4.1.3. For $\mathbf{c}, \mathbf{d} \in \mathcal{T}^n$ such that $\mathbf{c} = (\mathsf{c}^0, \mathsf{c}^1, \ldots, \mathsf{c}^{n-1})$ and $\mathbf{d} = (\mathsf{d}^0, \mathsf{d}^1, \ldots, \mathsf{d}^{n-1})$, their Euclidean product is defined as

$$\langle \mathbf{c}, \mathbf{d} \rangle_E = \sum_{i=1}^{n-1} \mathsf{c}^i \mathsf{d}^i.$$

We say that $\mathbf{c}$ is orthogonal to $\mathbf{d}$ if $\langle \mathbf{c}, \mathbf{d} \rangle_E = 0$ and we'll denote it as $\mathbf{c} \perp_E \mathbf{d}$.

Definition 4.1.4. If $q = p^{2e}$ is an even power of some prime, then for

$\mathsf{v} = \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} v_{i_1 i_2 \ldots i_r} \in \mathcal{T}$, its conjugate is defined as

$$\bar{\mathsf{v}} = \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} \overline{v}_{i_1 i_2 \ldots i_r} = \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} v_{i_1 i_2 \ldots i_r}^{p^e}.$$

For $\mathbf{c}, \mathbf{d} \in \mathcal{T}^n$ such that $\mathbf{c} = (\mathsf{c}^0, \mathsf{c}^1, \ldots, \mathsf{c}^{n-1})$ and $\mathbf{d} = (\mathsf{d}^0, \mathsf{d}^1, \ldots, \mathsf{d}^{n-1})$, their Hermitian product is defined as

$$\langle \mathbf{c}, \mathbf{d} \rangle_H = \sum_{i=1}^{n-1} \mathsf{c}^i \overline{\mathsf{d}^i}.$$

We say that $\mathbf{c}$ is Hermitian orthogonal to $\mathbf{d}$ if $\langle \mathbf{c}, \mathbf{d} \rangle_H = 0$ and we'll denote it as $\mathbf{c} \perp_H \mathbf{d}$.

Theorem 4.1.5. [14, Theorem 6] For any two $\mathbf{c}, \mathbf{d} \in \mathcal{T}^n$, $\mathbf{c} \perp_E \mathbf{d}$ if and only if $\Phi(\mathbf{c}) \perp_E \Phi(\mathbf{d})$. In other words, Φ preserves Euclidean orthogonality.

Theorem 4.1.6. For any two $\mathbf{c}, \mathbf{d} \in \mathcal{T}^n$, $\mathbf{c} \perp_H \mathbf{d}$ if and only if $\Phi(\mathbf{c}) \perp_H \Phi(\mathbf{d})$. In other words, Φ preserves Hermitian orthogonality.

Proof. Let $\mathbf{c}, \mathbf{d} \in \mathcal{T}^n$ such that $\mathbf{c} = (\mathsf{c}^0, \mathsf{c}^1, \ldots, \mathsf{c}^{n-1})$ and $\mathbf{d} = (\mathsf{d}^0, \mathsf{d}^1, \ldots, \mathsf{d}^{n-1})$, where $\mathsf{c}^i = \sum_{i_1 i_2 \ldots i_r} c^i_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r}$ and $\mathsf{d}^i = \sum_{i_1 i_2 \ldots i_r} d^i_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r}$, for $i = 0, 1, 2, \ldots,$ $n-1$. Now, using the definition of Hermitian product and properties of primitive orthogonal idempotents, we get

$$\begin{aligned}
\langle \mathbf{c}, \mathbf{d} \rangle_H &= \sum_{i=0}^{n-1} \mathsf{c}^i \overline{\mathsf{d}^i} \\
&= \sum_{i=0}^{n-1} \left(\sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} c^i_{i_1 i_2 \ldots i_r} \right) \cdot \left(\sum_{l_1 l_2 \ldots l_r} \eta_{l_1 l_2 \ldots l_r} \overline{d^i}_{i_1 i_2 \ldots i_r} \right) \\
&= \sum_{i=0}^{n-1} \left(\sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} c^i_{i_1 i_2 \ldots i_r} \overline{d^i}_{i_1 i_2 \ldots i_r} \right) \\
&= \sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} \left(\sum_{i=0}^{n-1} c^i_{i_1 i_2 \ldots i_r} \overline{d^i}_{i_1 i_2 \ldots i_r} \right). \qquad (4.2)
\end{aligned}$$

and

$$\begin{aligned}
\langle \Phi(\mathbf{c}), \Phi(\mathbf{d}) \rangle_H &= \Phi(\mathbf{c}) \overline{\Phi(\mathbf{d})}^T \\
&= \sum_{i=0}^{n-1} \phi(\mathsf{c}^i) \overline{\phi(\mathsf{d}^i)}^T
\end{aligned}$$

$$= \begin{bmatrix} \phi(\mathsf{c}^0) & \phi(\mathsf{c}^1) & \cdots & \phi(\mathsf{c}^{n-1}) \end{bmatrix} \begin{bmatrix} \overline{\phi(\mathsf{d}^0)}^T \\ \overline{\phi(\mathsf{d}^1)}^T \\ \vdots \\ \overline{\phi(\mathsf{d}^{n-1})}^T \end{bmatrix}$$

$$= \begin{bmatrix} (c^0_{i_1i_2\ldots i_r})_{i_1i_2\ldots i_r}M & \cdots & (c^{n-1}_{i_1i_2\ldots i_r})_{i_1i_2\ldots i_r}M \end{bmatrix} \begin{bmatrix} ((\overline{d^0}_{i_1i_2\ldots i_r})_{i_1i_2\ldots i_r})^T M \\ ((\overline{d^1}_{i_1i_2\ldots i_r})_{i_1i_2\ldots i_r})^T M \\ \vdots \\ ((\overline{d^{n-1}}_{i_1i_2\ldots i_r})_{i_1i_2\ldots i_r})^T M \end{bmatrix}$$

$$= \sum_{i=0}^{n-1} (c^i_{i_1i_2\ldots i_r})_{i_1i_2\ldots i_r} M M^T ((\overline{d^i}_{i_1i_2\ldots i_r})_{i_1i_2\ldots i_r})^T$$

$$= \lambda \sum_{i=0}^{n-1} \left(\sum_{i_1i_2\ldots i_r} c^i_{i_1i_2\ldots i_r} \overline{d^i}_{i_1i_2\ldots i_r} \right)$$

$$= \lambda \sum_{i_1i_2\ldots i_r} \sum_{i=0}^{n-1} c^i_{i_1i_2\ldots i_r} \overline{d^i}_{i_1i_2\ldots i_r} \tag{4.3}$$

Since, $\{\eta_{i_1i_2\ldots i_r} : i_j \in \{1,2,\ldots,l_j\}, j = 1,2,\ldots,r\}$ is a linearly independent set and $\lambda \in \mathbb{F}_q^*$, from (4.2) and (4.3), we conclude that $\langle \mathbf{c}, \mathbf{d} \rangle_H = 0$ if and only if $\langle \Phi(\mathbf{c}), \Phi(\mathbf{d}) \rangle_H = 0$, i.e. $\mathbf{c} \perp_H \mathbf{d}$ if and only if $\Phi(\mathbf{c}) \perp_H \Phi(\mathbf{d})$. □

4.1.3 Linear Codes over $\mathcal{T}$

A $\mathcal{T}$-submodule $\mathcal{C}$ of $\mathcal{T}^n$ is called a linear code of length n over $\mathcal{T}$. For a linear code $\mathcal{C} \subseteq \mathcal{T}^n$, and for $(i_1, i_2, \ldots, i_r)$ with $i_j \in \{1, 2, \ldots, l_j\}$, $j = 1, 2, \ldots, r$, we define

$$\mathcal{C}_{i_1i_2\ldots i_r} = \Big\{ w_{i_1i_2\ldots i_r} \in \mathbb{F}_q^n \ : \exists\ w_{k_1k_2\ldots k_r} \in \mathbb{F}_q^n, (i_1, i_2, \ldots, i_r) \neq (k_1, k_2, \ldots, k_r) \text{ such that } \sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r} w_{i_1i_2\ldots i_r} \in \mathcal{C} \Big\}.$$

Then, $\mathcal{C}_{i_1 i_2 \ldots i_r} \subseteq \mathbb{F}_q^n$ is a linear code of length n over $\mathbb{F}_q$, for all $i_j \in \{1, 2, \ldots, l_j\}$, $j = 1, 2, \ldots, r$. Moreover, $\mathcal{C} = \bigoplus_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} \mathcal{C}_{i_1 i_2 \ldots i_r}$ and $|\mathcal{C}| = \Pi_{i_1 i_2 \ldots i_r} |\mathcal{C}_{i_1 i_2 \ldots i_r}|$.

Definition 4.1.7. For a linear code $\mathcal{C}$ of length n over $\mathcal{T}$, its Euclidean dual $\mathcal{C}^{\perp_E}$ and Hermitian dual $\mathcal{C}^{\perp_H}$ are defined as

$$\mathcal{C}^{\perp_E} = \{\mathbf{w} \in \mathcal{T}^n : \langle \mathbf{w}, \mathbf{v} \rangle_E = 0, \forall\ \mathbf{v} \in \mathcal{C}\},$$

$$\mathcal{C}^{\perp_H} = \{\mathbf{w} \in \mathcal{T}^n : \langle \mathbf{w}, \mathbf{v} \rangle_H = 0, \forall\ \mathbf{v} \in \mathcal{C}\}.$$

Definition 4.1.8. $\mathcal{C}$ is Euclidean self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^{\perp_E}$, Euclidean self-dual if $\mathcal{C}^{\perp_E} = \mathcal{C}$ and Euclidean dual-containing if $\mathcal{C}^{\perp_E} \subseteq \mathcal{C}$.

$\mathcal{C}$ is Hermitian self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^{\perp_H}$, Hermitian self-dual if $\mathcal{C}^{\perp_H} = \mathcal{C}$ and Hermitian dual-containing if $\mathcal{C}^{\perp_H} \subseteq \mathcal{C}$.

Theorem 4.1.9. Let $\mathcal{C} = \bigoplus_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} \mathcal{C}_{i_1 i_2 \ldots i_r}$ be an (n, q^k, d_L) linear code over $\mathcal{T}$, then

(i) $\Phi(\mathcal{C})$ is a $[l_1 l_2 \ldots l_r n, k, d_H]$ linear code over $\mathbb{F}_q$, where $d_H = d_L$;

(ii) $\Phi(\mathcal{C})^\perp = \Phi(\mathcal{C}^\perp)$;

(iii) $\mathcal{C}^\perp = \bigoplus_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} \mathcal{C}^\perp_{i_1 i_2 \ldots i_r}$;

(iv) $\mathcal{C}$ is a self-orthogonal code if and only if $\Phi(\mathcal{C})$ is a self-orthogonal code over $\mathbb{F}_q$;

(v) $\mathcal{C}$ is a dual-containing code if and only if $\Phi(\mathcal{C})$ is a dual-containing code over $\mathbb{F}_q$;

(vi) $\mathcal{C}$ is a self-dual code if and only if $\Phi(\mathcal{C})$ is a self-dual code over $\mathbb{F}_q$.

Here, $\perp = \perp_E$ *or* $\perp_H$.

Proof. These results can be established using an approach similar to the proof of Theorem 3.3.3. □

Theorem 4.1.10. Let $\mathcal{C} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$ be a linear code of length n over $\mathcal{T}$. Furthermore, let $G_{i_1 i_2 \dots i_r}$ be a generator matrix of the $[n, k_{i_1 i_2 \dots i_r}]$ q-ary linear code $\mathcal{C}_{i_1 i_2 \dots i_r}$, $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$ and M be the matrix used in Gray map ϕ such that

$$G_{i_1 i_2 \dots i_r} = \begin{bmatrix} a_{00}^{i_1 i_2 \dots i_r} & a_{01}^{i_1 i_2 \dots i_r} & \dots & a_{0(n-1)}^{i_1 i_2 \dots i_r} \\ a_{10}^{i_1 i_2 \dots i_r} & a_{11}^{i_1 i_2 \dots i_r} & \dots & a_{1(n-1)}^{i_1 i_2 \dots i_r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(k_{i_1 i_2 \dots i_r}-1)0}^{i_1 i_2 \dots i_r} & a_{(k_{i_1 i_2 \dots i_r}-1)1}^{i_1 i_2 \dots i_r} & \dots & a_{(k_{i_1 i_2 \dots i_r}-1)(n-1)}^{i_1 i_2 \dots i_r} \end{bmatrix}_{k_{i_1 i_2 \dots i_r} \times n}$$

$$\text{and } M = \begin{bmatrix} m_{11\dots 11, 11\dots 11} & \dots & m_{11\dots 11, l_1 l_2 \dots l_r} \\ \vdots & \vdots & \vdots \\ m_{l_1 l_2 \dots l_r, 11\dots 11} & \dots & m_{l_1 l_2 \dots l_r, l_1 l_2 \dots l_r} \end{bmatrix} \in GL_{l_1 l_2 \dots l_r}(\mathbb{F}_q).$$

Then,

$$G = \begin{bmatrix} G_{11\dots 11} \otimes M_{R_{11\dots 11}} \\ G_{11\dots 12} \otimes M_{R_{11\dots 12}} \\ \vdots \\ G_{l_1 l_2 \dots l_{r-1} l_r} \otimes M_{R_{l_1 l_2 \dots l_{r-1} l_r}} \end{bmatrix}$$

is a generator matrix of $\Phi(\mathcal{C})$, where $M_{R_{i_1 i_2 \dots i_r}}$ denotes the $i_1 i_2 \dots i_r{}^{th}$ row of M, $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$.

Proof. Let $\mathbf{w} \in \Phi(\mathcal{C})$ be an arbitrary element. Since, Φ is bijective, there exists a unique $\mathbf{v} = \sum_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathbf{v}_{i_1 i_2 \dots i_r} \in \mathcal{C}$ such that $\Phi(\mathbf{v}) = \mathbf{w}$ and $\mathbf{v}_{i_1 i_2 \dots i_r} = (\mathsf{v}_{i_1 i_2 \dots i_r}^0, \mathsf{v}_{i_1 i_2 \dots i_r}^1, \dots, \mathsf{v}_{i_1 i_2 \dots i_r}^{n-1}) \in \mathcal{C}_{i_1 i_2 \dots i_r}$. Since, $G_{i_1 i_2 \dots i_r}$ is a generator matrix of

$\mathcal{C}_{i_1i_2\ldots i_r}$, there exist $\alpha_{i_1i_2\ldots i_r,0}, \alpha_{i_1i_2\ldots i_r,1}, \ldots, \alpha_{i_1i_2\ldots i_r,k_{i_1i_2\ldots i_r}-1}$ such that

$$\begin{aligned}
\mathbf{v}_{i_1i_2\ldots i_r} &= \sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} \left(a_{l0}^{i_1i_2\ldots i_r}, a_{l1}^{i_1i_2\ldots i_r}, \ldots, a_{l(n-1)}^{i_1i_2\ldots i_r}\right) \\
&= \left(\sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,,l} a_{l0}^{i_1i_2\ldots i_r}, \sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,,l} a_{l1}^{i_1i_2\ldots i_r}, \ldots, \right. \\
&\qquad \left. \sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,,l} a_{l(n-1)}^{i_1i_2\ldots i_r}\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{v} &= \sum_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r} \mathbf{v}_{i_1i_2\ldots i_r} \\
&= \sum_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r} \left(\sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} a_{l0}^{i_1i_2\ldots i_r}, \sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} a_{l1}^{i_1i_2\ldots i_r}, \ldots, \right. \\
&\qquad \left. \sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} a_{l(n-1)}^{i_1i_2\ldots i_r}\right) \\
&= \left(\sum_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r} \left(\sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} a_{l0}^{i_1i_2\ldots i_r}\right), \right. \\
&\qquad \sum_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r} \left(\sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} a_{l1}^{i_1i_2\ldots i_r}\right), \ldots, \\
&\qquad \left. \sum_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r} \left(\sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} a_{l(n-1)}^{i_1i_2\ldots i_r}\right)\right) \\
&= (\mathsf{v}^0, \mathsf{v}^1, \ldots, \mathsf{v}^{n-1}) \text{ (say)}.
\end{aligned}$$

Therefore, $\mathbf{w} = \Phi(\mathbf{v}) = (\phi(\mathsf{v}^0), \phi(\mathsf{v}^1), \ldots, \phi(\mathsf{v}^{n-1})) = (\mathtt{w}^0, \mathtt{w}^1, \ldots, \mathtt{w}^{n-1})$ (say). Then,

$$\mathtt{w}^i = \phi\left(\sum_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r} \left(\sum_{l=0}^{k_{i_1i_2\ldots i_r}-1} \alpha_{i_1i_2\ldots i_r,l} a_{li}^{i_1i_2\ldots i_r}\right)\right)$$

$$= \left(\sum_{l=0}^{k_{11\ldots 11}-1} \alpha_{11\ldots 11,l} a_{li}^{11\ldots 11}, \sum_{l=0}^{k_{11\ldots 12}-1} \alpha_{11\ldots 12,l} a_{li}^{11\ldots 12}, \ldots, \right.$$

$$\left. \sum_{l=0}^{k_{l_1 l_2 \ldots l_{r-1} l_r}-1} \alpha_{l_1 l_2 \ldots l_{r-1} l_r, l} a_{li}^{l_1 l_2 \ldots l_{r-1} l_r} \right) M$$

$$= \left(\sum_{i_1 i_2 \ldots i_r} m_{i_1 i_2 \ldots i_r, 11\ldots 11} \left(\sum_{l=0}^{k_{i_1 i_2 \ldots i_r}-1} \alpha_{i_1 i_2 \ldots i_r, l} a_{li}^{i_1 i_2 \ldots i_r} \right), \right.$$

$$\sum_{i_1 i_2 \ldots i_r} m_{i_1 i_2 \ldots i_r, 11\ldots 12} \left(\sum_{l=0}^{k_{i_1 i_2 \ldots i_r}-1} \alpha_{i_1 i_2 \ldots i_r, l} a_{li}^{i_1 i_2 \ldots i_r} \right), \ldots,$$

$$\left. \sum_{i_1 i_2 \ldots i_r} m_{i_1 i_2 \ldots i_r, l_1 l_2 \ldots l_r} \left(\sum_{l=0}^{k_{i_1 i_2 \ldots i_r}-1} \alpha_{i_1 i_2 \ldots i_r, l} a_{li}^{i_1 i_2 \ldots i_r} \right) \right)$$

$$= \mathbf{a}\mathcal{G}^i, \ \text{(say)}.$$

Then,

$$\mathbf{a} = \begin{bmatrix} \alpha_{11\ldots 11,0} & \ldots & \alpha_{11\ldots 11, k_{11\ldots 11}-1} & \ldots & \alpha_{l_1 l_2 \ldots l_{r-1} l_r, 0} & \ldots & \alpha_{l_1 l_2 \ldots l_{r-1} l_r, k_{l_1 l_2 \ldots l_{r-1} l_r}-1} \end{bmatrix}$$

$$\mathcal{G}^i = \begin{bmatrix} m_{11\ldots 11, 11\ldots 11} a_{0i}^{11\ldots 11} & \ldots & m_{11\ldots 11, l_1 l_2 \ldots l_{r-1} l_r} a_{0i}^{11\ldots 11} \\ m_{11\ldots 11, 11\ldots 11} a_{1i}^{11\ldots 11} & \ldots & m_{11\ldots 11, l_1 l_2 \ldots l_{r-1} l_r} a_{1i}^{11\ldots 11} \\ \vdots & \vdots & \vdots \\ m_{11\ldots 11, 11\ldots 11} a_{(k_{11\ldots 11}-1)i}^{11\ldots 11} & \ldots & m_{11\ldots 11, l_1 l_2 \ldots l_{r-1} l_r} a_{(k_{11\ldots 11}-1)i}^{11\ldots 11} \\ \vdots & \vdots & \vdots \\ m_{l_1 l_2 \ldots l_{r-1} l_r, 11\ldots 11} a_{0i}^{l_1 l_2 \ldots l_{r-1} l_r} & \ldots & m_{l_1 l_2 \ldots l_{r-1} l_r, l_1 l_2 \ldots l_{r-1} l_r} a_{0i}^{l_1 l_2 \ldots l_{r-1} l_r} \\ m_{l_1 l_2 \ldots l_{r-1} l_r, 11\ldots 11} a_{1i}^{l_1 l_2 \ldots l_{r-1} l_r} & \ldots & m_{l_1 l_2 \ldots l_{r-1} l_r, l_1 l_2 \ldots l_{r-1} l_r} a_{1i}^{l_1 l_2 \ldots l_{r-1} l_r} \\ \vdots & \vdots & \vdots \\ m_{l_1 l_2 \ldots l_{r-1} l_r, 11\ldots 11} a_{(k_{l_1 l_2 \ldots l_{r-1} l_r}-1)i}^{l_1 l_2 \ldots l_{r-1} l_r} & \ldots & m_{l_1 l_2 \ldots l_{r-1} l_r, l_1 l_2 \ldots l_{r-1} l_r} a_{(k_{l_1 l_2 \ldots l_{r-1} l_r}-1)i}^{l_1 l_2 \ldots l_{r-1} l_r} \end{bmatrix}$$

$$= \begin{bmatrix} G_{11\dots11,C_i} \otimes M_{R_{11\dots11}} \\ G_{11\dots12,C_i} \otimes M_{R_{11\dots12}} \\ \vdots \\ G_{l_1l_2\dots l_{r-1}l_r,C_i} \otimes M_{R_{l_1l_2\dots l_{r-1}l_r}} \end{bmatrix},$$

where $G_{i_1i_2\dots i_r,C_i}$ denotes the i^{th} column of $G_{i_1i_2\dots i_r}$, and $M_{R_{i_1i_2\dots i_r}}$ denotes the $i_1i_2\dots {i_r}^{th}$ row of M. Thus,

$$\begin{aligned} \mathbf{w} &= (\mathsf{w}^0, \mathsf{w}^1, \dots, \mathsf{w}^{n-1}) \\ &= \mathbf{a}\begin{bmatrix} \mathcal{G}^0 & \mathcal{G}^1 & \dots & \mathcal{G}^{n-1} \end{bmatrix} = \mathbf{a}G \text{ (say)}. \end{aligned}$$

Hence,

$$\begin{aligned} G &= \begin{bmatrix} \mathcal{G}^0 & \mathcal{G}^1 & \dots & \mathcal{G}^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} G_{11\dots11,C_0} \otimes M_{R_{11\dots11}} & \dots & G_{11\dots11,C_{n-1}} \otimes M_{R_{11\dots11}} \\ G_{11\dots12,C_0} \otimes M_{R_{11\dots12}} & \dots & G_{11\dots12,C_{n-1}} \otimes M_{R_{11\dots12}} \\ \vdots & \vdots & \vdots \\ G_{l_1l_2\dots l_{r-1}l_r,C_0} \otimes M_{R_{l_1l_2\dots l_{r-1}l_r}} & \dots & G_{l_1l_2\dots l_{r-1}l_r,C_{n-1}} \otimes M_{R_{l_1l_2\dots l_{r-1}l_r}} \end{bmatrix} \\ &= \begin{bmatrix} G_{11\dots11} \otimes M_{R_{11\dots11}} \\ G_{11\dots12} \otimes M_{R_{11\dots12}} \\ \vdots \\ G_{l_1l_2\dots l_{r-1}l_r} \otimes M_{R_{l_1l_2\dots l_{r-1}l_r}} \end{bmatrix} \end{aligned}$$

is a generator matrix of $\Phi(\mathcal{C})$. □

4.2 Skew Constacyclic Codes over $\mathcal{T}$

In this section, we study skew constacyclic codes over $\mathcal{T}$. We will first look at the automorphisms of $\mathcal{T}$ and then provide a few results on units of $\mathcal{T}$. Further, we discuss the structural properties of skew constacyclic codes over $\mathcal{T}$. Finally, we conclude this section with an important result on Gray images of skew constacyclic codes over $\mathcal{T}$.

4.2.1 Automorphisms of $\mathcal{T}$

Let $\Theta : \mathcal{T} \to \mathcal{T}$ be an automorphism. Let Then $\Theta|_{\mathbb{F}_q}$, the restriction map on $\mathbb{F}_q$ is an $\mathbb{F}_q$-automorphism. Therefore, $\Theta|_{\mathbb{F}_q} = \theta_t = Frob^t$ ($0 \leq t \leq m-1$, if $q = p^m$), where $Frob$ is the Frobenius automorphism defined as $a \longmapsto a^p,\ a \in \mathbb{F}_q$. Thus, for $\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r} v_{i_1i_2\ldots i_r} \in \mathcal{T}$, we have

$$\Theta\left(\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r} v_{i_1i_2\ldots i_r}\right) = \sum_{i_1,i_2,\ldots,i_r} \Theta(\eta_{i_1i_2\ldots i_r}) v_{i_1i_2\ldots i_r}^{p^t}.$$

From eq. 4.1, we conclude that $\{\eta_{i_1i_2\ldots i_r} : i_j \in \{1,2,\ldots,l_j\},\ j = 1,2,\ldots,r\}$ is a complete set in $\mathcal{T}$. Therefore, the set $\{\Theta(\eta_{i_1i_2\ldots i_r}) : i_j \in \{1,2,\ldots,l_j\}\ j = 1,2,\ldots,r\}$ is a permutation of the set $\{\eta_{i_1i_2\ldots i_r} : i_j \in \{1,2,\ldots,l_j\}\ j = 1,2,\ldots,r\}$. Hence, $\exists\ \gamma_j \in S_{l_j}$, the permutation group of $\{1,2,\ldots,l_j\},\ j = 1,2,\ldots,r$ such that $\Theta(\eta_{i_1i_2\ldots i_r}) = \eta_{\gamma_1(i_1)\gamma_2(i_2)\ldots\gamma_r(i_r)}$. Therefore,

$$\Theta\left(\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r} v_{i_1i_2\ldots i_r}\right) = \sum_{i_1,i_2,\ldots,i_r} \eta_{\gamma_1(i_1)\gamma_2(i_2)\ldots\gamma_r(i_r)} v_{i_1i_2\ldots i_r}^{p^t}.$$

Whenever γ_j represents the identity permutation for all $j = 1,2,\ldots,r$, and $\Theta|_{\mathbb{F}_q} = \theta_t$, then we'll denote Θ by Θ_t.

4.2.2 Units of $\mathcal{T}$

Lemma 4.2.1. Let $\alpha = \sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\alpha_{i_1i_2\ldots i_r} \in \mathcal{T}$. Then α is a unit in $\mathcal{T}$ if and only if $\alpha_{i_1i_2\ldots i_r}$ is a unit in $\mathbb{F}_q$, for all $i_j \in \{1,2,\ldots,l_j\}$, $j = 1,2,\ldots,r$.

Proof. Let $\alpha = \sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\alpha_{i_1i_2\ldots i_r}$ be a unit in $\mathcal{T}$. Then there exists $\mu \in \mathcal{T}$ such that $\alpha\mu = 1$. If $\mu = \sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\beta_{i_1i_2\ldots i_r}$, we have

$$\left(\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\alpha_{i_1i_2\ldots i_r}\right)\left(\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\beta_{i_1i_2\ldots i_r}\right) = 1.$$

Now, using the properties of primitive orthogonal idempotents $\eta_{i_1i_2\ldots i_r}$, $i_j \in \{1,2,\ldots,l_j\}$, $j = 1,2,\ldots,r$, we get

$$\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\alpha_{i_1i_2\ldots i_r}\beta_{i_1i_2\ldots i_r} = 1.$$

Since, $1 \in \mathcal{T}$ has a unique representation as linear combination of $\eta_{i_1i_2\ldots i_r}$, $i_j \in \{1,2,\ldots l_j\}$, $j = 1,2,\ldots,r$, we have $\alpha_{i_1i_2\ldots i_r}\beta_{i_1i_2\ldots i_r} = 1$, for all $i_j \in \{1,2,\ldots,l_j\}$, $j = 1,2,\ldots,r$. This shows that $\alpha_{i_1i_2\ldots i_r}$ is a unit in $\mathbb{F}_q$, for all $i_j \in \{1,2,\ldots,l_j\}$, $j = 1,2,\ldots,r$.

Conversely, let us suppose that $\alpha_{i_1i_2\ldots i_r}$ is a unit, for all $i_j \in \{1,2,\ldots l_j\}$, $j = 1,2,\ldots,r,$. Let us take $\mu = \sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\alpha^{-1}_{i_1i_2\ldots i_r}$. Then using the properties of primitive orthogonal idempotents, we have

$$\begin{aligned}
&\alpha\mu \\
&= \left(\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\alpha_{i_1i_2\ldots i_r}\right)\left(\sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\alpha^{-1}_{i_1i_2\ldots i_r}\right) \\
&= \sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r}\left(\alpha_{i_1i_2\ldots i_r}\alpha^{-1}_{i_1i_2\ldots i_r}\right) \\
&= \sum_{i_1,i_2,\ldots,i_r} \eta_{i_1i_2\ldots i_r} = 1
\end{aligned}$$

This proves that α is a unit in $\mathcal{T}$. □

Corollary 4.2.2. For a unit $\alpha = \sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\alpha_{i_1i_2\dots i_r}$ in $\mathcal{T}$, $\alpha^2 = 1$ if and only if $\alpha^2_{i_1i_2\dots i_r} = 1$, for all $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$.

Definition 4.2.3. Let $\mathbf{v} = (\mathsf{v}^0, \mathsf{v}^1, \dots, \mathsf{v}^{n-1}) \in \mathcal{T}^n$.For an automorphism Θ of $\mathcal{T}$ and for a unit α in $\mathcal{T}$ its (Θ, α)-skew constacyclic shift is defined as

$$\sigma_{(\Theta,\alpha)}(\mathbf{v}) = (\alpha\Theta(\mathsf{v}^{n-1}), \Theta(\mathsf{v}^0), \dots, \Theta(\mathsf{v}^{n-2})).$$

A code $\mathcal{C} \subseteq \mathcal{T}^n$ is called a skew (Θ, α)-constacyclic code of length n over $\mathcal{T}$, if it is linear and $\sigma_{(\Theta,\alpha)}(\mathbf{v}) \in \mathcal{C}$ whenever $\mathbf{v} \in \mathcal{C}$.

For an automorphism Θ of $\mathcal{T}$, $\mathcal{T}[y; \Theta]$ is a non-commutative ring (in general) under usual addition of polynomials and multiplication defined as $y * ay = \Theta(a)y^2$ and it is called skew-Θ polynomial ring. Moreover, for a vector $\mathbf{v} = (\mathsf{v}^0, \mathsf{v}^1, \dots, \mathsf{v}^{n-1}) \in \mathcal{T}^n, \mathbf{v} \mapsto \sum_{i=0}^{n-1} \mathsf{v}^i y^i$ is an isomorphism between $\mathcal{T}^n$ and $\mathcal{T}[y; \Theta]/\langle y^n - \alpha\rangle$. Under this isomorphism, a linear code C is a skew (Θ, α)-constacyclic code of length n if and only if it (its image) is a left submodule of $A_n = \mathcal{T}[y; \Theta]/\langle y^n - \alpha\rangle$. If $\Theta(\alpha) = \alpha$ and $o(\Theta)$ divides n, then $\langle y^n - \alpha\rangle$ is a two-sided ideal and A_n is a ring. Thus, in this case, a linear code $\mathcal{C}$ is a skew (Θ, α)-constacyclic code of length n if and only if it (its image) is a left ideal of A_n.

Theorem 4.2.4. Let $\Theta_t \in Aut(\mathcal{T})$ and $\alpha = \sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\alpha_{i_1i_2\dots i_r} \in \mathcal{U}(\mathcal{T})$. A linear code $\mathcal{C} = \bigoplus_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\mathcal{C}_{i_1i_2\dots i_r}$ of length n over $\mathcal{T}$ is a skew (Θ_t, α)-constacyclic code over $\mathcal{T}$ if and only if $\mathcal{C}_{i_1i_2\dots i_r}$ is a skew $(\theta_t, \alpha_{i_1i_2\dots i_r})$-constacyclic code over $\mathbb{F}_q$, for all $i_j \in \{1, 2, \dots l_j\}$, $j = 1, 2, \dots r$.

Proof. Let $\mathbf{c} = (\mathsf{c}^0, \mathsf{c}^1, \mathsf{c}^2, \dots, \mathsf{c}^{n-1}) \in \mathcal{C}$ be an arbitrary codeword. For $l \in \{0, 1, \dots, n-1\}$, let $\mathsf{c}^l = \sum_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r} c^l_{i_1i_2\dots i_r}$. Then, $\mathbf{c}_{i_1i_2\dots i_r} = \left(c^0_{i_1i_2\dots i_r}, c^1_{i_1i_2\dots i_r}, \dots, c^{n-1}_{i_1i_2\dots i_r}\right) \in \mathcal{C}_{i_1i_2\dots i_r}$. Now,

$\sigma_{(\Theta_t,\alpha)}(\mathbf{c})$

$$= \left(\alpha\Theta_t(\mathsf{c}^{n-1}), \Theta_t(\mathsf{c}^0), \Theta_t(\mathsf{c}^1), \dots, \Theta_t(\mathsf{c}^{n-2})\right)$$

$$= \Bigg(\Bigg(\sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\alpha_{i_1i_2\dots i_r}\Bigg)\Bigg(\sum_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\theta_t(c^{n-1}_{i_1i_2\dots i_r})\Bigg), \sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\theta_t(c^{0}_{i_1i_2\dots i_r}), \dots, \sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\theta_t(c^{n-2}_{i_1i_2\dots i_r})\Bigg)$$

$$= \Bigg(\sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}(\alpha_{i_1i_2\dots i_r}\theta_t(c^{n-1}_{i_1i_2\dots i_r})), \sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\theta_t(c^{0}_{i_1i_2\dots i_r}), \dots, \sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\theta_t(c^{n-2}_{i_1i_2\dots i_r})\Bigg)$$

$$= \sum_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\left(\alpha_{i_1i_2\dots i_r}\theta_t(c^{n-1}_{i_1i_2\dots i_r}), \theta_t(c^{0}_{i_1i_2\dots i_r}), \theta_t(c^{1}_{i_1i_2\dots i_r}), \dots, \theta_t(c^{n-2}_{i_1i_2\dots i_r})\right)$$

$$= \sum_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\left(\sigma_{(\theta_t,\alpha_{i_1i_2\dots i_r})}(c^{0}_{i_1i_2\dots i_r}, c^{1}_{i_1i_2\dots i_r}, \dots, c^{n-1}_{i_1i_2\dots i_r})\right)$$

$$= \sum_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\left(\sigma_{(\theta_t,\alpha_{i_1i_2\dots i_r})}(\mathbf{c}_{i_1i_2\dots i_r})\right).$$

By the unique representation of elements of $\mathcal{C}$ as a linear combination of elements of $\mathcal{C}_{i_1i_2\dots i_r}$, we conclude that $\sigma_{(\Theta_t,\alpha)}(\mathbf{c}) \in \mathcal{C}$ if and only if $\sigma_{(\theta_t,\alpha_{i_1i_2\dots i_r})}(\mathbf{c}_{i_1i_2\dots i_r}) \in \mathcal{C}_{i_1i_2\dots i_r}$, for all $i_j \in \{1, 2, \dots l_j\}$, $j = 1, 2, \dots r$. Hence, $\mathcal{C}$ is a skew (Θ_t, α)-constacyclic code over $\mathcal{T}$ if and only if $\mathcal{C}_{i_1i_2\dots i_r}$ is a skew $(\theta_t, \alpha_{i_1i_2\dots i_r})$-constacyclic code over $\mathbb{F}_q$, for all $i_j \in \{1, 2, \dots l_j\}$, $j = 1, 2, \dots r$. □

Theorem 4.2.5. Let $\Theta_t \in Aut(\mathcal{T})$ and $\alpha = \sum_{i_1,i_2,\dots,i_r} \eta_{i_1i_2\dots i_r}\alpha_{i_1i_2\dots i_r} \in \mathcal{U}(\mathcal{T})$. Furthermore, let $\mathcal{C} = \bigoplus_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\mathcal{C}_{i_1i_2\dots i_r}$ be a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$, where $\mathcal{C}_{i_1i_2\dots i_r} = \langle f_{i_1i_2\dots i_r}(y)\rangle$ is a skew $(\theta_t, \alpha_{i_1i_2\dots i_r})$-constacyclic codes of length n over $\mathbb{F}_q$ and $f_{i_1i_2\dots i_r}(y)$ is a monic right divisor of $y^n - \alpha_{i_1i_2\dots i_r}$, $i_j \in \{1, 2, \dots l_j\}$, $j = 1, 2, \dots r$. Then, there exists a polynomial $f(y)$ in $\mathcal{T}[y; \Theta_t]$ such that

(i) $\mathcal{C} = \langle f(y)\rangle$;

(ii) $f(y)$ is a right divisor of $y^n - \alpha$;

(iii) $|\mathcal{C}| = q^{l_1 l_2 \ldots l_r n - \sum_{i_1, i_2, \ldots, i_r} deg(f_{i_1 i_2 \ldots i_r}(y))}$.

Proof. (i) Since $\mathcal{C}_{i_1 i_2 \ldots i_r} = \langle f_{i_1 i_2 \ldots i_r}(y) \rangle,\ i_j \in \{1, 2, \ldots l_j\},\ j = 1, 2, \ldots r$, and $\mathcal{C} = \bigoplus_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} \mathcal{C}_{i_1 i_2 \ldots i_r}$, we have

$$\mathcal{C} = \left\{ c(y) = \sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} r_{i_1 i_2 \ldots i_r}(y) f_{i_1 i_2 \ldots i_r}(y) : r_{i_1 i_2 \ldots i_r}(y) \in \mathbb{F}_q[y; \theta_t] \right\}.$$

Hence, $\mathcal{C} \subseteq \left\langle \eta_{11 \ldots 11} f_{11 \ldots 11}(y), \eta_{11 \ldots 12} f_{11 \ldots 12}(y) \ldots, \eta_{l_1 l_2 \ldots l_{r-1} l_r} f_{l_1 l_2 \ldots l_{r-1} l_r}(y) \right\rangle = \Delta$. Conversely, for any $\sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} k_{i_1 i_2 \ldots i_r}(y) f_{i_1 i_2 \ldots i_r}(y) \in \Delta$ where $k_{i_1 i_2 \ldots i_r}(y) \in \mathbb{F}_q[y; \theta_t]/\langle y^n - 1 \rangle$, there exist $r_{i_1 i_2 \ldots i_r}(y) \in \mathbb{F}_q[y; \theta_t]$ such that $\eta_{i_1 i_2 \ldots i_r} k_{i_1 i_2 \ldots i_r}(y) = \eta_{i_1 i_2 \ldots i_r} r_{i_1 i_2 \ldots i_r}(y)$. Thus $\Delta \subseteq \mathcal{C}$. Hence $\Delta = \mathcal{C}$. Now, let us take $f(y) = \sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} f_{i_1 i_2 \ldots i_r}(y) \in \Delta$. Thus, $\langle f(y) \rangle \subseteq \mathcal{C}$. Since $\eta_{i_1 i_2 \ldots i_r} f_{i_1 i_2 \ldots i_r} = \eta_{i_1 i_2 \ldots i_r} f_{i_1 i_2 \ldots i_r}$, for all $i_j \in \{1, 2, \ldots, l_j\}, j = 1, 2, \ldots, r$ so $\mathcal{C} \subseteq \langle f(y) \rangle$. Hence, $\mathcal{C} = \langle f(y) \rangle$.

(ii) Further, as $f_{i_1 i_2 \ldots i_r}(y)$ is a monic right divisor of $y^n - \alpha_{i_1 i_2 \ldots i_r} \in \mathbb{F}_q[y; \theta_t]$, there exists $g_{i_1 i_2 \ldots i_r}(y) \in \mathbb{F}_q[y; \theta_t]$ such that $y^n - \alpha_{i_1 i_2 \ldots i_r} g_{i_1 i_2 \ldots i_r}(y) f_{i_1 i_2 \ldots i_r}(y),\ i_j \in \{1, 2, \ldots, l_j\},\ j = 1, 2, \ldots r$. Therefore,

$$\begin{aligned}
&\left(\sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} g_{i_1 i_2 \ldots i_r}(y) \right) f(y) \\
&= \left(\sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} g_{i_1 i_2 \ldots i_r}(y) \right) \left(\sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} f_{i_1 i_2 \ldots i_r}(y) \right) \\
&= \sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} \left(g_{i_1 i_2 \ldots i_r}(y) f_{i_1 i_2 \ldots i_r}(y) \right) \\
&= \sum_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} (y^n - \alpha_{i_1 i_2 \ldots i_r}) \\
&= y^n - \alpha \in \mathcal{T}[y; \Theta_t].
\end{aligned}$$

Hence, $f(y)$ divides $y^n - \alpha$ from right.

(iii) Finally, since $|\mathcal{C}| = \prod_{i_1 i_2 \dots i_r} |\mathcal{C}_{i_1 i_2 \dots i_r}|$, we get

$$\begin{aligned}|\mathcal{C}| = &\prod_{i_1 i_2 \dots i_r} q^{n-deg(f_{i_1 i_2 \dots i_r}(y))}\\ =&q^{\sum_{i_1 i_2 \dots i_r}(n-deg(f_{i_1 i_2 \dots i_r}(y))}\\ =&q^{l_1 l_2 \dots l_r n - \sum_{i_1, i_2, \dots, i_r} deg(f_{i_1 i_2 \dots i_r}(y))}.\end{aligned}$$

□

Definition 4.2.6. Let $\theta_t \in Aut(\mathbb{F}_q)$, $\beta_1, \beta_2, \dots, \beta_s \in \mathbb{F}_q^*$ and C be a linear code of length $n = s\ell$ over $\mathbb{F}_q$. Then, C is said to be a skew-θ_t quasi-$(\beta_1, \beta_2, \dots \beta_s)$-twisted code if

$$\begin{aligned}&\rho_{(\theta_t, (\beta_1, \beta_2, \dots, \beta_s))}(\mathbf{c})\\ &= (\beta_1 \theta_t(c_{1,\ell-1}), \theta_t(c_{1,0}), \dots, \theta_t(c_{1,\ell-2}) | \beta_2 \theta_t(c_{2,\ell-1}), \theta_t(c_{2,0}), \dots, \theta_t(c_{2,\ell-2}) | \dots\\ &\qquad | \beta_s \theta_t(c_{s,\ell-1}), \theta_t(c_{s,0}), \dots, \theta_t(c_{s,\ell-2})) \in C\end{aligned}$$

whenever $\mathbf{c} = (c_{1,0}, c_{1,1}, \dots, c_{1,\ell-1} | c_{2,0}, c_{2,1}, \dots, c_{2,\ell-1} | c_{s,0}, c_{s,1}, \dots, c_{s,\ell-1}) \in C$.

In particular, if $\beta_1 = \beta_2 = \dots = \beta_s = \beta$ then, C is a skew quasi β-twisted code.

Remark 4.2.7. Note that skew quasi-$(\beta_1, \beta_2, \dots, \beta_s)$-twisted codes are generalizations of skew quasi β-twisted codes, and skew multi-twisted codes introduced by Sharma et al. [86] are generalizations of skew quasi-$(\beta_1, \beta_2, \dots, \beta_s)$-twisted codes. In other words, the class of skew quasi-$(\beta_1, \beta_2, \dots, \beta_s)$-twisted codes lies between classes of skew quasi-twisted codes and skew multi-twisted codes.

Theorem 4.2.8. The Gray image $\Phi(C)$ of a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$ is permutation equivalent to a skew quasi-$(\alpha_{11\dots1}, \dots, \alpha_{l_1 l_2 \dots l_r})$-twisted code of length $l_1 l_2 \dots l_r n$ over $\mathbb{F}_q$, where $\alpha = \sum_{i_1, i_2, \dots, i_r} \eta_{i_1 i_2 \dots i_r} \alpha_{i_1 i_2 \dots i_r}$.

Proof. Let $\mathcal{C} = \bigoplus_{i_1 i_2 \ldots i_r} \eta_{i_1 i_2 \ldots i_r} \mathcal{C}_{i_1 i_2 \ldots i_r}$ be a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$. Then,

$$\mathcal{C} = \Bigg\{ \Bigg(\sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} c^0_{i_1 i_2 \ldots i_r}, \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} c^1_{i_1 i_2 \ldots i_r}, \ldots, \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} c^{n-1}_{i_1 i_2 \ldots i_r} \Bigg) : \left(c^0_{i_1 i_2 \ldots i_r}, c^1_{i_1 i_2 \ldots i_r}, \ldots, c^{n-1}_{i_1 i_2 \ldots i_r}\right) \in \mathcal{C}_{i_1 i_2 \ldots i_r} \Bigg\},$$

and

$$\Phi(\mathcal{C}) = \Big\{ \big((c^0_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} M, (c^0_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} M, \ldots, (c^0_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} M \big) : (c^0_{i_1 i_2 \ldots i_r}, c^1_{i_1 i_2 \ldots i_r}, \ldots, c^{n-1}_{i_1 i_2 \ldots i_r}) \in \mathcal{C}_{i_1 i_2 \ldots i_r} \Big\}.$$

Let a permutation equivalent code of $\Phi(C)$ be $\delta(\Phi(\mathcal{C}))$ such that

$\delta(\Phi(\mathcal{C})) = \{(d^0_{i_1 i_2 \ldots i_r}, d^1_{i_1 i_2 \ldots i_r}, \ldots, d^{n-1}_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} : (d^l_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} = (c^l_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} M\}$.

Now, for any $(d^0_{i_1 i_2 \ldots i_r}, d^1_{i_1 i_2 \ldots i_r}, \ldots, d^{n-1}_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} \in \delta(\Phi(\mathcal{C}))$, there exists

$$\Big(\textstyle\sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} c^0_{i_1 i_2 \ldots i_r}, \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} c^1_{i_1 i_2 \ldots i_r}, \ldots, \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} c^{n-1}_{i_1 i_2 \ldots i_r} \Big) \in \mathcal{C}$$

such that

$$(c^0_{i_1 i_2 \ldots i_r}, c^1_{i_1 i_2 \ldots i_r}, \ldots, c^{n-1}_{i_1 i_2 \ldots i_r}) \in \mathcal{C}_{i_1 i_2 \ldots i_r} \text{ and } (c^l_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} = (d^l_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} M^{-1}.$$

Since, $\mathcal{C}$ is a skew (Θ_t, α)-constacyclic code, we have

$$\sigma_{(\Theta_t, \alpha)}(\mathbf{c}) = \Bigg(\sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} (\alpha_{i_1 i_2 \ldots i_r} \theta_t(c^{n-1}_{i_1 i_2 \ldots i_r})), \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} \theta_t(c^0_{i_1 i_2 \ldots i_r}), \ldots, \sum_{i_1, i_2, \ldots, i_r} \eta_{i_1 i_2 \ldots i_r} \theta_t(c^{n-2}_{i_1 i_2 \ldots i_r}) \Bigg) \in \mathcal{C}.$$

Then,

$$\begin{aligned} &\Phi(\sigma_{(\Theta_t, \alpha)}(\mathbf{c})) \\ =&\big((\alpha_{i_1 i_2 \ldots i_r} \theta_t(c^{n-1}_{i_1 i_2 \ldots i_r}))_{i_1 i_2 \ldots i_r} M, (\theta_t(c^0_{i_1 i_2 \ldots i_r}))_{i_1 i_2 \ldots i_r} M, \ldots (\theta_t(c^{n-2}_{i_1 i_2 \ldots i_r}))_{i_1 i_2 \ldots i_r} M \big) \\ \in&\Phi(\mathcal{C}). \end{aligned}$$

Thus,

$$\begin{aligned} &\rho_{\theta_t, (\alpha_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r}} \big((d^0_{i_1 i_2 \ldots i_r}, d^1_{i_1 i_2 \ldots i_r}, \ldots, d^{n-1}_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r} \big) \\ =&(\alpha_{i_1 i_2 \ldots i_r} \theta_t(d^{n-1}_{i_1 i_2 \ldots i_r}), \theta_t(d^0_{i_1 i_2 \ldots i_r}), \ldots, \theta_t(d^{n-2}_{i_1 i_2 \ldots i_r}))_{i_1 i_2 \ldots i_r} \\ \in&\ \delta(\Phi(\mathcal{C})) \end{aligned}$$

This proves that $\delta(\Phi(\mathcal{C}))$ is a skew-θ_t quasi-$(\alpha_{i_1 i_2 \ldots i_r})_{i_1 i_2 \ldots i_r}$-twisted code and hence, the result follows. □

Remark 4.2.9. In [15, Theorem 15, 16], for a non-chain ring $\mathcal{R}$ which is a particular case of our class of rings, when $r = 2$, Bharadwaj and Raka proved that **"The Gray image $\Phi(\mathcal{C})$ of a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{R}$ is permutation equivalent to a skew quasi-α-twisted code of length $l_1 l_2 n$ over $\mathbb{F}_q$, where $\alpha = \sum_{i_1, i_2} \eta_{i_1 i_2} \alpha_{i_1 i_2}$, $l_1 = deg(f_1)$ and $l_2 = deg(f_2)$."** We observe that in the proof of Theorem 15 in [15], they applied Φ_π as $\mathcal{R}$-linear but it is just $\mathbb{F}_q$-linear. Our analysis shows that in this case, the Gray image $\Phi(\mathcal{C})$ is permutation equivalent to a **skew quasi-$(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{l_1 l_2})$-twisted code** of length $l_1 l_2 n$ over $\mathbb{F}_q$.

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