Chapter 3

$\mathcal{L}^p$-Approximation Using Fractal Functions on SG

Associated with $\mathcal{L}^p$ approximation, various research articles can be viewed, see for instance [84, 85, 86]. Their research motivates us to explore the approximation aspects in $\mathcal{L}^p$ space.

This chapter becomes more interesting in many aspects since most of the significant results will be obtained by applying the definition of a self-similar measure on SG.

3.1 Fractal Interpolation Function on SG

Let us define

$$\mathcal{L}^P(SG) = \{f : SG \to \mathbb{R} : f \text{ is } p-\text{integrable function on } SG\}.$$

In this chapter, we assume that $f, b \in \mathcal{L}^P(SG)$. We construct an IFS whose attractor is the graph of a p-integrable function denoted by f^α on SG such that $f^\alpha|_{V_N} = f|_{V_N}$. Let $K = SG \times \mathbb{R}$. Define maps $W_w : K \to K$ by

$$W_w(t,x) = \Big(L_w(t), F_w(t,x)\Big), \quad w \in \{1,2,3\}^N,$$

where $F_w(t,x) : SG \times \mathbb{R} \to \mathbb{R}$ are required to satisfy following conditions:

$$|F_w(.,x) - F_w(.,x')| \leq c|x - x'|$$

with $c < 1$, that is, F is a contraction map with respect to second variable, and $F_w(p_j, f(p_j)) = f(L_w(p_j))$. In particular, we take

$$F_w(t,x) = \alpha_w(t)x + f(L_w(t)) - \alpha_w(t)b(t),$$

where $b : SG \to \mathbb{R}$ is a p-integrable function such that $b(p_j) = f(p_j),\ j = 1,2,3$, and for each $w \in \{1,2,3\}^N$, $\alpha_w : SG \to \mathbb{R}$ is a continuous function with $\|\alpha_w\|_\infty < 1$. We get an IFS $\{K; W_w, w \in \{1,2,3\}^N\}$.

$$f_L^\alpha(t) = f(t) + \alpha(L_w^{-1}(t))\ f_L^\alpha\big(L_w^{-1}(t)\big) - \alpha(L_w^{-1}(t))\ f\big(L_w^{-1}(t)\big), \tag{3.1.1}$$

for $t \in L_w(SG),\ w \quad \in \{1,2,3\}^N$.

Theorem 3.1.1. *Let $f, b \in \mathcal{L}^P(SG)$. The IFS $\{K; W_w, w \in \{1,2,3\}^N\}$ defined above has a unique attractor $Gr(f^\alpha)$. Set $Gr(f^\alpha)$ is the graph of a p-integrable function $f^\alpha : SG \to \mathbb{R}$, which satisfies $f^\alpha|_{V_N} = f|_{V_N}$. Furthermore, if $\|\alpha\|_\infty < \frac{1}{3^{\frac{N}{p}}}$, then the then the α- fractal function f^α holds the following functional equation:*

$$f^\alpha(t) = f(t) + \alpha_w(L_w^{-1}(t))(f^\alpha - b)\big(L_w^{-1}(t)\big)\ \forall\ t \in L_w(SG), \quad w \in \{1,2,3\}^N. \tag{3.1.2}$$

Proof. Let $\mathcal{L}^p_f(SG) = \{g \in \mathcal{L}^p(SG) : g(p_j) = f(p_j),\ \forall\, j \in \{1,2,3\}\}$. We observe that set $\mathcal{L}^p_f(SG)$ is a closed subset of $\mathcal{L}^p(SG)$. Since $(\mathcal{L}^p(SG), \|.\|_p)$ is a Banach space, we get $\mathcal{L}^p_f(SG)$ is a complete metric space with respect to metric induced by norm $\|.\|_p$. We define a map $T : \mathcal{L}^p_f(SG) \to \mathcal{L}^p_f(SG)$ by

$$(Tg)(t) = f(t) + \alpha_w(L_w^{-1}(t))\ (g-b)(L_w^{-1}(t))$$

for all $t \in L_w(SG)$, where $w \in \{1,2,3\}^N$. It is obvious that T is well-defined. Let $g, h \in \mathcal{L}^p_f(SG)$, then

$$\begin{aligned}\big|(Tg)(t) - (Th)(t)\big| &= \Big|\alpha_w(L_w^{-1}(t))\ (g-b)(L_w^{-1}(t)) - \alpha_w(L_w^{-1}(t))\ (h-b)(L_w^{-1}(t))\Big| \\ &= |\alpha_w(L_w^{-1}(t))\ (g-h)(L_w^{-1}(t))| \\ &= |\alpha_w(L_w^{-1}(t))|\ |(g-h)(L_w^{-1}(t))| \\ &\le \|\alpha\|_\infty |(g-h)(L_w^{-1}(t))|,\end{aligned}$$

which is true for all $t \in L_w(SG)$ and for all $w \in \{1,2,3\}^N$. Further, we have

$$\int_{SG} \big|(Tg)(t) - (Th)(t)\big|^p d\mu_p(t) \le \|\alpha\|_\infty^p \sum_{w\in\{1,2,3\}^N} \int_{L_w(SG)} |(g-h)(L_w^{-1}(t))|^p d\mu_p(t),$$

now using Equation (1.8.1), we deduce

$$\begin{aligned}\int_{SG} \big|(Tg)(t) - (Th)(t)\big|^p d\mu_p(t) &\le \|\alpha\|_\infty^p \sum_{w\in\{1,2,3\}^N} \int_{L_w(SG)} |(g-h)(\tilde{t})|^p d\mu_p(\tilde{t}) \\ &\le 3^N \|\alpha\|_\infty^p \int_{SG} |(g-h)(\tilde{t})|^p d\mu_p(\tilde{t}).\end{aligned}$$

Therefore, we obtain $\|Tg - Th\|_p \le 3^{\frac{N}{p}} \|\alpha\|_\infty \|g-h\|_p$. Using $\|\alpha\|_\infty = \max_{w\in\{1,2,3\}^N} \|\alpha_w\|_\infty$ and $3^{\frac{N}{p}}\|\alpha\|_\infty < 1$, we get that T is a contraction map on $\mathcal{L}^p_f(SG)$. With the help of Banach contraction principle, we get a unique fixed point of T, namely $f^\alpha \in \mathcal{L}^p_f(SG)$. Finally, we check that the graph of f^α is an attractor of an IFS. We have $T(f^\alpha) = f^\alpha$, that is, $f^\alpha(t) = F_w\big(L_w^{-1}(t), f^\alpha(L_w^{-1}(t))\big)\ \forall\ t \in L_w(SG), w \in \{1,2,3\}^N$. Furthermore, we represent this as $f^\alpha(L_w(t)) = F_w(t, f^\alpha(t))$ for $w \in \{1,2,3\}^N$. Finally, we get

$$\cup_{w\in\{1,2,3\}^N} W_w(Gr(f^\alpha)) = Gr(f^\alpha),$$

which proves the result. □

Remark 3.1.1. *We recall Equation (3.1.2):*

$$f^\alpha(t) = f(t) + \alpha_w(L_w^{-1}(t))(f^\alpha - b)\big(L_w^{-1}(t)\big) \quad \forall\ \ t \in L_w(SG), \quad w \in \{1,2,3\}^N.$$

Now for every $t \in L_w(SG)$ and $w \in \{1,2,3\}^N$, we have

$$\begin{aligned} |f^\alpha(t) - f(t)| &= |\alpha_w(L_w^{-1}(t)).(f^\alpha - b)\big(L_w^{-1}(t)\big)| \\ &= |\alpha_w(L_w^{-1}(t))|\ |(f^\alpha - b)\big(L_w^{-1}(t)\big)| \\ &\le \|\alpha_w\|_\infty |(f^\alpha - b)\big(L_w^{-1}(t)\big)|. \end{aligned}$$

The above inequality implies that $\|f^\alpha - f\|_p \le 3^{\frac{N}{p}}\|\alpha\|_\infty \|f^\alpha - b\|_p$. Using triangle inequality, we obtain $\|f^\alpha - f\|_p \le 3^{\frac{N}{p}}\big[\|\alpha\|_\infty\|f^\alpha - f\|_p + \|\alpha\|_\infty\|f - b\|_p\big]$. Finally, we have $\|f^\alpha - f\|_p \le \frac{3^{\frac{N}{p}}\|\alpha\|_\infty}{1-3^{\frac{N}{p}}\|\alpha\|_\infty}\|f - b\|_p$. Furthermore, we have

$$\|f^\alpha\|_p - \|f\|_p \le \|f^\alpha - f\|_p \le \frac{3^{\frac{N}{p}}\|\alpha\|_\infty}{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty}\|f - b\|_p.$$

With a bounded linear operator $L : \mathcal{L}^p(SG) \to \mathcal{L}^p(SG)$ satisfying $(Lg)(p_1) = g(p_1), (Lg)(p_2) = g(p_2)$ and $(Lg)(p_3) = g(p_3)$, we define a fractal operator $\mathcal{F}^\alpha : \mathcal{L}^p(SG) \to \mathcal{L}^p(SG)$ by $\mathcal{F}^\alpha(f) = f^\alpha$, where f^α is fractal perturbation of f corresponding to base function $b = Lf$ and scale vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{L}^p(SG))^3$. The upcoming theorem contains some elementary properties of fractal operator $\mathcal{F}^\alpha$. We find fractal operator associated with univariate and bivariate functions in [10, 43].

Theorem 3.1.2. *Let Id be the identity operator on $\mathcal{L}^p(SG)$. Denote $\|\alpha\|_\infty = \max\{\|\alpha_w\|_\infty : w \in \{1,2,3\}^N\}$, then the following statements hold:*

1. *Let $f \in \mathcal{L}^p(SG)$ be arbitrary. Then the perturbation error is of the form:*

$$\|f^\alpha - f\|_p \le \frac{3^{\frac{N}{p}}\|\alpha\|_\infty}{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty}\|f - Lf\|_p.$$

   Note that for $\|\alpha\|_\infty = 0$ or $L = Id$, we have $\mathcal{F}^\alpha = Id$.

2. *If the norm defined on $\mathcal{L}^p(SG)$ is a p-norm, then the fractal operator $\mathcal{F}^\alpha$ is a bounded linear operator, moreover, the norm fulfils:*

$$\|\mathcal{F}^\alpha\| \le 1 + \frac{3^{\frac{N}{p}}\|\alpha\|_\infty \ \ \|Id - L\|}{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty}.$$

3. *For $3^{\frac{N}{p}}\|\alpha\|_\infty < \|L\|^{-1} < 1$, $\mathcal{F}^\alpha$ is a topological isomorphism. Moreover,*

$$\frac{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty}{1 + 3^{\frac{N}{p}}\|\alpha\|_\infty\|L\|} \le \|(\mathcal{F}^\alpha)^{-1}\| \le \frac{1 + 3^{\frac{N}{p}}\|\alpha\|_\infty}{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty\|L\|}.$$

4. *For $\|\alpha\|_\infty \neq 0$, the fixed points of L are the fixed points of $\mathcal{F}^\alpha$. If α_w are nowhere zero, then the fixed points of $\mathcal{F}^\alpha$ are the fixed points of L.*

5. *If the point spectrum of L contains 1, then $1 \le \|\mathcal{F}^\alpha\|$.*

6. *For $3^{\frac{N}{p}}\|\alpha\|_\infty < \|L\|^{-1}$, the fractal operator $\mathcal{F}^\alpha$ is not a compact operator.*

7. *If $3^{\frac{N}{p}}\|\alpha\|_\infty < \|L\|^{-1}$, then $\mathcal{F}^\alpha$ is Fredholm, and its index is 0.*

8. *There exists a non-trivial closed invariant subspace for the fractal operator $\mathcal{F}^\alpha$.*

Proof. 1. From the self-referential equation, we can write:

$$f^\alpha(t)-f(t)=\alpha_w(L_w^{-1}(t))(f^\alpha-Lf)\big(L_w^{-1}(t)\big)\quad \forall\ t\in L_w(SG),\ \ w\in\{1,2,3\}^N.$$

Note that

$$\begin{aligned}\big\|f^\alpha-f\big\|_p^p &\leq \int_{SG}\big|f^\alpha(t)-f(t)\big|^p d\mu_p(t)\\ &=\sum_{w\in\{1,2,3\}^N}\int_{L_w(SG)}\big|f^\alpha(t)-f(t)\big|^p d\mu_p(t).\end{aligned}$$

Using the functional equation given for f^α, we find

$$\begin{aligned}\big\|f^\alpha-f\big\|_p^p &= \sum_{w\in\{1,2,3\}^N}\int_{L_w(SG)}\big|\alpha_w(L_w^{-1}(t))(f^\alpha-Lf)\big(L_w^{-1}(t)\big)\big|^p d\mu_p(t)\\ &\leq \sum_{w\in\{1,2,3\}^N}\int_{L_w(SG)}\|\alpha_w\|_\infty^p\big|(f^\alpha-Lf)\big(L_w^{-1}(t)\big)\big|^p d\mu_p(t) \qquad (3.1.3)\\ &\leq \|\alpha\|_\infty^p\sum_{w\in\{1,2,3\}^N}\int_{L_w(SG)}\big|(f^\alpha-Lf)\big(L_w^{-1}(t)\big)\big|^p d\mu_p(t).\end{aligned}$$

Using Equation (1.8.1) and Equation (3.1.3), we obtain

$$\begin{aligned}\|f^\alpha-f\|_p^p &\leq \sum_{w\in\{1,2,3\}^N}\int_{SG}\|\alpha_w\|_\infty^p\big|(f^\alpha-Lf)\big(\tilde{t}\big)\big|^p d\mu_p(\tilde{t})\\ &\leq \|\alpha\|_\infty^p\sum_{w\in\{1,2,3\}^N}\int_{SG}\big|(f^\alpha-Lf)\big(\tilde{t}\big)\big|^p d\mu_p(\tilde{t})\\ &\leq 3^N\ \ \|\alpha\|_\infty^p\|f^\alpha-Lf\|_p^p.\end{aligned}$$

Consequently, we get

$$\big\|f^\alpha-f\big\|_p\leq 3^{\frac{N}{p}}\ \ \|\alpha\|_\infty\ \ \|f^\alpha-Lf\|_p.$$

Using triangle inequality, we obtain

$$\left\|f^\alpha - f\right\|_p \leq 3^{\frac{N}{p}} \ \|\alpha\|_\infty \left[\|f^\alpha - f\|_p + \|f - Lf\|_p\right].$$

Therefore, the following is the bound for the perturbation error:

$$\left\|f^\alpha - f\right\|_p \leq \frac{3^{\frac{N}{p}}\|\alpha\|_\infty}{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty}\|f - Lf\|_p.$$

2. The proof immediately follows from item (1).

3. By Remark 3.1.1, we have

$$\begin{aligned}\|f - \mathcal{F}^\alpha(f)\|_p &\leq 3^{\frac{N}{p}}\|\alpha\|_\infty\|\mathcal{F}^\alpha(f) - Lf\|_p \\ &\leq 3^{\frac{N}{p}} \ \|\alpha\|_\infty\left\|\mathcal{F}^\alpha(f)\right\|_p + 3^{\frac{N}{p}}\|\alpha\|_\infty\|L\|\|f\|_p.\end{aligned}$$

Since $3^{\frac{N}{p}} \ \|\alpha\|_\infty < \|L\|^{-1} < 1$, Lemma 1.12.1 yields that the fractal operator $\mathcal{F}^\alpha$ is a topological isomorphism. Moreover, we have the following bounds:

$$\frac{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty}{1 + 3^{\frac{N}{p}}\|\alpha\|_\infty\|L\|} \leq \|(\mathcal{F}^\alpha)^{-1}\| \leq \frac{1 + 3^{\frac{N}{p}}\|\alpha\|_\infty}{1 - 3^{\frac{N}{p}}\|\alpha\|_\infty\|L\|}.$$

4. From item (1), we know that

$$\left\|\mathcal{F}^\alpha(f) - f\right\|_p \leq \frac{3^{\frac{N}{p}} \ \|\alpha\|_\infty}{1 - 3^{\frac{N}{p}} \ \|\alpha\|_\infty}\|f - Lf\|_p, \quad \forall \ f \in \mathcal{L}^p(SG).$$

Now this implies that if $Lg = g$, then $\mathcal{F}^\alpha(g) = g$. Recall that

$$\mathcal{F}^\alpha(f)(t) - f(t) = \alpha_w(L_w^{-1}(t))(\mathcal{F}^\alpha(f) - Lf)\big(L_w^{-1}(t)\big) \quad \forall \ t \in L_w(SG), \ \ w \in \{1,2,3\}^N.$$

Let $\mathcal{F}^\alpha(g) = g$. Then

$$\alpha_w(L_w^{-1}(t))(g - Lg)\big(L_w^{-1}(t)\big) = 0 \quad \forall \ t \in L_w(SG), \ \ w \in \{1, 2, 3\}^N.$$

Since α_w is nowhere zero function, we have

$$(g - Lg)\big(L_w^{-1}(t)\big) = 0 \quad \forall \ t \in L_w(SG), \ \ w \in \{1, 2, 3\}^N,$$

that is, $Lg = g$, completing the claim.

5. Choose $g \in \mathcal{L}^p(SG)$ such that $Lg = g$ and $\|g\|_p = 1$ by the previous part of theorem $\mathcal{F}^\alpha(g) = g$, and hence $\|\mathcal{F}^\alpha(g)\|_p = \|g\|_p$. The definition of the operator norm now yields $1 \leq \|\mathcal{F}^\alpha\|_p$.

6. For $3^{\frac{N}{p}}\|\alpha\|_\infty < \|L\|^{-1}$, we know that $\mathcal{F}^\alpha : \mathcal{L}^p(SG) \to \mathcal{L}^p(SG)$ is one-one. Define the inverse map $(\mathcal{F}^\alpha)^{-1} : \mathcal{F}^\alpha(\mathcal{L}^p(SG)) \to \mathcal{L}^p(SG)$. With this choice of α, $\mathcal{F}^\alpha$ is bounded below, and hence it follows that $(\mathcal{F}^\alpha)^{-1}$ is a bounded linear operator. Assume that $\mathcal{F}^\alpha$ is a compact operator. From [87], we deduce that the operator $T = \mathcal{F}^\alpha(\mathcal{F}^\alpha)^{-1} : \mathcal{F}^\alpha(\mathcal{L}^p(SG)) \to \mathcal{L}^p(SG)$ is a compact operator, which is a contradiction to the infinite dimensionality of the space $\mathcal{F}^\alpha(\mathcal{L}^p(SG))$. Therefore, $\mathcal{F}^\alpha$ is not a compact operator.

7. Under the hypothesis, the range space of $\mathcal{F}^\alpha$ is closed. Furthermore, $\mathcal{F}^\alpha$ is invertible. Recall that [87] if $T : X \to Y$ is invertible, then T^* is also invertible. Therefore, $(\mathcal{F}^\alpha)^*$ is invertible. As a consequence, $\mathcal{F}^\alpha$ is Fredholm. The index of a Fredholm operator is defined as

$$index(\mathcal{F}^\alpha) = dim\big(ker(\mathcal{F}^\alpha)\big) - dim\big(ker(\mathcal{F}^\alpha)^*\big).$$

Hence, the index is zero.

8. The proof follows from [43, Theorem 3.5].

$\square$

3.2 Dimension Preserving Approximation in $\mathcal{L}^p$-norm

This part of our chapter is motivated by the research work published in [7, 27].

It is well-known that the set of Lipschitz functions on SG, which we denote by $\mathcal{L}ip(SG)$, is a dense subset of $\mathcal{L}^p(SG)$ when the latter is endowed with $\|\cdot\|_p$. We use this fact to prove the following theorem.

Since box dimension, Hausdorff dimension, and packing dimension are all Lipschitz invariant and therefore the next theorem is also valid for these dimensions.

Theorem 3.2.1. *Let* $\frac{\log 3}{\log 2} \leq \beta \leq \frac{\log 3}{\log 2} + 1$. *Then the set* $\mathcal{S}_\beta := \{f \in \mathcal{L}^p(SG) : \dim(Gr(f)) = \beta\}$ *is dense in* $\mathcal{L}^p(SG)$.

Proof. Let $f \in \mathcal{L}^p(SG)$ and $\epsilon > 0$. Using the denseness of $\mathcal{L}ip(SG)$ in $\mathcal{L}^p(SG)$, there exists g in $\mathcal{L}ip(SG)$ such that

$$\|f - g\|_p < \frac{\epsilon}{2}.$$

Further, we consider a non-vanishing bounded function $h \in \mathcal{S}_\beta$. Let $h_* = g + \frac{\epsilon}{2\|h\|_p} h$, which immediately gives

$$\|g - h_*\|_p \leq \frac{\epsilon}{2}.$$

This together with Lemma 1.2.3 implies that $\dim(Gr(h_*)) = \dim(Gr(h)) = \beta$. Hence, we have $h_* \in \mathcal{S}_\beta$ and

$$\|f - h_*\|_p \leq \|f - g\|_p + \|g - h_*\|_p < \epsilon.$$

Thus the proof of theorem is complete. □

The proofs of the following theorems 3.2.2-3.2.3 are adapted from the original idea presented in [27] with density of Lipscitz space in L^p space. Here tools and techniques are also taken from [27].

Theorem 3.2.2. *Let $f \in \mathcal{L}^p(SG)$. Then for $\epsilon > 0$, there exists $g \in \mathcal{S}_\beta$ such that*

$$g(t) \leq f(t) \ \forall \ t \in SG \ \textit{and} \ \|f - g\|_p < \epsilon.$$

Proof. Since $f \in \mathcal{L}^p(SG)$ and $\epsilon > 0$, Theorem 3.2.1 generates a member $h \in \mathcal{S}_\beta$ such that

$$\|f - h\|_p < \frac{\epsilon}{2}.$$

Choose $g(t) := h(t) - \frac{\epsilon}{2}, \quad \forall \ \ t \in SG$. Then

$$g(t) = h(t) - f(t) + f(t) - \frac{\epsilon}{2} \leq \|f - h\|_p + f(t) - \frac{\epsilon}{2} < f(t).$$

Furthermore,

$$\|f - g\|_p \leq \|f - h\|_p + \|h - g\|_p < \epsilon,$$

establishing the proof. □

Theorem 3.2.3. *Let $f \in \mathcal{L}^P(SG)$ be such that $f(t) \geq 0$, $\forall \ t \in SG$. Then for $\epsilon > 0$, and for $\alpha_w \in \mathcal{C}(SG)$, where $w \in \{1,2,3\}^N$ satisfying $\|\alpha\|_\infty < 1$, there exists an*

α-fractal function g_L^α, which satisfies

$$g_L^\alpha(t) \geq 0, \quad \forall\ t \in SG \quad \textit{and}\ \|f - g_L^\alpha\|_p < \epsilon.$$

Proof. Note that the linear operator L fixes the constant function 1, that is, $L(1) = 1$, where $1(t) = 1$ on SG. Consider $\alpha_w \in \mathcal{C}(SG)$ such that $\|\alpha\|_\infty < 1$. From Equation (3.1.1), we deduce

$$\|g_L^\alpha - g\|_p \leq \|\alpha\|_\infty \|g_L^\alpha - Lg\|_p,\ \forall\ g \in \mathcal{L}^P(SG).$$

Choose $g = 1$, then the above inequality gives

$$\|f_L^\alpha - 1\|_p \leq \|\alpha\|_\infty \|f_L^\alpha - 1\|_p,$$

and this further yields $\|f_L^\alpha - 1\|_p = 0$. Therefore, $f_L^\alpha = 1$, that is, $\mathcal{F}_L^\alpha(1) = 1$. For $\epsilon > 0$, $\alpha_w \in \mathcal{C}(SG)$ and $f \in \mathcal{L}^P(SG)$. Using Theorem 3.2.1, there exists a function h_L^α such that

$$\|f - h_L^\alpha\|_p < \frac{\epsilon}{2}, \quad \text{where } \mathcal{F}_L^\alpha(h) = h_L^\alpha.$$

Define $g_L^\alpha(t) = h_L^\alpha(t) + \frac{\epsilon}{2}$ for all $t \in SG$. Since $\mathcal{F}_L^\alpha(1) = 1$,

$$g_L^\alpha(t) = h_L^\alpha(t) + \frac{\epsilon}{2}1(t) = h_L^\alpha(t) + \frac{\epsilon}{2}1^\alpha(t).$$

Further, since $\mathcal{F}_L^\alpha$ is a linear operator

$$g_L^\alpha = h_L^\alpha + \frac{\epsilon}{2}1^\alpha = \mathcal{F}_L^\alpha\left(h + \frac{\epsilon}{2}1\right).$$

Moreover,

$$\begin{aligned} g_L^\alpha(t) &= h_L^\alpha(t) + \frac{\epsilon}{2} \\ &= h_L^\alpha(t) + \frac{\epsilon}{2} - f(t) + f(t) \\ &\geq f(t) + \frac{\epsilon}{2} - \|h_L^\alpha - f\|_p \\ &\geq 0. \end{aligned}$$

Further, we get

$$\begin{aligned} \|f - g_L^\alpha\|_p &\leq \|f - h_L^\alpha\|_p + \|h_L^\alpha - g_L^\alpha\|_p \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

completing the proof. □

Theorem 3.2.4. *The multi-valued mapping $\mathcal{W}^\alpha : \mathcal{L}^p(SG) \rightrightarrows \mathcal{L}^p(SG)$ defined by*

$$\begin{aligned} \mathcal{W}^\alpha(f) = \{f_L^\alpha : L \text{ is a bounded linear operator such that} \\ (Lg)_{V_0} = g|_{V_0} \ \forall\ g \in \mathcal{L}^p(SG) \text{ and } \|L\| \leq q\} \end{aligned} \tag{3.2.1}$$

is a Lipschitz process, where q is a fixed positive real number.

Proof. Using the linearity of $\mathcal{F}_L^\alpha$, we have

$$\begin{aligned} \mathcal{W}^\alpha(\lambda f) &= \{(\lambda f)_L^\alpha : (Lg)_{V_0} = g|_{V_0} \ \forall\ g \in \mathcal{L}^p(SG) \text{ and } \|L\| \leq q\} \\ &= \lambda \mathcal{W}^\alpha(f),\ \forall\ f \in \mathcal{L}^p(SG),\ \lambda > 0. \end{aligned}$$

Again by linearity of $\mathcal{F}_L^\alpha$, it is evident that $\mathcal{W}^\alpha(0) = \{0\}$. Therefore, $\mathcal{W}^\alpha$ is a process.

Let $f, g \in \mathcal{C}(SG)$. On applying Equation (3.1.2), we have

$$\begin{aligned}\left|f_L^\alpha(t)-g_L^\alpha(t)\right| \le \|f-g\|_p + \|\alpha\|_\infty \|f_L^\alpha - g_L^\alpha\|_p \\ + \|\alpha\|_\infty \|L(g)-L(f)\|_p,\end{aligned}$$

for any $t \in SG$. Further, we conclude

$$\|f_L^\alpha - g_L^\alpha\|_p \le \frac{1+\|\alpha\|_\infty \|L\|}{1-\|\alpha\|_\infty}\|f-g\|_p.$$

Applying $\|L\| \le q$, we have

$$\|f_L^\alpha - g_L^\alpha\|_p \le \frac{q(1+\|\alpha\|_\infty)}{1-\|\alpha\|_\infty}\|f-g\|_p.$$

Therefore,

$$\mathcal{W}^\alpha(g) \subseteq \mathcal{W}^\alpha(f) + \frac{1+\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\ \|f-g\|_p U_{\mathcal{L}^p(SG)},$$

proving the Lipschitzness of $\mathcal{W}^\alpha$, and hence the result. □

Theorem 3.2.5. *Let f be a continuous function on SG. The multi-valued mapping $\mathcal{T}_L : \mathcal{L}^p(SG) \rightrightarrows \mathcal{L}^p(SG)$ defined by*

$$\mathcal{T}_L(f) = \{f_L^\alpha : \|\alpha\|_\infty \le q < 1\},$$

is Lipschitz.

Proof. Let $f, g \in \mathcal{C}(SG)$. Equation (3.1.1) yields

$$\begin{aligned}\left|f_L^\alpha(t)-g_L^\alpha(t)\right| \le& \|f-g\|_p + \|\alpha\|_\infty \|f_L^\alpha - g_L^\alpha\|_p \\ &+ \|\alpha\|_\infty \|Lg-Lf\|_p,\end{aligned}$$

for every $t \in SG$. This further concludes that

$$\|f_L^\alpha - g_L^\alpha\| \leq \frac{1 + \|\alpha\|_\infty \|L\|}{1 - \|\alpha\|_\infty} \|f - g\|_p.$$

Since $\|\alpha\|_\infty \leq q$, we get

$$\|f_L^\alpha - g_L^\alpha\| \leq \frac{1 + q}{1 - q} \|f - g\|_p.$$

Choosing $l = \frac{1+q}{1-q}$, we have

$$\mathcal{T}_L(g) \subset \mathcal{T}_L(f) + l \, \|f - g\|_p U_{\mathcal{L}^p(SG)},$$

proving the assertion. □

3.3 Conclusions

This chapter has focused on function approximation using fractal functions with respect to the $\mathcal{L}^p$-norm on SG. We defined α-fractal functions in $\mathcal{L}^p$ space and explored properties such as topological isomorphism and those closely related to the fractal operator. Furthermore, we also gave a hint for the existence of a non-trivial closed invariant subspace for the fractal operator. The latter part of this chapter discussed set-valued mappings and examined some useful properties.

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