CHAPTER 3

Higher Order Computational Approach for Generalized Time-Fractional Diffusion Equation

3.1 Introduction

The objective of this chapter is to develop higher order convergence schemes to find the numerical solution to the mathematical problem (3.1)-(3.2).

Consider the following mathematical problem:

$$\partial_{0,t}^{\alpha,\omega(t)}\zeta = \mathcal{L}\zeta + \phi(x,t), \qquad x \in (0,1), \quad t \in (0,T] \tag{3.1}$$

$$\zeta(0,t)=0,\quad \zeta(1,t)=0,\quad t\in[0,T],\quad \zeta(x,0)=\zeta_0(x),\quad x\in[0,1] \tag{3.2}$$

where,

$$\mathcal{L}\zeta=\frac{\partial}{\partial x}\left(m(x,t)\frac{\partial\zeta}{\partial x}\right)-p(x,t)\zeta, \tag{3.3}$$

and

$$\partial_{0,t}^{\alpha,\omega(t)}\zeta(x,t)=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^t\frac{\omega(t-\xi)}{(t-\xi)^{\alpha}}\frac{\partial\zeta}{\partial\xi}(x,\xi)\mathrm{d}\xi, \tag{3.4}$$

is the fractional derivative of order α, $0<\alpha<1$ in the generalized Caputo sense with the weight function $\omega(t)\in C^2[0,T]$, $H=\{(x,t):0\le x\le l,0\le t\le T\}$, $\omega(t)>0$ and $\omega'(t)\le 0$ for all $t\in[0,T]$, and $0<c_1\le m(x,t)\le c_2$ and $p(x,t)\ge 0$ for all $(x,t)\in H$.

To design the proposed schemes, we consider three different classes of weighting function $\omega(t)$ resulting in three different discrete analogs of generalized derivative (3.4) namely, Formula 1, 2, and 3. We begin with developing the first difference analog of the generalized Caputo FD Formula 1 defined in (3.8) which uses second order approximation of the weight function $\omega(t)\in C^2[0,T]$. Then we upgrade Formula 1 by using third order approximation of the function $\omega(t)\in C^3[0,T]$. This upgraded analog can be called as the L2 formula (3.10) for the generalized Caputo derivative having an approximation order of $\mathcal{O}(\tau^{3-\alpha})$. Finally, we impose a very weak regularity condition on the weight function as $\omega(t)\in C[0,T]$ and develop formula 3 defined in (3.12) using the exact weighting function without any approximation.

The arrangement of the remaining chapter is as follows

- Sect. 3.2 : Development of three different formulae viz. Formula 1 defined in (3.8), L2 formula (3.10), and Formula 3 defined in equation (3.12) based on above discussed approaches.
- Sect. 3.3 : A complete theoretical analysis is presented for the L2 scheme (3.14)-(3.15).
- Sect. 3.4 : The numerical experiments are performed on three different test problems along with few important discussions of the outcomes.
- Sect. 3.5 : This section contains some concluding remarks.

Remark 3.1. Among all the three developed formulae, the L2 formula (Formula 2) is the most efficient having an accuracy of $\mathcal{O}(\tau^{3-\alpha})$. Although, Formula 3 has an advantage of using a weak regularity condition on weight function $\omega(t) \in C[0,T]$, yet has the drawback that using exact weighting function, the analytic integration of the coefficients of the difference operator may not always be feasible.

3.2 Formulae and the numerical schemes

This section contains the derivation of three different discrete analogs of the Caputo FD in generalized sense along with the L2 difference scheme for GTFDE (3.1)-(3.2) considering Formula 2 (L2 Formula).

3.2.1 Formulae derivations

In order to design discrete analogs for the Caputo FD having generalized memory kernel, we use a piecewise quadratic polynomial for interpolating $\zeta(t)$ instead of the linear interpolation used in the traditional L1 formula. For this, we use the formula

$\Pi_{2,s}\zeta(t)$ of $\zeta(t)$ which uses three points $(t_{s-1}, \zeta(t_{s-1}))$, $(t_s, \zeta(t_s))$ and $(t_{s+1}, \zeta(t_{s+1}))$ on the interval $[t_s, t_{s+1}]$.

$$\Pi_{2,s}\zeta(t) = \zeta(t_{s-1})\frac{(t-t_s)(t-t_{s+1})}{2\tau^2} - \zeta(t_s)\frac{(t-t_{s-1})(t-t_{s+1})}{\tau^2} + \zeta(t_{s+1})\frac{(t-t_{s-1})(t-t_s)}{2\tau^2}, \tag{3.5}$$

and

$$(\Pi_{2,s}\zeta(t))' = \zeta_{t,s} + \zeta_{\bar{t}t,s}(t - t_{s+1/2}), \quad t_{s+1/2} = t_s + 0.5\tau,$$

with

$$\zeta_{\bar{t}t,s} = \frac{\zeta_{s+1} - 2\zeta_s + \zeta_{s-1}}{\tau^2}, \qquad t \in [t_{s-1}, t_{s+1}], \ \ \zeta_s = \zeta(t_s),$$

where,

$$\zeta(t) - \Pi_{2,s}\zeta(t) = \frac{\zeta'''(\eta_s)}{6}(t-t_{s-1})(t-t_s)(t-t_{s+1}) \ , \quad t_{s-1} < \eta_s < t_{s+1}.$$

For the generalized Caputo derivative of $\zeta(t) \in C^3[0,T]$ of order α, $0 < \alpha < 1$, with weight $\omega(t)$ we consider the uniform mesh $\chi_\tau = \{t_j = j\tau, j = 0, 1, \ldots, n-1,$ $1 \le n \le N; T = \tau N\}$. For the fixed point $t_{j+1}, j \in 1, 2, \ldots, N-1$, the following equality holds

$$\begin{aligned} \partial_{0,t_{j+1}}^{\alpha,\omega(t)}\zeta(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{j+1}} \frac{\zeta'(\xi)\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^\alpha}d\xi \\ &= \frac{1}{\Gamma(1-\alpha)}\left[\int_0^{t_2} \frac{\zeta'(\xi)\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^\alpha}d\xi + \sum_{s=2}^{j}\int_{t_s}^{t_{s+1}} \frac{\zeta'(\xi)\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^\alpha}d\xi\right]. \end{aligned}$$

We use the quadratic interpolation defined in (3.5) in order to approximate $\zeta(t)$ on the interval $[t_s, t_{s+1}]$, $(1 \leq s \leq j)$. Thus, we have

$$\begin{aligned}
\partial_{0,t_{j+1}}^{\alpha,\omega(t)}\zeta(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}\frac{\zeta'(\xi)\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi + \frac{1}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{\zeta'(\xi)\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi \\
&\approx \frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}\frac{(\Pi_{2,1}\zeta(\xi))'\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi \\
&+ \frac{1}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{(\Pi_{2,s}\zeta(\xi))'\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}\frac{[\zeta_{t,1}+\zeta_{\bar{t}t,1}(\xi-t_{3/2})]\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi \\
&+ \frac{1}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{[\zeta_{t,s}+\zeta_{\bar{t}t,s}(\xi-t_{s+1/2})]\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi.
\end{aligned}$$

3.2.1.1 Formula 1

This subsection includes the derivation of a difference analog where we use the quadratic interpolation formula $\Pi_{2,s}\zeta(t)$ of $\zeta(t)$ defined in (3.5) along with the second order approximation of the weighting function $\omega(t) \in C^2[0,T]$ defined by the following relation

$$\omega(t_{j+1}-\xi) = \omega_{j-s+1/2} + \frac{(\omega_{j-s}-\omega_{j-s+1})}{\tau}(\xi - t_{s+1/2}) + \mathcal{O}(\tau^2), \quad t_s < \xi < t_{s+1}.$$

Using this interpolation of $\omega(t_{j+1}-\xi)$, the above approximation of the FD $\partial_{0,t_{j+1}}^{\alpha,\omega(t)}\zeta(t)$ gets transformed into the following form:

$$\partial_{0,t_{j+1}}^{\alpha,\omega(t)}\zeta(t) \approx \frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}\frac{(\zeta_{t,1}+\zeta_{\bar{t}t,1}(\xi-t_{3/2}))\big[\omega_{j-1/2}+(\omega_{j-1}-\omega_j)(\xi-t_{3/2})/\tau\big]}{(t_{j+1}-\xi)^{\alpha}}d\xi$$
$$+\frac{1}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{(\zeta_{t,s}+\zeta_{\bar{t}t,s}(\xi-t_{s+1/2}))\big[\omega_{j-s+1/2}+(\omega_{j-s}-\omega_{j-s+1})(\xi-t_{s+1/2})/\tau\big]}{(t_{j+1}-\xi)^{\alpha}}d\xi.$$

Since,

$$\int_{t_s}^{t_{s+1}}\frac{d\xi}{(t_{j+1}-\xi)^{\alpha}} = \frac{\tau^{1-\alpha}}{1-\alpha}a_{j-s}$$
$$a_l = (l+1)^{1-\alpha}-l^{1-\alpha}, \qquad l \geq 0; \tag{3.6}$$

$$\int_{t_s}^{t_{s+1}}\frac{(\xi-t_{s+1/2})}{(t_{j+1}-\xi)^{\alpha}}d\xi = \frac{\tau^{2-\alpha}}{1-\alpha}b_{j-s}, \qquad 0\leq s\leq j,$$
$$b_l = \frac{1}{2-\alpha}\big[(l+1)^{2-\alpha}-l^{2-\alpha}\big]-\frac{1}{2}\big[(l+1)^{1-\alpha}+l^{1-\alpha}\big], \qquad l\geq 0. \tag{3.7}$$

Taking into account the above equalities (3.6)-(3.7), the first difference analog of the generalized Caputo FD of order α $(0<\alpha<1)$ for the function $\zeta(t)$ (Formula 1), at the points t_{j+1} $(j = 1, 2, \ldots, N-1)$, has the following representation:

$$\begin{aligned}\partial_{0,t_{j+1}}^{\alpha,\omega(t)}\zeta(t) \approx \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\Big[&\zeta_{t,0}\big[\omega_{j-1/2}(a_j-b_j-b_{j-1})\big] + \zeta_{t,1}\big[\omega_{j-1/2}(b_j+b_{j-1}+a_{j-1}) \\ &- \omega_{j-3/2}b_{j-2} + (\omega_{j-1}-\omega_j)(b_j+b_{j-1}-a_j)\big] + \sum_{s=2}^{j-1}\zeta_{t,s}\Big\{\omega_{j-s+1/2}(a_{j-s}+b_{j-s}) \\ &+ b_{j-s}(\omega_{j-s}-\omega_{j-s+1}) - \omega_{j-s-1/2}b_{j-s-1}\Big\} + \zeta_{t,j}\big[\omega_{1/2}(a_0+b_0)+b_0(\omega_0-\omega_1)\big]\Big] \\ &= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{j} d^1_{j-s}\zeta_{t,s}, \end{aligned} \tag{3.8}$$

where for $j = 1$

$$d^1_s = \begin{cases} (b_1+b_0+a_0)\omega_{1/2} + (\omega_0-\omega_1)(b_1+b_0-a_1), & s = 0, \\ \omega_{1/2}(a_1-b_1-b_0), & s = 1, \end{cases}$$

for $j = 2$

$$d^1_s = \begin{cases} \omega_{1/2}(a_0+b_0) + b_0(\omega_0-\omega_1), & s = 0, \\ \omega_{3/2}(b_2+b_1+a_1) - \omega_{1/2}b_0 + (\omega_1-\omega_2)(b_2+b_1-a_2), & s = 1, \\ \omega_{3/2}(a_2-b_2-b_1), & s = 2, \end{cases}$$

and for $j \geq 3$,

$$d^1_s = \begin{cases} \omega_{1/2}(a_0+b_0) + b_0(\omega_0-\omega_1), & s = 0, \\ \omega_{s+1/2}(a_s+b_s) + b_s(\omega_s-\omega_{s+1}) - b_{s-1}\omega_{s-1/2}, & 1 \leq s \leq j-2, \\ \omega_{j-1/2}(b_j+b_{j-1}+a_{j-1}) - \omega_{j-3/2}b_{j-2} + (\omega_{j-1}-\omega_j)(b_j+b_{j-1}-a_j), & s = j-1, \\ \omega_{j-1/2}(a_j-b_j-b_{j-1}), & s = j. \end{cases}$$

3.2.1.2 Formula 2 (The L2 formula)

In this subsection, the designed formula (3.8) has been upgraded using third order approximation of the weight function $\omega(t) \in C^3[0,T]$ and hence we use the following representation for $\omega(t_{j+1}-\xi)$,

$$\omega(t_{j+1}-\xi) = \Psi(\xi) + [\omega(t_{j+1}-\xi) - \Psi(\xi)], \tag{3.9}$$

where,

$$\begin{aligned}\Psi(\xi) = \omega_{j-s+1/2} + \frac{(\omega_{j-s}-\omega_{j-s+1})}{\tau}(\xi - t_{s+1/2})\\ + \frac{2(\omega_{j-s+1}-2\omega_{j-s+1/2}+\omega_{j-s})}{\tau^2}(\xi - t_{s+1/2})^2,\end{aligned}$$

Thus, using the defined representation of the function $\omega(t_{j+1}-\xi)$ and ignoring higher order terms of $(\xi - t_{s+1/2})^3$, the derived L2 difference analog of the generalized Caputo FD of order α $(0 < \alpha < 1)$ for the function $\zeta(t)$, at the points t_{j+1}, $(j =$

$1, 2, \dots, N-1)$, has the following form:

$$\begin{aligned}
\partial^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}\frac{[\zeta_{t,1}+\zeta_{\bar{t}t,1}(\xi-t_{3/2})]\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi \\
&+ \frac{1}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{[\zeta_{t,s}+\zeta_{\bar{t}t,s}(\xi-t_{s+1/2})]\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi + R_1 + R_2 \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}\frac{[\zeta_{t,1}+\zeta_{\bar{t}t,1}(\xi-t_{3/2})]\Psi(\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi \\
&+ \frac{1}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{[\zeta_{t,s}+\zeta_{\bar{t}t,s}(\xi-t_{s+1/2})]\Psi(\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi + R_1 + R_2 + R_3 \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\Bigg[\zeta_{t,0}\Bigg\{(a_j-b_j-b_{j-1})\omega_{j-1/2}+(\omega_j-\omega_{j-1})(c_j+c_{j-1}-2b_j+a_j)\Bigg\} \\
&+ \zeta_{t,1}\Bigg\{(a_{j-1}+b_j+b_{j-1})\omega_{j-1/2}+(\omega_{j-1}-\omega_j)(c_j+c_{j-1}-b_j+b_{j-1}) \\
&+ 2(\omega_j-2\omega_{j-1/2}+\omega_{j-1})(c_j+c_{j-1}-2b_j+a_j)+(\omega_{j-1}-\omega_{j-2})c_{j-2}-\omega_{j-3/2}b_{j-2}\Bigg\} \\
&+ \sum_{s=2}^{j-1}\zeta_{t,s}\Bigg\{\omega_{j-s+1/2}a_{j-s}+(\omega_{j-s}+\omega_{j-s+1/2}-\omega_{j-s+1})b_{j-s} \\
&+ (\omega_{j-s+1}+3\omega_{j-s}-4\omega_{j-s+1/2})c_{j-s}-\omega_{j-s-1/2}b_{j-s-1}+(\omega_{j-s}-\omega_{j-s-1})c_{j-s-1}\Bigg\} \\
&+ \zeta_{t,j}\Bigg\{\omega_{1/2}a_0+(\omega_0+\omega_{1/2}-\omega_1)b_0+(\omega_1+3\omega_0-4\omega_{1/2})c_0\Bigg\}\Bigg] \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{j}d_{j-s}\zeta_{t,s} = \Delta^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta(t), \qquad (3.10)
\end{aligned}$$

where a_l and b_l are identical to equations (3.6)-(3.7) and we have a new coefficient c_l defined below

$$\int_{t_s}^{t_{s+1}}\frac{(\xi-t_{s+1/2})^2}{(t_{j+1}-\xi)^{\alpha}}d\xi = \frac{\tau^{3-\alpha}}{1-\alpha}c_{j-s},$$

$$c_l = \frac{2}{(2-\alpha)(3-\alpha)}\left[(l+1)^{3-\alpha}-l^{3-\alpha}\right]-\frac{1}{2-\alpha}\left[(l+1)^{2-\alpha}+l^{2-\alpha}\right]+\frac{1}{4}\left[(l+1)^{1-\alpha}-l^{1-\alpha}\right],\ l \geq 0; \tag{3.11}$$

for $j = 1$

$$d_s = \begin{cases} \omega_{1/2}(b_1+b_0+a_0)+(\omega_0-\omega_1)(c_1+c_0-b_1+b_0) \\ +2(\omega_1-2\omega_{1/2}+\omega_0)(c_1+c_0-2b_1+a_1), \quad s=0, \\ \\ (a_1-b_1-b_0)\omega_{1/2}+(\omega_1-\omega_0)(c_1+c_0-2b_1+a_1), \quad s=1, \end{cases}$$

for $j = 2$

$$d_s = \begin{cases} \omega_{1/2}a_0+(\omega_0+\omega_{1/2}-\omega_1)b_0+(\omega_1+3\omega_0-4\omega_{1/2})c_0, \quad s=0, \\ \\ \omega_{3/2}(b_1+b_2+a_1)-\omega_{1/2}b_0+(\omega_1-\omega_0)c_0+(\omega_1-\omega_2) \\ (c_1+c_2-b_2+b_1)+2(\omega_2-2\omega_{3/2}+\omega_1)(c_2+c_1-2b_2+a_2), \quad s=1, \\ \\ (a_2-b_2-b_1)\omega_{3/2}+(\omega_2-\omega_1)(c_2+c_1-2b_2+a_2), \quad s=2, \end{cases}$$

and for $j \geq 3$,

$$d_s = \begin{cases} \omega_{1/2}a_0 + (\omega_0 + \omega_{1/2} - \omega_1)b_0 + (\omega_1 + 3\omega_0 - 4\omega_{1/2})c_0, & s = 0, \\ \\ \omega_{s+1/2}a_s + (\omega_s + \omega_{s+1/2} - \omega_{s+1})b_s - \omega_{s-1/2}b_{s-1} \\ +(\omega_{s+1} + 3\omega_s - 4\omega_{s+1/2})c_s + (\omega_s - \omega_{s-1})c_{s-1}, & 1 \leq s \leq j-2, \\ \\ \omega_{j-1/2}(b_j + b_{j-1} + a_{j-1}) - \omega_{j-3/2}b_{j-2} + (\omega_{j-1} - \omega_{j-2})c_{j-2} + (\omega_{j-1} - \omega_j) \\ (c_j + c_{j-1} - b_j + b_{j-1}) + 2(\omega_j - 2\omega_{j-1/2} + \omega_{j-1})(c_j + c_{j-1} - 2b_j + a_j), & s = j-1, \\ \\ \omega_{j-1/2}(a_j - b_j - b_{j-1}) + (\omega_j - \omega_{j-1})(c_j + c_{j-1} - 2b_j + a_j), & s = j, \end{cases}$$

with

$$\begin{aligned} R_1 &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{(\zeta(\xi) - \Pi_{2,1}\zeta(\xi))'\omega(t_{j+1} - \xi)}{(t_{j+1} - \xi)^\alpha} d\xi \ , \\ R_2 &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\zeta(\xi) - \Pi_{2,s}\zeta(\xi))'\omega(t_{j+1} - \xi)}{(t_{j+1} - \xi)^\alpha} d\xi \ , \\ R_3 &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{(\Pi_{2,1}\zeta(\xi))'(\omega(t_{j+1} - \xi) - \Psi(\xi))}{(t_{j+1} - \xi)^\alpha} d\xi \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\Pi_{2,s}\zeta(\xi))'(\omega(t_{j+1} - \xi) - \Psi(\xi))}{(t_{j+1} - \xi)^\alpha} d\xi \ . \end{aligned}$$

The derived difference analog (3.10) can be called as the L2 formula for the generalized Caputo fractional derivative.

3.2.1.3 Formula 3 (The exact formulation)

In the above formulation of the scheme, we remove the strong regularity conditions on weight function and impose a weak condition such that $\omega(t) \in C[0,T]$. Hence, instead of using the third order approximation formula (3.9) for $\omega(t_{j+1}-\xi)$, we use the exact weight function $\omega(t_{j+1}-\xi)$ for the computation of coefficients $\bar{a}_s$ and $\bar{b}_s$ defined below. The analytical integration of these coefficients may not always be feasible but can be computed by numerical integration with small truncation errors.

$$\bar{a}_{j-s} = \int_{t_s}^{t_{s+1}} \frac{\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \ ,$$

$$\bar{b}_{j-s} = \frac{1}{\tau}\int_{t_s}^{t_{s+1}} \frac{\omega(t_{j+1}-\xi)(\xi - t_{s+1/2})}{(t_{j+1}-\xi)^{\alpha}} d\xi \ .$$

Thus,

$$\bar{\Delta}^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta(t) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{j} \bar{d}_{j-s}\zeta_{t,s}, \tag{3.12}$$

where for $j = 1$

$$\bar{d}_s = \begin{cases} \bar{a}_0 + \bar{b}_0 + \bar{b}_1, & s = 0, \\ \bar{a}_1 - \bar{b}_1 - \bar{b}_0, & s = 1, \end{cases}$$

for $j = 2$

$$\bar{d}_s = \begin{cases} \bar{a}_0 + \bar{b}_0, & s = 0, \\ \bar{a}_1 + \bar{b}_1 + \bar{b}_2 - \bar{b}_0, & s = 1, \\ \bar{a}_2 - \bar{b}_2 - \bar{b}_1 \ , & s = 2, \end{cases}$$

and for $j \geq 3$,

$$\bar{d}_s = \begin{cases} \bar{a}_0 + \bar{b}_0, \quad s = 0, \\ \bar{a}_s + \bar{b}_s - \bar{b}_{s-1}, \quad 1 \leq s \leq j-2, \\ \bar{a}_{j-1} + \bar{b}_{j-1} + \bar{b}_j - \bar{b}_{j-2}, \quad s = j-1, \\ \bar{a}_j - \bar{b}_j - \bar{b}_{j-1}, \quad s = j. \end{cases}$$

3.2.2 The L2 difference scheme for GTFDEs

This subsection contains the development of a stable L2 difference scheme for GTFDEs with an order of accuracy of $\mathcal{O}(\tau^{3-\alpha} + h^2)$. Let $\zeta(x,t) \in C^{4,3}([0,1] \times [0,T])$ be the exact solution of model (3.1)-(3.2), then using Lemma 3.1 from [47] to discretize the spatial operator $\mathcal{L}$ defined in (**??**) gives

$$\mathcal{L}\zeta|_{(x_i,t_{j+1})} = \wedge\zeta(x_i, t_{j+1}) + \mathcal{O}(h^2),$$

where for any function $v(x) \in C^4[0,1]$, the difference operator $\wedge$ can be defined as

$$(\wedge v)_i = ((av_{\bar{x}}) - dv)_i = \frac{a_{i+1}v_{i+1} - (a_{i+1} + a_i)v_i + a_i v_{i-1}}{h^2} - d_i v_i, \quad i = 1, 2, \ldots, M-1, \tag{3.13}$$

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h}, \quad v_{x,i} = \frac{v_{i+1} - v_i}{h},$$

with the coefficients $a_i^{j+1} = m(x_{i-1/2}, t_{j+1})$ and $d_i^{j+1} = p(x_i, t_{j+1})$ where $m(x,t)$ and $p(x,t)$ are the given functions.

In the rectangle $H = [0,1] \times [0,T]$, we consider the mathematical model (3.1)-(3.2) at the point (x_i, t_{j+1}), with $1 \leq i \leq M-1$, $j = 1, 2, \dots, N-1$. Then, let $\varpi(x,t) \in C^{4,3}(H)$ be the numerical solution of (3.1)-(3.2) in the mesh $\chi_h \times \chi_\tau$, applying the L2 formula (3.10) and the spatial discretization (3.13) for the GTFDEs (3.1)-(3.2) with $\phi_i^{j+1} = \phi(x_i, t_{j+1})$, we obtain the following difference scheme with the approximation order of $\mathcal{O}(\tau^{3-\alpha} + h^2)$.

$$\Delta_{0,t_{j+1}}^{\alpha,\omega(t)} \varpi_i = \wedge \varpi_i^{j+1} + \phi_i^{t_{j+1}}, \qquad 1 \leq i \leq M-1, \quad 1 \leq j \leq N-1, \tag{3.14}$$

$$\varpi(0,t) = 0, \quad \varpi(1,t) = 0, \quad t \in \chi_\tau, \quad \varpi(x,0) = \zeta_0(x), \quad x \in \chi_h. \tag{3.15}$$

3.3 Theoretical analysis of L2 scheme

3.3.1 Stability of L2 scheme

Lemma 3.3.1. For $s = 1, 2, 3, \dots$ and $\forall\ \alpha \in (0,1)$, the coefficients a_s, b_s defined in (3.6)-(3.7) and the coefficient c_s defined in (3.11) have the following bounds.

$$\frac{(1-\alpha)}{(s+1)^\alpha} < a_s < \frac{(1-\alpha)}{s^\alpha}, \tag{3.16}$$

$$\frac{\alpha(1-\alpha)}{(s+2)^{\alpha+1}} < a_s - a_{s+1} < \frac{\alpha(1-\alpha)}{s^{\alpha+1}}, \tag{3.17}$$

$$\frac{\alpha(1-\alpha)}{12(s+1)^{\alpha+1}} < b_s < \frac{\alpha(1-\alpha)}{12 s^{\alpha+1}}, \tag{3.18}$$

$$(1-\alpha)\left(\frac{1}{4(s+1)^\alpha} - \frac{1}{6s^\alpha}\right) < c_s < (1-\alpha)\left(\frac{1}{4s^\alpha} - \frac{1}{6(s+1)^\alpha}\right). \tag{3.19}$$

Proof. The proof for the bounds of a_s and b_s is identical to Lemma 2.2 in [47]. Hence, we prove the validity of the bounds for the coefficient c_s defined in (3.11).

$$c_s = \frac{1-\alpha}{4}\int_0^1 \frac{d\xi}{(\xi+s)^\alpha} - \frac{1-\alpha}{2^{2-\alpha}}\int_0^1 \xi d\xi \int_{2s+1-\xi}^{2s+1+\xi}\frac{d\eta}{\eta^\alpha},$$

Let

$$k(s) = \frac{1-\alpha}{4}\int_0^1 \frac{d\xi}{(\xi+s)^\alpha} - \frac{1-\alpha}{2^{2-\alpha}}\int_0^1 \xi d\xi \int_{2s+1-\xi}^{2s+1+\xi}\frac{d\eta}{\eta^\alpha},$$

$$k'(s) = \frac{1-\alpha}{4}[(s+1)^{-\alpha} - s^{-\alpha}] + \frac{1-\alpha}{2^{2-\alpha}}\int_0^1 2\xi\big((2s+1-\xi)^{-\alpha} - (2s+1+\xi)^{-\alpha}\big)d\xi.$$

Let $f(\xi) = (2s+1-\xi)^{-\alpha} - (2s+1+\xi)^{-\alpha}$. Clearly $f(\xi)$ is an increasing function as $f'(\xi) \geq 0$. Thus, the maximum value of $f(\xi)$ for the interval $\xi \in [0,1]$ is $(2s)^{-\alpha} - (2s+2)^{-\alpha}$. Hence

$$k'(s) \leq \frac{1-\alpha}{4}[(s+1)^{-\alpha} - s^{-\alpha}] - \frac{1-\alpha}{4}[(s+1)^{-\alpha} - s^{-\alpha}] = 0.$$

Thus, $c_s = k(s)$ is decreasing for $\forall \;\; s \in \mathbb{N}$. Clearly

$$\begin{aligned}\frac{1-\alpha}{4(s+1)^\alpha} &< \int_0^1 \frac{d\xi}{(\xi+s)^\alpha} < \frac{1-\alpha}{s^\alpha} \quad , \\ \frac{\alpha(1-\alpha)}{12(s+1)^\alpha} &< \frac{1-\alpha}{2^{2-\alpha}}\int_0^1 \xi \int_{2s+1-\xi}^{2s+1+\xi}\frac{d\xi}{\xi^\alpha} < \frac{\alpha(1-\alpha)}{12 s^\alpha} \quad .\end{aligned}$$

Using the above inequalities bounds for coefficient c_s are obtained. □

Lemma 3.3.2. The following inequalities are valid for any $\alpha \in (0,1)$, $\tau < \tau_0$ and $(0 \leq s \leq j,\ j \geq 2)$.

$$d_0 > d_2 > d_3 > \ldots > d_{j-2} > d_{j-1} > d_j, \tag{3.20}$$

$$d_0 + 3d_1 - 4d_2 > 0. \tag{3.21}$$

Proof.

$$d_0 - d_2 \geq a_0 - a_2 + b_0 - b_2 + b_1 - b_3 > 0.$$

For $j \geq 5,\ 2 \leq s \leq j-3$,

Let $d_s^* = a_s + b_s - b_{s-1}$

$$\begin{aligned} d_s = {} & \omega_{s+1/2} d_s^* + (\omega_s - \omega_{s+1}) b_s + (\omega_{s+1/2} - \omega_{s-1/2}) b_{s-1} \\ & + (\omega_{s+1} + 3\omega_s - 4\omega_{s+1/2}) c_s + (\omega_s - \omega_{s-1}) c_{s-1}, \end{aligned}$$

Using the bound for b_s proved in Lemma (3.3.1), we get

$$d_s = \omega_{s+1/2} d_s^* + \frac{\mathcal{O}(\tau)}{s^{\alpha+1}} + (\omega_{s+1} + 3\omega_s - 4\omega_{s+1/2}) c_s + (\omega_s - \omega_{s-1}) c_{s-1},$$

Similarly, we have

$$d_{s+1} = \omega_{s+3/2} d_{s+1}^* + \frac{\mathcal{O}(\tau)}{s^{\alpha+1}} + (\omega_{s+2} + 3\omega_{s+1} - 4\omega_{s+3/2}) c_{s+1} + (\omega_{s+1} - \omega_s) c_s,$$

$$\begin{aligned} d_s - d_{s+1} &= \omega_{s+1/2} d_s^* - \omega_{s+3/2} d_{s+1}^* + \frac{\mathcal{O}(\tau)}{s^{\alpha+1}} + (4\omega_s - 4\omega_{s+1/2})c_s \\ &\quad + (\omega_s - \omega_{s-1})c_{s-1} - (\omega_{s+2} + 3\omega_{s+1} - 4\omega_{s+3/2})c_{s+1}, \end{aligned}$$

Consider

$$\begin{aligned} &(4\omega_s - 4\omega_{s+1/2})c_s + (\omega_s - \omega_{s-1})c_{s-1} - (\omega_{s+2} + 3\omega_{s+1} - 4\omega_{s+3/2})c_{s+1} \\ &= (2\omega_s - 4\omega_{s+1/2} + 2\omega_{s+1})c_s + (2\omega_s - 2\omega_{s+1})c_s + (\omega_s - \omega_{s-1})c_{s-1} \\ &\quad - (2\omega_{s+2} - 4\omega_{s+3/2} + 2\omega_{s+1})c_{s+1} - (\omega_{s+1} - \omega_{s+2})c_{s+1} \\ &= \mathcal{O}(\tau^2)c_s + (2\omega_s - 2\omega_{s+1})c_s + (\omega_s - \omega_{s-1})c_{s-1} \\ &\quad + \mathcal{O}(\tau^2)c_{s+1} - (\omega_{s+1} - \omega_{s+2})c_{s+1} \qquad \big(\omega_s - 2\omega_{s+1/2} + \omega_{s+1} = \mathcal{O}(\tau^2)\big), \\ &= 2q(s) - q(s-1) - q(s+1) + \frac{\mathcal{O}(\tau)}{s^{\alpha+1}}, \ \Bigg(\text{using } c_s \text{ bound from equation (3.18) and} \\ &q(z) = (\omega(z\tau) - \omega(z\tau + \tau))r(z), \quad r(z) = \frac{2}{(2-\alpha)(3-\alpha)}\big[(z+1)^{3-\alpha} - z^{3-\alpha}\big] - \\ &\frac{1}{2-\alpha}\big[(z+1)^{2-\alpha} + z^{2-\alpha}\big] + \frac{1}{4}\big[(z+1)^{1-\alpha} - z^{1-\alpha}\big]\Bigg) \\ &= -q''(s+\theta) + \frac{\mathcal{O}(\tau)}{s^{\alpha+1}}, \qquad 0 < \theta < 1. \end{aligned}$$

Now,

$$\begin{aligned} q''(z) &= \tau^2(\omega''(\tau z) - \omega''(\tau z + z))r(z) + (\omega(z\tau) - \omega(z + \tau z))r''(z) \\ &\quad + 2\tau(\omega'(\tau z) - \omega'(\tau z + \tau))r'(z), \end{aligned}$$

using the bounds of a_s, b_s and c_s from Lemma (3.3.1), we have

$$q(z) \propto \frac{1}{z^{\alpha}}, \quad q'(z) \propto \frac{1}{z^{\alpha+1}}, \quad q''(z) \propto \frac{1}{z^{\alpha+2}},$$

Therefore,

$$d_s - d_{s+1} \geq \omega_{s+1/2}(d_s^* - d_{s+1}^*) + \frac{\mathcal{O}(\tau)}{s^{\alpha+1}},$$

From Lemma 2.3 in [47],

$$d_s^* - d_{s+1}^* \geq \frac{\alpha(1-\alpha)}{2(s+2)^{\alpha+1}},$$

Thus,

$$\begin{aligned} d_s - d_{s+1} &\geq \omega_{s+1/2}\left(\frac{\alpha(1-\alpha)}{2(s+2)^{\alpha+1}}\right) + \frac{\mathcal{O}(\tau)}{s^{\alpha+1}}, \\ &> \frac{1}{(s+2)^{\alpha+1}}\big(\omega(T)\gamma + \mathcal{O}(\tau)\big) \qquad \gamma = \frac{\alpha(1-\alpha)}{2}, \end{aligned}$$

Hence, there exist a $\tau_1 > 0$, such that $\omega(T)\gamma + \mathcal{O}(\tau) > 0$, for all $\tau < \tau_1$. Thus,

$$d_s - d_{s+1} > 0, \quad \forall \tau < \tau_1.$$

Using a similar approach, we prove inequality (3.20) for $j = 3, 4$. Hence, inequality (3.20) is valid for all $j \geq 2$, $\tau < \tau_1$.

Next, we prove the inequality (3.21). For $j \geq 3$,

$$\begin{aligned} d_0 &= \omega_{1/2}a_0 + (\omega_0 + \omega_{1/2} - \omega_1)b_0 + (\omega_1 + 3\omega_0 - 4\omega_{1/2})c_0 \\ &= \omega_0 d_0^* + \mathcal{O}(\tau), \qquad d_0^* = a_0 + b_0, \\ d_1 &= \omega_{3/2}a_1 + (\omega_1 + \omega_{3/2} - \omega_2)b_1 - \omega_{1/2}b_0 + (\omega_2 + 3\omega_1 - 4\omega_{3/2})c_1 + (\omega_1 - \omega_0)c_0 \\ &= \omega_0 d_1^* + \mathcal{O}(\tau), \qquad d_1^* = a_1 + b_1 - b_0, \\ d_2 &= \omega_{5/2}a_2 + (\omega_2 + \omega_{5/2} - \omega_3)b_2 - \omega_{3/2}b_1 + (\omega_3 + 3\omega_2 - 4\omega_{5/2})c_2 + (\omega_2 - \omega_1)c_1 \\ &= \omega_0 d_2^* + \mathcal{O}(\tau) \qquad d_2^* = a_2 + b_2 - b_1, d_0 + 3d_1 - 4d_2 = \omega_0(d_0^* + 3d_1^* - 4d_2^*) + \mathcal{O}(\tau), \end{aligned}$$

Since $d_0^* + 3d_1^* - 4d_2^* > 0$ from [47], hence there exist a $\tau_2 > 0$, such that $d_0 + 3d_1 - 4d_2 > 0, \quad \forall\tau < \tau_2$. Similarly, we can prove for $j = 2$.

Now, let $\tau_0 = min\{\tau_1, \tau_2\}$. Thus, Lemma (3.2) holds for $\forall\tau < \tau_0$. □

Theorem 3.3.1. The difference scheme (3.14)-(3.15) is unconditionally stable $\forall\tau < \tau_0$ with its solution satisfying the following inequality

$$\sum_{j=1}^{N-1} (||\varpi^{j+1}||_0^2 + ||\varpi_{\bar{x}}^{j+1}]|_0^2)\tau \leq C\Bigg(||\varpi^1||_0^2 + ||\varpi^0||_0^2 + \sum_{j=1}^{N-1} ||\phi^{j+1}||_0^2\tau\Bigg), \tag{3.22}$$

where $||\varpi]|_0^2 = \sum\limits_{i=1}^{M} \varpi_i^2 h$, $C > 0$ is a known constant independent of τ and h.

Proof. The inner product of the equation (3.14) with ϖ^{j+1} yields

$$\Bigg(\varpi^{j+1}, \Delta_{0,t_{j+1}}^{\alpha,\omega(t)}\varpi\Bigg) = (\varpi^{j+1}, \wedge\varpi^{j+1}) + (\varpi^{j+1}, \phi^{j+1}). \tag{3.23}$$

Using Lemma (2.4)-(2.6) from [47], we obtain

$$\Bigg(\varpi^{j+1}, \Delta_{0,t_{j+1}}^{\alpha,\omega(t)}\varpi\Bigg) \geq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(E_{j+1} - E_j) + \frac{1}{2}\Delta'^{\alpha,\omega(t)}_{0,t_{j+1}}||\varpi||_0^2$$
$$= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(\mathcal{E}_{j+1} - \mathcal{E}_j) - \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}d'_j||\varpi^0||_0^2, \quad j = 1, 2, \ldots, N-1,$$

where

$$\Delta'^{\alpha,\omega(t)}_{0,t_{j+1}}\varpi = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{j} d'_{j-s}(\varpi_{s+1} - \varpi_s), \quad j = 1, 2, \ldots, N,$$

$$d'_0 = d_2, \quad d'_1 = d_2, \quad d'_s = d_s, \quad s = 2, 3, ..., j,$$

and

$$\begin{aligned} E_j = &\left(\frac{1}{2}\sqrt{\frac{d_0-d_1}{2}}+\frac{1}{2}\sqrt{\frac{d_0+3d_1-4d_2}{2}}\right)^2 ||\varpi^j||_0^2 \\ &+\left|\left|\sqrt{\frac{d_0-d_1}{2}}\varpi^j-\left(\frac{1}{2}\sqrt{\frac{d_0-d_1}{2}}+\frac{1}{2}\sqrt{\frac{d_0+3d_1-4d_2}{2}}\right)\varpi^{j-1}\right|\right|_0^2, \end{aligned}$$

$$\mathcal{E}_j = E_j + \frac{1}{2}\sum_{s=0}^{j-1} d'_{j-1-s}||\varpi^{s+1}||_0^2.$$

For the difference operator $\wedge$ using Green's first difference formula for the functions vanishing at $x=0$ and $x=1$, we get $(-\wedge\varpi,\varpi)\geq k_1||\varpi_{\bar{x}}]|_0^2$. Using this with (3.23)

$$(\varpi^{j+1},\phi^{j+1}) \leq k_1||\varpi^{j+1}||_0^2+\frac{1}{4k_1}||\phi^{j+1}||_0^2 \leq \frac{k_1}{2}||\varpi_{\bar{x}}^{j+1}]|_0^2+\frac{1}{4k_1}||\phi^{j+1}||_0^2,$$

we obtain the inequality

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(\mathcal{E}_{j+1}-\mathcal{E}_j)+\frac{k_1}{2}||\varpi_{\bar{x}}^{j+1}]|_0^2 \leq \frac{1}{4k_1}||\phi^{j+1}||_0^2+\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}d'_j||\varpi^0||_0^2. \tag{3.24}$$

In the formula (3.10), for all $j\in\mathbb{N}$,

$$\begin{aligned} d_j &= \omega_{j-1/2}(a_j-b_j-b_{j-1})+(\omega_j-\omega_{j-1})(c_j+c_{j-1}-2b_j+a_j) \\ &< \omega_{j-1/2}(a_j-b_j-b_{j-1}) < \omega_0 a_j, \quad \text{as} \ \ (\omega_j-\omega_{j-1})(c_j+c_{j-1}-2b_j+a_j)<0. \end{aligned}$$

Therefore,

$$d_j < \frac{1-\alpha}{j^\alpha}\omega_0. \tag{3.25}$$

Multiplying equation (3.24) by τ and then using summation of the resultant equation from 1 to $N-1$ and also considering inequality (3.25), one obtain the inequality (3.22). Hence, the developed scheme (3.14)-(3.15) is unconditionally stable. $\square$

3.3.2 Convergence analysis of the L2 scheme

Lemma 3.3.3. Let $\zeta(t) \in C^3[0, t_{j+1}]$ and $\Delta^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta$ be a difference operator of scheme (3.14)-(3.15), then for any $\alpha \in (0,1)$ and $j = 1, 2, \ldots, N-1$, it is true that

$$\partial^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta = \Delta^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta + \mathcal{O}(\tau^{3-\alpha}),$$

where weight function $\omega(t) \in C^3[0, t_{j+1}]$ with $\omega(t) > 0, \omega'(t) \leq 0, \forall\ t \in [0,T]$.

Proof. Consider

$$\partial^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta = \Delta^{\alpha,\omega(t)}_{0,t_{j+1}}\zeta + R_1 + R_2 + R_3,$$

Now, first we estimate the errors R_1 and R_2

$$\begin{aligned}
R_1 &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{(\zeta(\xi) - \Pi_{2,1}\zeta(\xi))'\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{(\zeta(\xi) - \Pi_{2,1}\zeta(\xi))\omega'(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \\
&- \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{(\zeta(\xi) - \Pi_{2,1}\zeta(\xi))\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha+1}} d\xi \\
&= \frac{1}{6\Gamma(1-\alpha)} \int_0^{t_2} \frac{\zeta'''(\eta_1)\xi(\xi-t_1)(\xi-t_2)\omega'(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \\
&- \frac{\alpha}{6\Gamma(1-\alpha)} \int_0^{t_2} \frac{\zeta'''(\eta_1)\xi(\xi-t_1)(\xi-t_2)\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha+1}} d\xi.
\end{aligned}$$

$$
\begin{aligned}
|R_1| \leq & \frac{1}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\zeta'''(\eta_1)\xi(\xi-t_1)(\xi-t_2)\omega'(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi\right| \\
& + \frac{\alpha}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\zeta'''(\eta_1)\xi(\xi-t_1)(\xi-t_2)\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha+1}}d\xi\right| \\
\leq & \frac{M_3m_1}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\xi(\xi-t_1)(\xi-t_2)}{(t_{j+1}-\xi)^{\alpha}}d\xi\right| + \frac{\alpha M_3\omega(0)}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\xi(\xi-t_1)(\xi-t_2)}{(t_{j+1}-\xi)^{\alpha+1}}d\xi\right|.
\end{aligned}
$$

For $j = 1$ we have

$$
\begin{aligned}
|R_1| \leq & \frac{M_3m_1}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\xi(\xi-t_1)(\xi-t_2)}{(t_2-\xi)^{\alpha}}d\xi\right| + \frac{\alpha M_3\omega(0)}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\xi(\xi-t_1)(\xi-t_2)}{(t_2-\xi)^{\alpha+1}}d\xi\right| \\
\leq & \frac{M_3m_1\tau^2}{3\Gamma(1-\alpha)}\int_0^{t_2}(t_2-\xi)^{1-\alpha}d\xi + \frac{\alpha M_3\omega(0)\tau^2}{3\Gamma(1-\alpha)}\int_0^{t_2}(t_2-\xi)^{-\alpha}d\xi \\
\leq & \frac{2^{2-\alpha}\tau^{4-\alpha}M_3m_1}{3(2-\alpha)\Gamma(1-\alpha)} + \alpha\frac{2^{1-\alpha}\tau^{3-\alpha}M_3\omega(0)}{3\Gamma(2-\alpha)}.
\end{aligned}
$$

For $j \geq 2$

$$
\begin{aligned}
|R_1| \leq & \frac{M_3m_1}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\xi(\xi-t_1)(\xi-t_2)}{(t_{j+1}-\xi)^{\alpha}}d\xi\right| + \frac{\alpha M_3\omega(0)}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\frac{\xi(\xi-t_1)(\xi-t_2)}{(t_{j+1}-\xi)^{\alpha+1}}d\xi\right| \\
\leq & \frac{2\sqrt{3}M_3m_1\tau^3}{54\Gamma(1-\alpha)}\int_0^{t_2}(t_{j+1}-\xi)^{-\alpha}d\xi + \frac{\sqrt{3}\alpha M_3\omega(0)}{27\Gamma(1-\alpha)}\int_0^{t_2}(t_{j+1}-\xi)^{-\alpha-1}d\xi \\
= & \frac{\sqrt{3}M_3m_1\tau^{4-\alpha}}{27\Gamma(2-\alpha)} + \frac{\sqrt{3}M_3\omega(0)\tau^{3-\alpha}}{27\Gamma(1-\alpha)}\big((j-1)^{-\alpha}-(j+1)^{-\alpha}\big) \\
\leq & \frac{\sqrt{3}M_3m_1\tau^{4-\alpha}}{27\Gamma(2-\alpha)} + \frac{\sqrt{3}(1-3^{-\alpha})M_3\omega(0)\tau^{3-\alpha}}{27\Gamma(1-\alpha)}.
\end{aligned}
$$

$M_3 = \underset{0\leq t\leq t_{j+1}}{max}|\zeta'''(t)|$ and $m_1 = \underset{0\leq t\leq t_{j+1}}{max}|\omega'(t)|$.

$$
\begin{aligned}
R_2 &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\zeta(\xi) - \Pi_{2,s}\zeta(\xi))'\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\zeta(\xi) - \Pi_{2,s}\zeta(\xi))\omega'(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \\
&- \frac{\alpha}{\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\zeta(\xi) - \Pi_{2,s}\zeta(\xi))\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha+1}} d\xi \\
&= \frac{1}{6\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{\zeta'''(\eta_s)(\xi - t_{s-1})(\xi - t_s)(\xi - t_{s+1})\omega'(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \\
&- \frac{\alpha}{6\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{\zeta'''(\eta_s)(\xi - t_{s-1})(\xi - t_s)(\xi - t_{s+1})\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha+1}} d\xi.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|R_2| &\leq \frac{1}{6\Gamma(1-\alpha)} \left| \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{\zeta'''(\eta_s)(\xi - t_{s-1})(\xi - t_s)(\xi - t_{s+1})\omega'(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha}} d\xi \right| \\
&+ \frac{\alpha}{6\Gamma(1-\alpha)} \left| \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{\zeta'''(\eta_s)(\xi - t_{s-1})(\xi - t_s)(\xi - t_{s+1})\omega(t_{j+1}-\xi)}{(t_{j+1}-\xi)^{\alpha+1}} d\xi \right| \\
&\leq \frac{M_3 m_1}{6\Gamma(1-\alpha)} \left| \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\xi - t_{s-1})(\xi - t_s)(\xi - t_{s+1})}{(t_{j+1}-\xi)^{\alpha}} d\xi \right| \\
&+ \frac{\alpha M_3 \omega(0)}{6\Gamma(1-\alpha)} \left| \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\xi - t_{s-1})(\xi - t_s)(\xi - t_{s+1})}{(t_{j+1}-\xi)^{\alpha+1}} d\xi \right| \\
&\leq \frac{2\sqrt{3} M_3 m_1 \tau^3}{54\Gamma(1-\alpha)} \sum_{s=2}^{j-1} \int_{t_s}^{t_{s+1}} (t_{j+1}-\xi)^{-\alpha} d\xi + \frac{M_3 m_1 \tau^2}{3\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1}} (t_{j+1}-\xi)^{1-\alpha} d\xi \\
&+ \frac{2\sqrt{3}\alpha M_3 \omega(0) \tau^3}{54\Gamma(1-\alpha)} \sum_{s=2}^{j-1} \int_{t_s}^{t_{s+1}} (t_{j+1}-\xi)^{-\alpha-1} d\xi + \frac{\alpha M_3 \omega(0) \tau^2}{3\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1}} (t_{j+1}-\xi)^{-\alpha} d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\sqrt{3}M_3 m_1 \tau^3}{54\Gamma(1-\alpha)} \int_{t_2}^{t_j} (t_{j+1}-\xi)^{-\alpha} d\xi + \frac{M_3 m_1 \tau^2}{3\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1}} (t_{j+1}-\xi)^{1-\alpha} d\xi \\
&+ \frac{2\sqrt{3}\alpha M_3 \omega(0)\tau^3}{54\Gamma(1-\alpha)} \int_{t_2}^{t_j} (t_{j+1}-\xi)^{-\alpha-1} d\xi + \frac{\alpha M_3 \omega(0)\tau^2}{3\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1}} (t_{j+1}-\xi)^{-\alpha} d\xi \\
&= \frac{\sqrt{3}M_3 m_1 \tau^3}{27\Gamma(2-\alpha)} \big(t_{j-1}^{1-\alpha} - \tau^{1-\alpha}\big) + \frac{M_3 m_1 \tau^{4-\alpha}}{3(2-\alpha)\Gamma(1-\alpha)} \\
&+ \frac{\sqrt{3}M_3 \omega(0)\tau^3}{27\Gamma(1-\alpha)} (\tau^{-\alpha} - t_{j-1}^{-\alpha}) + \frac{\alpha M_3 \omega(0)\tau^{3-\alpha}}{3\Gamma(2-\alpha)} \\
&\leq \frac{\sqrt{3}M_3 m_1 \tau^3}{27\Gamma(2-\alpha)} t_{j-1}^{1-\alpha} + \frac{M_3 m_1 \tau^{4-\alpha}}{3(2-\alpha)\Gamma(1-\alpha)} + \frac{\sqrt{3}M_3 \omega(0)\tau^{3-\alpha}}{27\Gamma(1-\alpha)} + \frac{\alpha M_3 \omega(0)\tau^{3-\alpha}}{3\Gamma(2-\alpha)} = \mathcal{O}(\tau^{3-\alpha}).
\end{aligned}
$$

Finally, we calculate bound for $|R_3|$.

$$
\begin{aligned}
|R_3| \leq & \left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{(\Pi_{2,1}\zeta(\xi))'(\omega(t_{j+1}-\xi) - \Psi(\xi))}{(t_{j+1}-\xi)^\alpha} d\xi \right| \\
&+ \left| \frac{1}{\Gamma(1-\alpha)} \sum_{s=2}^{j} \int_{t_s}^{t_{s+1}} \frac{(\Pi_{2,s}\zeta(\xi))'(\omega(t_{j+1}-\xi) - \Psi(\xi))}{(t_{j+1}-\xi)^\alpha} d\xi \right|.
\end{aligned}
$$

The two expressions for $|R_3|$ can be named as A and B, respectively.

$$
\begin{aligned}
A &= \left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{(\Pi_{2,1}\zeta(\xi))'(\omega(t_{j+1}-\xi) - \Psi(\xi))}{(t_{j+1}-\xi)^\alpha} d\xi \right| \\
&\leq \frac{(M_1 + 0.5M_2\tau)}{\Gamma(1-\alpha)} \int_0^{t_2} \frac{|\omega(t_{j+1}-\xi) - \Psi(\xi)|}{(t_{j+1}-\xi)^\alpha} d\xi \\
&\leq \frac{(M_1 + 0.5M_2\tau) m_3 \tau^3}{48\Gamma(1-\alpha)} \int_0^{t_2} (t_{j+1}-\xi)^{-\alpha} \\
&= \frac{(M_1 + 0.5M_2\tau) m_3 \tau^3}{48\Gamma(2-\alpha)} (t_{j+1}^{1-\alpha} - t_{j-1}^{1-\alpha}) \\
&\leq \frac{(M_1 + 0.5M_2\tau) m_3 \tau^3}{48\Gamma(2-\alpha)} (t_{j+1}^{1-\alpha}) = \mathcal{O}(\tau^3).
\end{aligned}
$$

$$
M_1 = \underset{0\leq t\leq t_{j+1}}{max} |\zeta'(t)|, \quad M_2 = \underset{0\leq t\leq t_{j+1}}{max} |\zeta''(t)| \text{ and } m_3 = \underset{0\leq t\leq t_{j+1}}{max} |\omega'''(t)|.
$$

$$\begin{aligned}
B &= \left|\frac{1}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{(\Pi_{2,s}\zeta(\xi))'(\omega(t_{j+1}-\xi)-\Psi(\xi))}{(t_{j+1}-\xi)^{\alpha}}d\xi\right| \\
&\leq \frac{(M_1+0.5M_2\tau)}{\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}\frac{|\omega(t_{j+1}-\xi)-\Psi(\xi)|}{(t_{j+1}-\xi)^{\alpha}}d\xi \\
&\leq \frac{(M_1+0.5M_2\tau)m_3\tau^3}{48\Gamma(1-\alpha)}\sum_{s=2}^{j}\int_{t_s}^{t_{s+1}}(t_{j+1}-\xi)^{-\alpha}d\xi \\
&= \frac{(M_1+0.5M_2\tau)m_3\tau^3}{48\Gamma(2-\alpha)}t_{j-1}^{1-\alpha} = \mathcal{O}(\tau^3).
\end{aligned}$$

Combining the bounds of R_1, R_2 and R_3, we complete the proof of the theorem. □

Theorem 3.3.2. For any weighting function $\omega(t) \in C[0,T]$, if the coefficients $\bar{d}_s$ defined in equation (3.12) satisfies the condition of Lemma (3.3.2), then the numerical scheme corresponding to the difference operator (3.12) is unconditionally stable and its solution satisfies a priori estimate (3.22) given in Theorem (3.3.1).

Example of a C-class weight function: Consider an example of the multi-term fractional diffusion equation having a weighting function $\omega(t) \in C[0,T]$ of the following multivariate form: (see [29]).

$$\omega(t) = \omega_0 + \omega_1 t^{\alpha-\alpha_1} + \omega_2 t^{\alpha-\alpha_2} + \ldots + \omega_m t^{\alpha-\alpha_m},$$

where

$$0 < \alpha_m < \alpha_{m-1} < \alpha_{m-2} \ldots < \alpha_1 < \alpha < 1,$$

and $\omega_0, \omega_1, \ldots, \omega_m$ and m are positive constants. Then the coefficients $\bar{a}_s$ and $\bar{b}_s$ defined in Formula 3 can be evaluated as

$$\begin{aligned}
\bar{a}_s &= \omega_0 \frac{\tau^{1-\alpha}}{1-\alpha} a_s^{\alpha} + \sum_{i=1}^{m} \frac{\omega_i \tau^{1-\alpha_i}}{1-\alpha_i} a_s^{\alpha_i}, \\
a_s^{\alpha_i} &= (l+1)^{1-\alpha_i} - l^{1-\alpha_i}, \ i = 1, 2, \ldots, m,
\end{aligned}$$

and

$$\begin{aligned}
\bar{b}_s &= \omega_0 \frac{\tau^{1-\alpha}}{1-\alpha} b_s^{\alpha} + \sum_{i=1}^{m} \frac{\omega_i \tau^{1-\alpha_i}}{1-\alpha_i} b_s^{\alpha_i}, \\
b_s^{\alpha_i} &= \frac{1}{2-\alpha_i}\left[(l+1)^{2-\alpha_i} - l^{2-\alpha_i}\right] - \frac{1}{2}\left[(l+1)^{1-\alpha_i} + l^{1-\alpha_i}\right], \ i = 1, 2, \ldots, m, \quad l > 0,
\end{aligned}$$

$a_s^{\alpha} = a_s$ and $b_s^{\alpha} = b_s$ defined in (3.6)-(3.7).

Remark 3.2. The stability and convergence of the above multi-term fractional diffusion equation can be investigated similar to [29]. This is a special case of the generalized Caputo derivative used in the present paper.

3.4 Results and discussion

In order to verify the theoretical findings, an algorithm is developed in this section for the numerical implementation of the designed schemes. Three different test functions are investigated using the developed algorithm. In the first test problem, all the three proposed formulations are executed and the obtained computational results show the efficiency of the difference schemes. The numerical experiments are conducted on two more examples to test the accuracy of the designed L2 scheme. The convergence order is calculated in $||\cdot||_0$ and $||\cdot||_{H(\chi_{h\tau})}$ norms, respectively, where $||\varpi||_{H(\chi_{h\tau})} = \max\limits_{(x_i,t_j)\in\chi_{h\tau}} |\varpi|$ and $||\cdot||_0$ is the L_2 norm. The errors $(w = \zeta - \varpi)$ and order of convergence are listed in Tables below. The temporal and spatial convergence order are computed by CO= $\log_{\frac{\tau_1}{\tau_2}} \frac{||w_{h\tau_1}||}{||w_{h\tau_2}||}$ and CO= $\log_{\frac{h_1}{h_2}} \frac{||w_{\tau h_1}||}{||w_{\tau h_2}||}$, respectively.

```
Algorithm 1: To evaluate the numerical solution of the model (3.1)-(3.2)
Input: The number $\alpha \in (0,1)$, weight function $\omega(t)$; final time level T; the
        functions $m(x,t)$, $p(x,t)$ and the source term $\phi(x,t)$.
Output: Numerical solution: $\varpi$.
for Numerical solution of (3.1)-(3.2), do
    • Step 1.1 Start with the uniform mesh $\chi_\tau$ and $\chi_h$.
    • Step 1.2 Evaluate the numerical solution $\varpi$ for the time level $n = 1$ by
      applying the L1 scheme [24] for GTFDE defined in (3.1)-(3.2).
    • Step 1.3 Using the solution obtained from step 2, apply one of the difference
      analog of the generalized FD designed in section 3.2, along with the spatial
      discretization (3.13) to obtain the numerical solution $\varpi$ for the remaining
      time levels: $n = 2, 3, \ldots, N$.
    • Step 1.4 Convert the system into the matrix form $AX = B$ and get the
      numerical solution $\varpi$.
end
```

Example 3.4.1. We construct an example with $\omega(t) = 1/(1+t)$, $m(x,t) = 2 - cos(xt)$, $p(x,t) = 1 - sin(xt)$, $T = 1$ and $\zeta(x,t) = t^\beta sin(\pi x)$ is the exact solution of equations (3.1)-(3.2). The numerical experiments are performed using the discrete formulae (3.8), (3.10) and (3.12), respectively and their outcomes are examined which results in the following observations.

- In Table 3.1, the results of all the formulae are presented for a fixed $M = 3000$, $\alpha = 0.5$ and $\beta = 3$.

- It can be clearly observed that the numerical application of (3.8) (Formula 1) results in second order of convergence whereas both the formulae 2 and 3 defined in equations (3.10) and (3.12), respectively gives $\mathcal{O}(\tau^{3-\alpha})$ order of convergence.

- The results of the developed L2 scheme are equivalent to the results obtained from the exact formulation (3.12) in Table 3.1 proving the numerical efficiency of the L2 scheme.

TABLE 3.1: Errors and CO for $M = 3000$, $\beta = 3$ and $\alpha = 0.5$ for Ex. 3.4.1

Formula	τ	$\max\limits_{0\leq n\leq M} \|\|w^n\|\|_0$	CO	$\|\|w\|\|_{Q(\chi_{h\tau})}$	CO
(3.8)	1/10	3.8100e-04	-	5.4230e-04	-
	1/20	8.3246e-05	2.1944	1.1849e-04	2.1944
	1/40	1.8622e-05	2.1603	2.6505e-05	2.1604
	1/80	4.2913e-06	2.1176	6.1066e-06	2.1178
(3.10)	1/10	1.4677e-04	-	2.0851e-04	-
	1/20	2.6637e-05	2.4620	3.7792e-05	2.4640
	1/40	4.7812e-06	2.4780	6.7812e-06	2.4785
	1/80	8.6540e-07	2.4659	1.2305e-06	2.4623
(3.12)	1/10	1.4603e-04	-	2.0728e-04	-
	1/20	2.6497e-05	2.4624	3.7590e-05	2.4631
	1/40	4.7661e-06	2.4749	6.7592e-06	2.4754
	1/80	8.6307e-07	2.4653	1.2272e-06	2.4615

Example 3.4.2. This example is from [47], where $\zeta(x,t) = (t^2 + t^{3+\alpha} + 1)\sin(\pi x)$ is the exact solution of the mathematical model (3.1)-(3.2) with $\omega(t) = e^{-bt}$, $b \geq 0$, and the coefficients $m(x,t) = 1 - sin(xt)$, $p(x,t) = 2 - \cos(xt)$ and T=1.

The outcomes of testing this example can be discussed through the following bullet points.

- In Table 3.2, the maximum errors and L_2 errors and their respective order of convergence are calculated for Ex. 3.4.2 by increasing step sizes in time and space keeping $h^2 = \tau^{3-\alpha}$.
- From Table 3.3, it is observed that if we fix the time step size $\tau = 1/1000$ and increase the spatial step size, the maximum error decreases and we get a CO of two in space.
- If the spatial step size is fixed to attain CO of $\mathcal{O}(\tau^{3-\alpha})$, then the computing cost will be very high. Thus, we use $h^2 = \tau^{3-\alpha}$ and the formula CO= $\log_{\frac{\tau_1}{\tau_2}} \frac{||w_{h_1\tau_1}||}{||w_{h_2\tau_2}||}$ to obtain the desired order of convergence in the temporal direction.

TABLE 3.2: Errors and convergence order with $h^2 = \tau^{3-\alpha}$ for Ex. 3.4.2.

α	τ	h	$\underset{0\leq n\leq N}{max}\|\|w^n\|\|_0$	CO	$\|\|w\|\|_{H(\chi_{h\tau})}$	CO
0.1	1/10	1/29	1.7000e-03	-	2.4000e-03	-
	1/20	1/78	2.3440e-04	2.8545	3.3048e-04	2.8536
	1/40	1/211	3.2045e-05	2.8708	4.5182e-05	2.8707
	1/80	1/575	4.3720e-06	2.8919	6.0872e-06	2.8919
0.5	1/10	1/18	4.6000e-03	-	6.4000e-03	-
	1/20	1/43	8.0198e-04	2.5087	1.1000e-03	2.5043
	1/40	1/101	1.4538e-04	2.4638	2.0496e-04	2.4633
	1/80	1/240	2.5772e-05	2.4959	3.6334e-05	2.4959
0.9	1/10	1/12	1.1800e-02	-	1.6700e-02	-
	1/20	1/24	2.9000e-03	2.0116	4.1000e-03	2.0117
	1/40	1/49	7.0229e-04	2.0626	9.8986e-04	2.0605
	1/80	1/100	1.6797e-04	2.0639	2.3685e-04	2.0633

Table 3.3: Errors and Spatial CO for Ex. 3.4.2 with $\tau = 1/1000$.

α	h	$\max\limits_{0\leq n\leq N} \lvert\lvert w^n \rvert\rvert_0$	CO	$\lvert\lvert w \rvert\rvert_{H(\chi_{h\tau})}$	CO
0.3	1/10	1.4100e-02	-	1.9800e-02	-
	1/20	3.5000e-02	2.0026	4.9000e-03	1.9992
	1/40	8.7747e-04	2.0007	1.2000e-03	1.9979
	1/80	2.1934e-04	2.0002	3.0922e-04	2.0002
0.7	1/10	1.3700e-02	-	1.9300e-02	-
	1/20	3.4000e-03	2.0025	4.8000e-03	1.9987
	1/40	8.5681e-04	2.0006	1.2000e-03	1.9981
	1/80	2.1420e-04	2.0000	3.0192e-04	2.0000
0.9	1/10	1.3500e-02	-	1.8900e-02	-
	1/20	3.4000e-03	2.0023	4.7000e-03	1.9983
	1/40	8.4245e-04	2.0004	1.2000-03	1.9980
	1/80	2.1072e-04	1.9993	2.9697e-04	1.9993

Example 3.4.3. This example is from [24, 69], where $\zeta(x,t) = \left(\frac{6-e^{-bt}(6bt+3b^2t^2+b^3t^3+6)}{b^4} + 1\right)$ $\sin(\pi x)$ is the exact solution of problem (3.1)-(3.2) with $\omega(t) = e^{-bt}$, $b \geq 0$, and the coefficients $m(x,t) = 1 - sin(xt)$, $p(x,t) = 2 - \cos(xt)$ and T=1.

The outcomes of this test problem are similar to the Example 3.4.2. It can be noted from Table 3.4 that the temporal convergence order for this example has significantly increased in this paper as compared to the work in [24].

b	α	τ	$\max\limits_{0\leq n\leq N} \|\|w^n\|\|_0$	CO	$\|\|w\|\|_{H(\chi_{h\tau})}$	CO
3	0.2	1/10	7.7565e-04	-	1.1000e-03	-
		1/20	1.1679e-04	2.7315	1.6465e-04	2.7297
		1/40	1.7121e-05	2.7702	2.4137e-05	2.7701
		1/80	2.4566e-06	2.8010	3.4634e-06	2.8010
2	0.5	1/10	1.6000e-03	-	2.3000e-03	-
		1/20	2.8845e-04	2.5141	4.0656e-04	2.5100
		1/40	5.2276e-05	2.4641	7.3709e-05	2.4635
		1/80	9.2573e-06	2.4975	1.3053e-05	2.4975
1	0.9	1/10	4.0000e-03	-	5.6000e-03	-
		1/20	9.9011e-04	2.0071	1.4000e-03	2.0048
		1/40	2.3710e-04	2.0621	3.3423e-04	2.0603
		1/80	5.6872e-05	2.0597	8.0174e-05	2.0596

TABLE 3.4: Temporal errors and convergence order for $h^2 = \tau^{3-\alpha}$ for Ex. 3.4.3.

3.5 Conclusion

The Caputo fractional derivative with a non-unity weighting function has an immense application in the field of science and technology. Hence, three different discrete analogs of such generalized Caputo have been presented in this chapter viz. (3.8), (3.10) and (3.12). The major highlight is the L2 scheme (3.14)-(3.15) having a CO of $\mathcal{O}(\tau^{3-\alpha})$ in the temporal direction. The stability and convergence of the proposed L2 scheme are proved theoretically in Lemma (3.3.2)-(3.3.3) and Theorem (3.3.1). All the three formulations are implemented and their results are compared on one test problem (See Table 3.1). The numerical accuracy of the L2 scheme is proved by implementing it on two more examples (Tables (3.2)-(3.4).

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