On the Solution of the Nonlinear Fractional Diffusion-Wave Equation with Absorption: a Homotopy Approach

Vivek Mishra[a], Kumar Vishal[b], Subir Das[a], and Seng Huat Ong[c]

[a] Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi-221005, India
[b] Department of Mathematics and Basic Science, NIIT University, Neemrana, Rajasthan-301705, India
[c] Institute of Mathematical Sciences, University of Malaya, Kuala Lumpur-50603, Malaysia

Reprint requests to S. D.; E-mail: subir_das08@hotmail.com

Z. Naturforsch. **69a**, 135 – 144 (2014) / DOI: 10.5560/ZNA.2013-0084
Received April 4, 2013 / revised November 8, 2013 / published online January 22, 2014

In this article, the homotopy analysis method is used to obtain approximate analytic solutions of the time-fractional diffusion-wave equation with given initial conditions. A special effort has been given to show the effect of reaction term with long term correlation to the diffusion-wave solutions for various values of anomalous exponent to constitute a good mathematical model useful for various engineering and scientific systems. Effects of parameters on the solution profile are calculated numerically and presented through graphs for different particular cases. Sub-diffusion and super-diffusion phenomena for birth and death processes are also shown through figures.

Key words: Fractional Diffusion-Wave Equation; Caputo Derivative; Homotopy Analysis Method.
Mathematics Subject Classification 2010: 26A33, 34A08, 60G22, 65Gxx, 35R11

1. Introduction

In this article, a sincere attempt has been taken to solve the nonlinear fractional diffusion equation with reaction term as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(u^n(x,t)\frac{\partial u(x,t)}{\partial x}\right) - \int_0^t a(t-\xi)u(x,\xi)\,\mathrm{d}\xi, \tag{1}$$

where $u(x,t)$ is a field variable and assumed to vanish for $t < 0$. The absorbent term related to the reaction diffusion process is described as $a(t) = a_0\frac{t^{-\beta}}{\Gamma(1-\beta)}, 0 < \beta < 1$, which possesses a long time correlation with the exponent β, which may be determined by dynamical mechanism of the physical process. The equation will represent a death process for the sink term as $a_0 > 0$ and a birth process for a source term as $a_0 < 0$ [1]. Equation (1) is said to be a fractional diffusion equation for $0 < \alpha < 1$ and a fractional wave equation for $1 < \alpha < 2$. The difference between these two cases can be seen in the formula for the Laplace transform of the Caputo fractional derivative of order α $(m-1 < \alpha \le m, m \in N)$ as

$$L\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right] = s^\alpha L[u(x,t)] - \sum_{k=0}^{m-1} s^{\alpha-k-1}\left.\frac{\partial^k u(x,t)}{\partial t^k}\right|_{t=0+}. \tag{2}$$

For the case $0 < \alpha \le 1$, we have the initial condition

$$u(x,0) = x^k. \tag{3}$$

For the case $1 < \alpha \le 2$, we have the initial conditions

$$u(x,0) = x^k, \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0. \tag{4}$$

Due to the presence of the reaction term, (1) may be useful to investigate several situations by choosing an appropriate $a(t)$, for example, catalytic processes in regular, heterogeneous, or disordered systems [2 – 4]. Another example is an irreversible first-order reaction of transported substance so that the rate of removal is $\alpha\rho$ [5]. The above type of anomalous diffusion is a ubiquitous phenomenon in nature and appears in different branches of science and engineering. Equation (1) for $\alpha = 1$ and without absorption represents a model of plasma diffusion for $n = -1/2$, thermal limit approximation of Carlemans model of the Boltzmann equation for $n = -1$, diffusive in higher polymer systems for $n = -2$, isothermal percolations of perfect gas through a micro-porous medium for $n = 1$, and process of melting and evaporation of metals for $n = 2$ (Wazwaz [6]). Equation (1) for $n = 1$, $a_0 = 0$, i.e. the nonlinear time-fractional diffusion equation in absence of absorption, has the exact solution $u(x,t) = x + t$. Similarly for $n = 2, a_0 = 0$ the exact solution is $u(x,t) = \frac{x}{\sqrt{1-4t}}$ [7].

For $\alpha = 1$, (1) represents a Fickian or normal diffusion process. When $0 < \alpha < 1$, $1 < \alpha < 2$, (1) describes a diffusion process which is temporally non-Fickian but specially Gaussian. For $\alpha = 2$, equation represents a wave equation, which is also known as Ballistic diffusion.

Einstein's theory of Brownian motion reveals that the mean square displacement of a particle moving randomly is proportional to time. But after the advancement of fractional calculus, it is seen that the mean square displacement for an anomalous diffusion equation having time fractional derivative grows slowly with time, i.e. $\langle X^2(t)\rangle \sim t^\alpha$, where $0 < \alpha < 1$ is the anomalous diffusion exponent. When $n = 0$, (1) reduces to the linear fractional-order diffusion equation as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - \int_0^t a(t-\xi)u(x,\xi)\,\mathrm{d}\xi \qquad (5)$$

for $0 < \alpha < 1$. In this case after a lengthy mathematical calculation it is seen that $M_{2k}(t) \sim t^{\alpha k} E_{\alpha-\beta+1,k\alpha+1}(-rat^{\alpha-\beta+1})$, where $r^n = \begin{bmatrix} k \\ n \end{bmatrix}$ and $M_{2k+1}(t) = 0$. Thus the mean square displacement $\langle X^2(t)\rangle \sim t^\alpha E_{\alpha-\beta+1,\alpha+1}(-rt^{\alpha-\beta+1})$, the Mittag–Leffler function, is defined by $E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha+\beta)}$. Replacing the integer order with the fractional order, the time derivative changes the fundamental concept of time and with it the concept of evolution in the foundations of physics. The fractional order α can be identified and has a physical meaning related to the statistics of waiting times in the Montroll–Weiss theory. The relation was established in two steps. First, it was shown in [8] that Montroll–Weiss continuous time random walks with a Mittag–Leffler waiting time density are rigorously equivalent to a fractional master equation. Then, in [9] this underlying random walk model was connected to the fractional time diffusion equation in the usual asymptotic sense of long times and large distances.

A simple model for simulating diffusive phenomena is the random walk approach. A random walker can be regarded as a diffusing particle, performing a random motion, similar to the Brownian motion, on an appropriate discrete lattice in discrete time steps. However, diffusion then is a stochastic process of many moving particles. So we have to simulate not only one diffusing particle, but a large number of particles. Both, the diffusive process and its simulation, can be characterized by the time development of their mean square displacement. It is already mentioned that the anomalous diffusion is characterized by a diffusion constant and the mean square displacement of diffusing species in the form $\langle X^2(t)\rangle \sim t^\alpha$, and the phenomena of anomalous diffusion is usually divided into anomalous subdiffusion for $0 < \alpha < 1$ and anomalous super-diffusion for $1 < \alpha < 2$. The strictly time fractional diffusion of order α, $0 < \alpha < 1$, generates a class of symmetric densities whose moments of order $2m$ are proportional to the $m\alpha$ power of time [10]. We thus obtain a class of non-Markovian stochastic processes (they possess a memory!) which exhibit a variance consistent with slow anomalous diffusion. In 1999, applying a fractional order Fokker–Plank equation approach, Metzler et al. [11] have shown that anomalous diffusion is based upon Boltzman statistics. Many researchers have used fractional equations to describe Levy flights or diverging diffusion.

The integer-order model can be viewed as a special case from the more general fractional-order model since it can be retrieved by putting all fractional orders of the derivatives equal to unity. In other words, the ultimate behaviour of the fractional-order system response must converge to the response of the integer-order version of the model. This shows that the fractional calculus is the extension of classical mathematics where derivatives are taken as rational, irrational, and complex orders. In the last two decades, fractional differential equations have been widely used by the researchers not only in science and engineering but also in economics and finance. It is also a powerful tool in modelling multi-scale problems, characterized by wide time or length scale. The attribute of fractional-order systems for which they have gained popularity in the investigation of dynamical systems is that they allow greater flexibility in the model. An integer-order differential operator is a local operator. Whereas the fractional-order differential operator is non-local in the sense that it takes into account the fact that the future state not only depends upon the present state but also upon all of the history of its previous states. An important characteristic of these evolution equations is that they generate the fractional Brownian motion which is a generalization of Brownian motion. For physical systems, one should have to keep in mind two things for an application of fractional order in the system for making a decisive step for the penetration of mathematics of fractional calculus into a body of natural sciences. Firstly, to analyze the importance and physical influence of the memory effects on time or space or both. Secondly, to give proper interpretation of general meaning of the non-integer operator. The main advantage of the fractional calculus is that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

Fractional derivatives and integrals are useful to explore the characteristic features of anomalous diffusion, transport, and fractal walks through setting up of fractional kinetic equations, master equations etc. Fractional kinetic equations have proved particularly useful in the context of anomalous sub-diffusion (Metzler and Klafter [12]). The fractional diffusion equation, which demonstrates the prevalence of anomalous sub-diffusion, has led to an intensive effort in recent years to find the solution accurately in straight forward manner (Langlands and Henry [13]). The fractional diffusion equation is valuable for describing reactions in the dispersive transport media [14]. Anomalous diffusion processes occur in many physical systems for various reasons including disorder in terms of energy or space or both [15, 16]. Fractional reaction-diffusion equations or continuous time random walk models are also introduced for the description of nonlinear reactions, propagating fronts, and two species reactions in sub-diffusive transport media (Henry and Wearne [17]). In 2007, Chen et al. [18] proposed an implicit difference approximation scheme (ISAS) for solving fractional diffusion equations, where the stability and convergence of the method had been analyzed by the Fourier method. Schot et al. [19] have given an approximate solution of the diffusion equation in terms of Fox H-function. Zahran [20] has offered a closed form solution in Fox H-function of the generalized fractional reaction-diffusion equation subjected to an external linear force field, one that is used to describe the transport processes in disorder systems. It is to be noted that some works on fractional diffusion equations have already been done by Angulo et al. [21], Pezat and Zabczyk [22], Schneider and Wyss [23], Yu and Zhang [24], Mainardi [25], Mainardi et al. [26], Anh and Leonenko [27]. Recently, Das [28] has solved the fractional-order nonlinear reaction diffusion equation using the mathematical tool variational iteration method and has shown that sub-diffusions occur even for cubic-order nonlinearity and also cubic order of x in the initial condition. The theory of fractional time evolutions describes novel three parameters susceptibility functions which contain only a single stretching exponent. It shows two widespread characteristics of relaxation spectra in the glass forming materials [29]. Hilfer has shown that a power-law tail in the waiting time density is not sufficient to guarantee the emergence of the propagator of fractional diffusion in the continuum limit [30]. Another work of Hilfer underlines that fractional relaxation equations provide a promising mathematical framework for slow and glassy dynamics. In particular, fractional susceptibilities seem to reproduce not only broadening or stretching of the relaxation peaks but also the high-frequency wing and shallow minima observed in the experiment [31].

The homotopy analysis method (HAM), proposed by Liao [32], is based on homotopy, a fundamental concept in topology and differential geometry. It is an analytical approach to get the series solutions of linear and nonlinear differential equations. The difference with the other perturbation methods is that this method is independent of small/large physical parameters. Another important advantage of this method as compared to other existing perturbation and non-perturbation methods lies in the flexibility to choose a proper base function to get better approximate solutions of the problems. This method offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy and requires large computer memory and time. This computational method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time. This method has been successfully applied by many researchers for solving linear and nonlinear partial differential equations [33 – 37].

Reaction-diffusion appears during the propagation of flames and migration of biological species. Tumor growths are the examples of such phenomena. Therefore, the authors have made an effort to see the nature of these types of equations with memory effect due to the presence of fractional-order time derivatives after solving the equation using HAM technique. The salient feature of the article is the graphical presentations and numerical discussions of the damping behaviours of the field variable $u(x,t)$ in order to obtain sub-diffusion of the time fractional nonlinear equations due to the presence of various parameters of physical interest.

2. Basics of Fractional Calculus

In this section, some definitions and properties of the fractional calculus are provided. There exists several definitions of fractional differentiation. In mathematical treatises on fractional differential equations, the Riemann–Liouville approach to the notion of the fractional derivative of order $(m-1<\alpha\le m, m\in\mathbb{N})$ is normally used, the Caputo fractional derivative often appears in applications [38].

Definition 1. A real function $f(t), t>0$, is said to be in the space $C_\mu, \mu\in\mathbb{R}$, if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^m, if and only if $f^{(m)}\in C_\mu, m\in\mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integral operator J_t^α of order $\alpha>0$, of a function $f\in C_\mu, \mu\ge -1$, is defined as

$$J_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}f(\xi)\,\mathrm{d}\xi\,, \quad \alpha>0,\ t>0, \quad J_t^0 f(t)=f(t)\,, \tag{6}$$

where $\Gamma(\alpha)$ is the well-known gamma function. Some of the properties of the operator $J_t^\alpha f(t)$, which are needed here, are as follows:
For $f\in C_\mu$, $\mu\ge -1$, $\alpha,\beta\ge 0$, and $\gamma\ge -1$,
1. $J_t^\alpha J_t^\beta f(t)=J_t^{\alpha+\beta}f(t)$,
2. $J_t^\alpha J_t^\beta f(t)=J_t^\beta J_t^\alpha f(t)$,
3. $J_t^\alpha t^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$.

Definition 3. The fractional derivative D_t^α of $f(t)$ in the Caputo sense [39, 40], is defined as

$$D_t^\alpha f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\xi)^{m-\alpha-1}f^m(\xi)\,\mathrm{d}\xi \tag{7}$$

for $m-1<\alpha<m$, $m\in\mathbb{N}$, $t>0$, $f\in C_{-1}^m$.

The followings are two basic properties of the Caputo fractional order derivative:
1. Let $f\in C_{-1}^m$, $m\in\mathbb{N}$, and $D_t^\alpha f$, $0<\alpha\le m$ is well defined and $D_t^\alpha f\in C_{-1}$.
2. Let $m-1\le\alpha\le m$, $m\in\mathbb{N}$, and $f\in C_\mu^m$, $\mu\ge -1$. Then

$$(J_t^\alpha D_t^\alpha)f(t)=f(t)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{t^k}{k!}\,. \tag{8}$$

3. Solution of the Problem by the Homotopy Analysis Method

Taking the Laplace transform on both sides of (1), we get

$$L\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right]=L\left[\frac{\partial}{\partial x}\left(u^n(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]-L\left[\int_0^t a(t-\xi)u(x,\xi)\,\mathrm{d}\xi\right]. \tag{9}$$

Now for $0<\alpha\le 1$, we have

$$L[u(x,t)]=\frac{1}{s}u(x,0)+\frac{1}{s^\alpha}L\left[\frac{\partial}{\partial x}\left(u^n(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]-\frac{1}{s^\alpha}L\left[\int_0^t a(t-\xi)u(x,\xi)\,\mathrm{d}\xi\right], \tag{10}$$

and for $1<\alpha\le 2$, we have

$$L[u(x,t)]=\frac{1}{s}u(x,0)+\frac{1}{s^2}\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}+\frac{1}{s^\alpha}L\left[\frac{\partial}{\partial x}\left(u^n(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]-\frac{1}{s^\alpha}L\left[\int_0^t a(t-\xi)u(x,\xi)\,\mathrm{d}\xi\right]. \tag{11}$$

In the view of (3) and (4), (10) and (11) reduce to

$$L[u(x,t)]=\frac{1}{s}x^k+\frac{1}{s^\alpha}L\left[\frac{\partial}{\partial x}\left(u^n(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]-\frac{1}{s^\alpha}L\left[\int_0^t a(t-\xi)u(x,\xi)\,\mathrm{d}\xi\right]. \tag{12}$$

Now taking the inverse Laplace transform, we have

$$u(x,t) = x^k + L^{-1}\left[\frac{1}{s^\alpha} L\left[\frac{\partial}{\partial x}\left(u^n(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]\right. \\ \left. - \frac{1}{s^\alpha} L\left[\int_0^t a(t-\xi)u(x,\xi)\,\mathrm{d}\xi\right]\right]. \qquad (13)$$

To solve (13) by HAM, we choose the linear auxiliary operator

$$\tilde{L}[\phi(x,t;q)] = \phi(x,t;q), \qquad (14)$$

where $\phi(x,t;q)$ is an unknown function. Furthermore, in the view of (10), we have defined the nonlinear operator as

$$N[\phi(x,t;q)] = \phi(x,t;q) - x^k \\ - L^{-1}\left[\frac{1}{s^\alpha} L\left[\frac{\partial}{\partial x}\left(\phi^n(x,t;q)\frac{\partial \phi(x,t;q)}{\partial x}\right)\right]\right] \\ + L^{-1}\left[\frac{1}{s^\alpha} L\left[\int_0^t a(t-\xi)\phi(x,\xi;q)\,\mathrm{d}\xi\right]\right]. \qquad (15)$$

Now we construct the zero-order deformation equation as

$$(1-q)\tilde{L}[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)]. \qquad (16)$$

It is obvious that for the embedding parameter $q = 0$ and $q = 1$, (16) becomes $\phi(x,t;0) = u_0(x,t)$, $\phi(x,t;1) = u(x,t)$, respectively.

Thus as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{k=1}^{\infty} q^k\, u_k(x,t), \qquad (17)$$

$$\text{where } u_k(x,t) = \frac{1}{k!}\left[\frac{\partial^k \phi(x,t;q)}{\partial q^k}\right]_{q=0}. \qquad (18)$$

If the auxiliary linear operator, the initial guess, and the convergence control parameters are properly chosen, the series (17) converges at $q = 1$. In this case one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t), \qquad (19)$$

which must be one of the solutions of the original equation, as proven by Liao [32].

Differentiating the zero-order equation (16) m-times with respect to q and then dividing it by $m!$ and finally setting $q = 0$, one has the so called mth-order deformation equation as

$$\tilde{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(u_{m-1}(x,t)) \qquad (20)$$

with the initial condition

$$u_m(x,t) = 0, \qquad (21)$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \qquad (22)$$

and $\hbar$ is a non-zero auxiliary parameter.

For $n = 1$,

$$R_m(u_{m-1}) = u_{m-1} - (1-\chi_m)x^k \\ + L^{-1}\left[\frac{1}{s^\alpha} L\left[\int_0^t a(t-\xi)u_{m-1}(x,\xi)\,\mathrm{d}\xi\right]\right] \\ - L^{-1}\left[\frac{1}{s^\alpha} L\left[\sum_{i=0}^{m-1}\frac{\partial u_i}{\partial x}\frac{\partial u_{m-1-i}}{\partial x}\right]\right. \\ \left. + L\left[\sum_{i=0}^{m-1} u_i \frac{\partial^2 u_{m-1-i}}{\partial x^2}\right]\right], \qquad (23)$$

and for $n = 2$,

$$R_m(u_{m-1}) = u_{m-1} - (1-\chi_m)x^k \qquad (24) \\ + L^{-1}\left[\frac{1}{s^\alpha} L\left[\int_0^t a(t-\xi)u_{m-1}(x,\xi)\,\mathrm{d}\xi\right]\right] \\ - L^{-1}\left[\frac{1}{s^\alpha} L\left[2\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i-1}\frac{\partial u_j}{\partial x}\frac{\partial u_{i-j}}{\partial x}\right)u_{m-1-i}\right]\right] \\ - L^{-1}\left[\frac{1}{s^\alpha} L\left[\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i-1} u_j u_{i-j}\right)\frac{\partial^2 u_{m-1-i}}{\partial x^2}\right]\right].$$

Applying the idea of HAM, we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar R_m(u_{m-1}(x,t)) + c, \qquad (25)$$

where the integration constant c is determined by the initial condition (21).

Case 1. (For $n = 1$)

Now from (25), the values $u_m(x,t)$ for $m = 1,2,3,\ldots$ can be obtained as

$$u_1(x,t) = \hbar\left(-k(2k-1)x^{2k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right),$$

$$u_2(x,t) = \hbar(1+\hbar)\left(-k(2k-1)x^{2k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right) + 3\hbar^2k\left(6k^3-13k^2+9k-2\right)x^{3k-4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (a_0\hbar)^2x^k\frac{t^{2\alpha-2\beta+2}}{\Gamma(2\alpha-2\beta+3)} - 3a_0\hbar^2k(2k-1)x^{2k-2}\frac{t^{2\alpha-\beta+1}}{\Gamma(2\alpha-\beta+2)},$$

and so on.

Case 2. (For $n = 2$)

From (25), the values $u_m(x,t)$ for $m = 1,2,3,\ldots$ can be obtained as

$$u_1(x,t) = \hbar\left(-k(3k-1)x^{3k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right),$$

$$u_2(x,t) = \hbar(1+\hbar)\left(-k(3k-1)x^{3k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right.$$

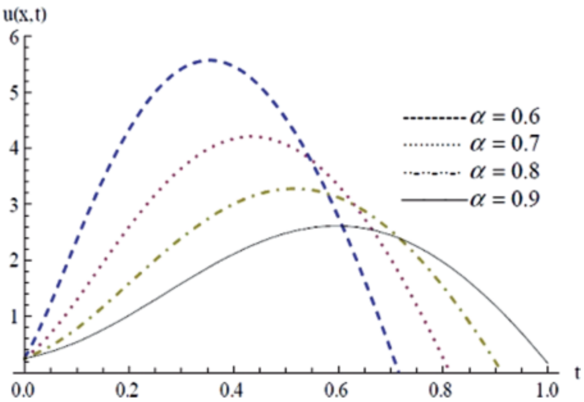

Fig. 1 (colour online). Plots of $u(x,t)$ versus t for $a_0 = 10$, $\beta = 0.5, k = 2, n = 1, x = 0.5$, and for different values of α.

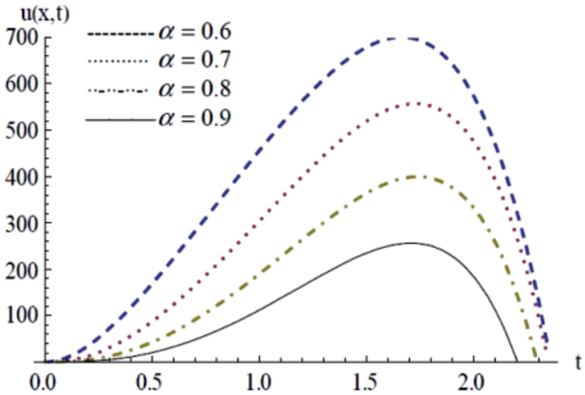

Fig. 2 (colour online). Plots of $u(x,t)$ versus t for $a_0 = 10$, $\beta = 0.5, k = 2, n = 2, x = 0.5$, and for different values of α.

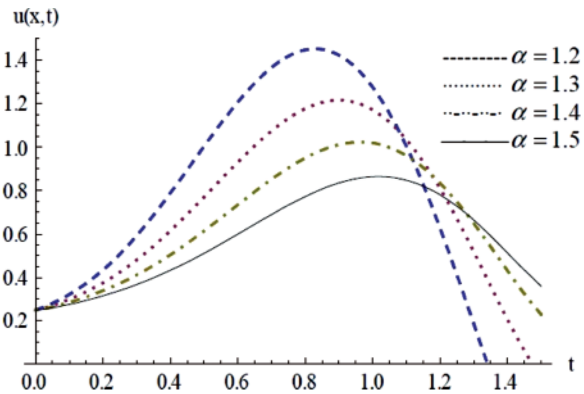

Fig. 3 (colour online). Plots of $u(x,t)$ versus t for $a_0 = 10, \beta = 0.5, k = 2, n = 1, x = 0.5$, and for different values of α.

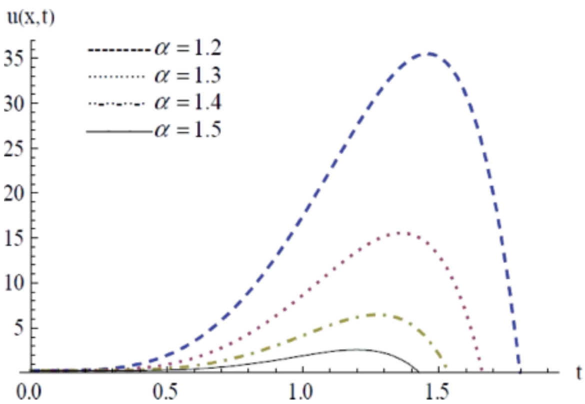

Fig. 4 (colour online). Plots of $u(x,t)$ versus t for $a_0 = 10, \beta = 0.5, k = 2, n = 2, x = 0.5$, and for different values of α.

$$+a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\Bigg)$$

$$+k\hbar^2\left(75k^3-100k^2+43k-6\right)x^{5k-4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+(a_0\hbar)^2x^k\frac{t^{2\alpha-2\beta+2}}{\Gamma(2\alpha-2\beta+3)}$$

$$-4a_0\hbar^2k(3k-1)x^{3k-2}\frac{t^{2\alpha-\beta+1}}{\Gamma(2\alpha-\beta+2)},$$

and so on.

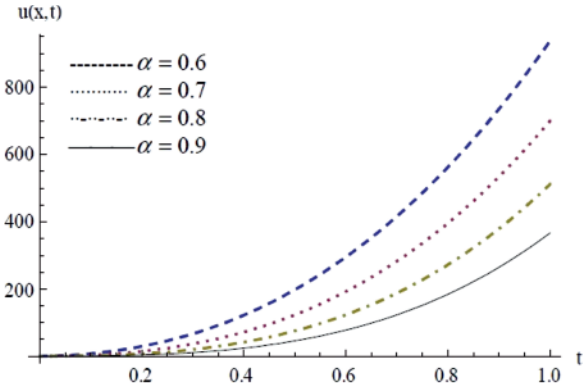

Fig. 5 (colour online). Plots of $u(x,t)$ versus t for $a_0=-10$, $\beta=0.5, k=2, n=1, x=0.5$, and for different values of α.

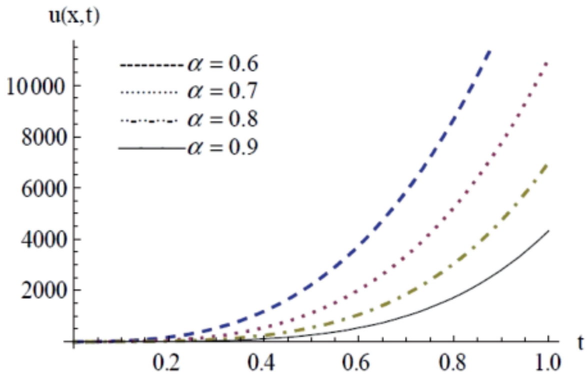

Fig. 6 (colour online). Plots of $u(x,t)$ versus t for $a_0=-10$, $\beta=0.5, k=2, n=2, x=0.5$, and for different values of α.

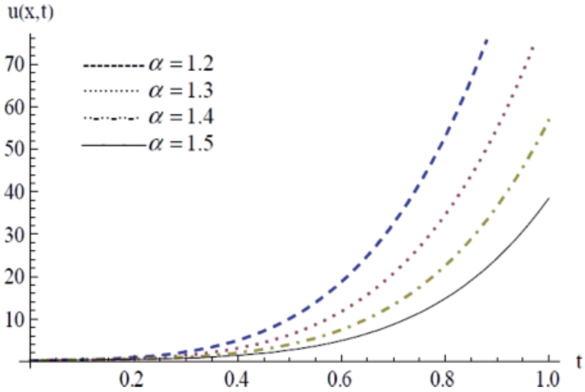

Fig. 7 (colour online). Plots of $u(x,t)$ versus t for $a_0=-10$, $\beta=0.5, k=2, n=1, x=0.5$, and for different values of α.

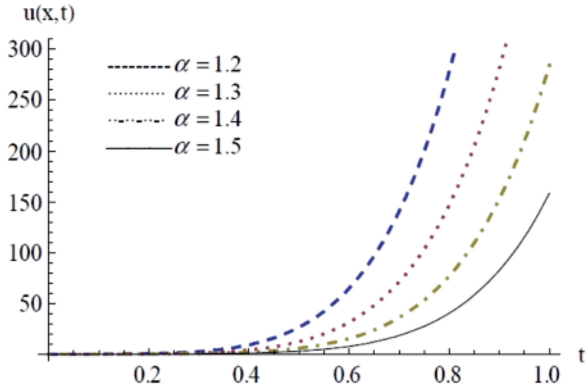

Fig. 8 (colour online). Plots of $u(x,t)$ versus t for $a_0=-10$, $\beta=0.5, k=2, n=2, x=0.5$, and for different values of α.

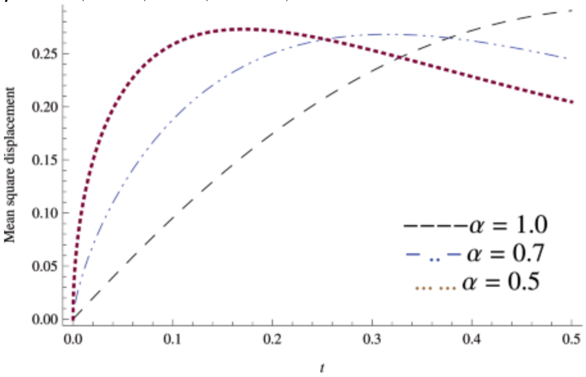

Fig. 9 (colour online). Plots of $\langle X^2(t)\rangle$ versus t for $a_0=5, \beta=0.5$, and for different values of α.

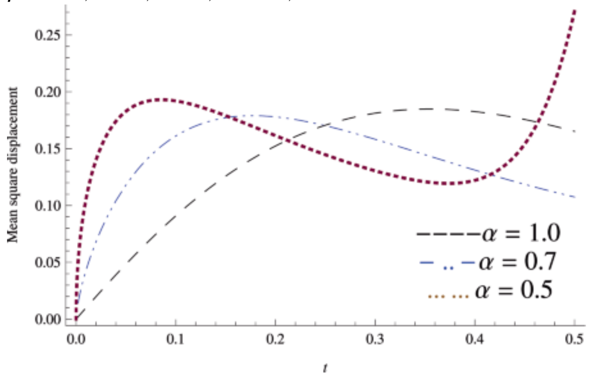

Fig. 10 (colour online). Plots of $\langle X^2(t)\rangle$ versus t for $a_0=10, \beta=0.5$, and for different values of α.

Finally, the mth-order approximation series solution is given as

$$\tilde{u}_m(x,t) = \sum_{k=0}^{m} u_k(x,t)\,. \tag{26}$$

4. Numerical Results and Discussion

In this section, numerical results of the field variable $u(x,t)$ for different values of fractional-order derivative α are calculated for the parameters' values $\beta = 0.5$, $k = 2$, $\hbar = -1$ at $x = 0.5$, and these results are depicted through Figures 1 and 2 at $\alpha = 0.6(0.1)0.9$ for $n = 1$ and $n = 2$, respectively, Figures 3 and 4 at $\alpha = 1.2(0.1)1.5$ for $n = 1$ and $n = 2$ when $a_0 = 10$, and also through Figures 5–8 with similar conditions when $a_0 = -10$. When the degree of nonlinearity is one, i.e. $n = 1$, it is seen from Figure 3 that even for $\alpha > 1$ due to the effect of sink term $(a_0 > 0)$ the sub-diffusions are observed with lesser overshoots than those for $\alpha < 1$ (Fig. 1). If the degree of nonlinearity increases, the similar types of results are found from Figures 2 and 4 with much greater overshoots of sub-diffusion. It is also observed from Figures 5–8 that even for $\alpha < 1$ the super-diffusions are found due to the effect of the source term $(a_0 < 0)$.

Figures 9–12 demonstrate the variations of the mean square displacement $(\langle X^2(t)\rangle)$ with time t for linear fractional-order system $(n = 0)$ in the presence of sink (death) and source (birth) terms. It is seen from the figures that there are behavioural changes of $\langle X^2(t)\rangle$ for death and birth processes. The figures clearly justify the occurrence of an anomalous behaviour of the linear diffusion equation in the fractional-order system.

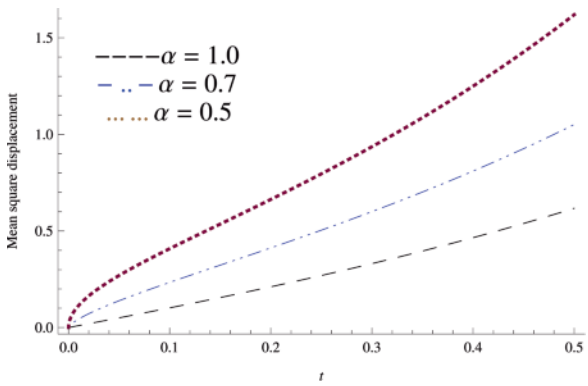

Fig. 11 (colour online). Plots of $\langle X^2(t)\rangle$ versus t for $a_0 = -2, \beta = 0.5$, and for different values of α.

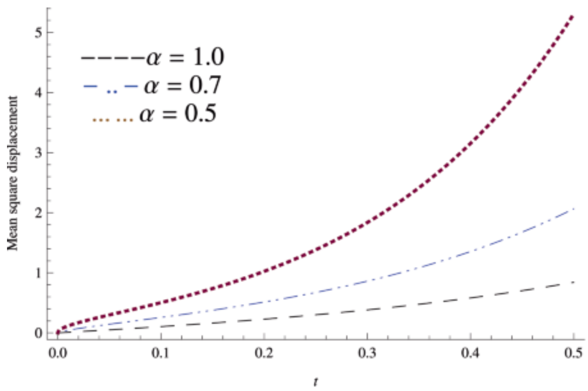

Fig. 12 (colour online). Plots of $\langle X^2(t)\rangle$ versus t for $a_0 = -5, \beta = 0.5$, and for different values of α.

Table 1. Comparison of the HAM solution with the exact solution for $n = 1$.

t	x	$u(x,t)$ at $\alpha = 1/2$	$u(x,t)$ at $\alpha = 1$	Exact Solution at $\alpha = 1$
0.0	−1.0	−1.0	−1.0	−1.0
0.0	−0.5	−0.5	−0.5	−0.5
0.0	0.0	0.0	0.0	0.0
0.0	0.5	0.5	0.5	0.5
0.0	1.0	1.0	1.0	1.0
0.5	−1.0	−0.2021	−0.5	−0.5
0.5	−0.5	0.2979	0.0	0.0
0.5	0.0	0.7979	0.5	0.5
0.5	0.5	1.2979	1.0	1.0
0.5	1.0	1.7979	1.5	1.5
1.0	−1.0	0.1284	0.0	0.0
1.0	−0.5	0.6284	0.5	0.5
1.0	0.0	1.1284	1.0	1.0
1.0	0.5	1.6284	1.5	1.5
1.0	1.0	2.1284	2.0	2.0

Table 2. Comparison of the HAM solution with the exact solution for $n = 2$.

t	x	$u(x,t)$ at $\alpha = 1/2$	$u(x,t)$ at $\alpha = 1$	Exact Solution at $\alpha = 1$
0.0	−1.0	−1.0	−1.0	−1.0
0.0	−0.5	−0.5	−0.5	−0.5
0.0	0.0	0.0	0.0	0.0
0.0	0.5	0.5	0.5	0.5
0.0	1.0	1.0	1.0	1.0
0.1	−1.0	−52.945	−1.2905	−1.2910
0.1	−0.5	−26.473	−0.6452	−0.6455
0.1	0.0	0.0	0.0	0.0
0.1	0.5	26.473	0.6452	0.6455
0.1	1.0	52.945	1.2905	1.2910
0.2	−1.0	−341.268	−2.0518	−2.2361
0.2	−0.5	−170.634	−1.0259	−1.1180
0.2	0.0	0.0	0.0	0.0
0.2	0.5	170.634	1.0259	1.1180
0.2	1.0	341.268	2.0518	2.2361

Tables 1 and 2 show a comparison between the approximate and exact values for $n = 1$ and $n = 2$ in the absence of a reaction term, i.e. $a_0 = 0$, which clearly exhibit the fact that even six order terms of the approximation of the solutions are sufficient to get good approximation to the exact solution. It is evident that the accuracy can further be enhanced by computing a few more terms of the approximate solutions.

5. Conclusion

There are four important goals that have been achieved through the study of the present article. First one is the successful presentation of the effects of the reaction term on the nonlinear fractional-order diffusion-wave solutions. Second one is the graphical presentations of the sub-diffusion and super-diffusion for different particular cases for both birth and death processes. Third one is the study of mean square displacement which justifies the anomalous nature of fractional-order diffusion processes for linear as well as nonlinear cases. Fourth one is the tabular presentation of the comparison of the approximate solutions for some particular cases with the exact solutions, which clearly reveals the reliability and effectiveness of our considered method HAM.

Acknowledgement

The authors of this article express their heartfelt thanks to the reviewers for their valuable suggestions for the improvement of the article. S. Das and S. H. Ong wish to acknowledge the financial support from the Ministry of Education FRGS grant FP010-2013A and University of Malaya's UMRGS grant RP009A-13AFR.

[1] C. N. Angstmann, I. C. Donnelly, and B. I. Henry, Math. Model. Nat. Phenom. **8**, 17 (2013).
[2] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).
[3] B. P. Lee, J. Phys. A **27**, 2633 (1994).
[4] P. A. Alemany, D. H. Zanette, and H. S. Wio, Phys. Rev. E **50**, 3646 (1994).
[5] J. Crank, The Mathematics of Diffusion, Oxford Univ. Press, London 1956.
[6] A. M. Wazwaz, Appl. Math. Comput. **123**, 109 (2001).
[7] S. Das, K. Vishal, and P. K. Gupta, Appl. Math. Model. **35**, 3970 (2011).
[8] R. Hilfer and L. Anton, Phys. Rev. E **51**, 848 (1995).
[9] R. Hilfer, Fractals **3**, 211 (1995).
[10] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, and P. Paradisi, Chem. Phys. **284**, 521 (2002).
[11] R. Metzler, E. Barkari, and J. Klafter, Phys. Rev. Let. **82**, 3565 (1999).
[12] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[13] T. A. M. Langlands and B. I. Henry, J. Comp. Phys. **205**, 719 (2005).
[14] S. B. Yuste and K. Lindenberg, Phys. Rev. Lett. **87**, 118301 (2001).
[15] G. H. Weiss, Aspects and Applications of the Random Walk, North-Holland, Amsterdam 1994.
[16] B. D. Hughes, Random Walks and Random Environments, Vol. 1, Clarendon Press, Oxford 1995.
[17] B. I. Henry and S. L. Wearne, Physica A **276**, 448 (2000).
[18] C. M. Chen, F. Liu, I. Turner, and V. Anh, J. Comput. Phys. **227**, 886 (2007).
[19] A. Schot, M. K. Lenzi, L. R. Evangelista, L. C. Malacarne, R. S. Mendis, and E. K. Lenzi, Phys. Lett. A **366**, 346 (2007).
[20] M. A. Zahran, App. Math. Model. **33**, 3088 (2009).
[21] J. M. Angulo, M. D. Ruiz-Medina, V. V. Anh, and W. Grecksch, Adv. Appl. Prob. **32**, 1077 (2000).
[22] S. Pezat and J. Zabczyk, Probab. Theory Relat. fields **116**, 421 (2000).
[23] W. R. Schneider and W. Wyss, J. Math. Phys. **30**, 134 (1989).
[24] R. Yu and H. Zhang, Chaos Solit. Fract. **30**, 946 (2006).
[25] F. Mainardi, Appl. Math. Lett. **9**, 23 (1996).
[26] F. Mainardi, Y. Luchko, and G. Pagnini, Fract. Calculus Appl. Anal. **4**, 153 (2001).
[27] V. V. Anh and N. N. Leonenko, Appl. Math. Comput. **141**, 77 (2003).
[28] S. Das, Int. Rev. Chem Eng. **4**, 501 (2012).
[29] R. Hilfer, Chemical Phys. **284**, 399 (2002).
[30] R. Hilfer, Phys. A **329**, 35 (2003).
[31] R. Hilfer, Fractals **11**, 251 (2003).
[32] S. J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[33] S. J. Liao, Commun. Nonlin. Sci. Numer. Simul. **14**, 983 (2009).
[34] S. J. Liao, Int. J. Nonlin. Mech. **30**, 371 (1995).
[35] S. J. Liao, Int. J. Nonlin. Mech. **32**, 815 (1997).
[36] S. J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall, CRC Press, Boca Raton 2003.

[37] S. Das, K. Vishal, P. K. Gupta, and S. S. Ray, Int. J. Appl. Math. Mech. **7**, 28 (2011).
[38] R. Hilfer, Application of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore 2000.
[39] M. Caputo, Geophys. J. R. Astr. Soc. **13**, 529 (1967).
[40] M. Caputo and F. Mainardi, Riv. Nuovo Cimento. **1**, 161 (1971).