Chapter 2

An Integral representation of Pseudo-Differential Operators involving the Weinstein transform

2.1 Introduction

Pseudo-differential operators are the generalization of partial differential operators. Significant works regarding pseudo-differential operators have been done by Hörmander [26, 27], Kato [31], Kohn and Nirenberg [32], Nagase [44], Rhuzhansky and Turunen [61, 62], Treves [72], Wong [81], Zaidman [83] and others. They gave important contributions to pseudo-differential operators by exploiting the theory of Fourier transform. Motivated by the aforesaid developments, exploiting the theory of Weinstein transform our main objective of the present chapter is to define pseudo-differential operators on the Schwartz space $S_*(\mathbb{R}_+^{n+1})$ and to study its various properties.

The organization of this chapter is as follows:

Section 2.1 is introductory. In Section 2.2, pseudo-differential operators involving the Weinstein transform are introduced and discussed the continuity property of pseudo-differential operators involving the Weinstein transform on the Schwartz space $S_*(\mathbb{R}^{n+1}_+)$. In Section 2.3, an integral representation of pseudo-differential operators T_σ associated to a symbol $\sigma \in S^m$ and various estimates are found. In Section 2.4, boundedness of pseudo-differential operators on weighted $L^p_\alpha(\mathbb{R}^{n+1}_+)$ -type Sobolev space $L^p_{\alpha,s}(\mathbb{R}^{n+1}_+)$ and other properties are established by using the aforesaid theory. With the help of this theory an application of pseudo-differential operators in heat equation involving Weinstein transform is given in Section 2.5.

2.2 Pseudo-differential operators associated with the Weinstein transform

In this section, definitions and various properties related to pseudo-differential operators T_σ associated with symbol $S^m, m \in \mathbb{R}$ on Schwartz space $S_*(\mathbb{R}^{n+1}_+)$ are discussed by utilizing the Weinstein transform.

Consider linear partial differential operator $P(x, \Delta_{\alpha,n,x})$ on $\mathbb{R}^{n+1}_+$, which is given by

$$P(x, \Delta_{\alpha,n,x}) = \sum_{\gamma=0}^{N} a_\gamma(x)\Delta^\gamma_{\alpha,n,x}, \tag{2.2.1}$$

where $a_\gamma(x)$ are $C^\infty-$ functions having bounded derivatives of all orders. If we replace the operator $\Delta_{\alpha,n,x}$ by a monomial $-\|\xi\|^2$ in (2.2.1), we get

$$P(x, \xi) = \sum_{\gamma=0}^{N} a_\gamma(x)(-1)^\gamma\|\xi\|^{2\gamma}, \tag{2.2.2}$$

called symbol of the operator (2.2.1). To obtain the integral representation of $P(x,\Delta_{\alpha,n,x})$, let $u\in S_*(\mathbb{R}^{n+1}_+)$, then by (2.2.1) we have

$$P(x,\Delta_{\alpha,n,x})u(x)=\sum_{\gamma=0}^{N}a_\gamma(x)\Delta^{\gamma}_{\alpha,n,x}u(x).$$

Using (1.4.1) and (1.4.2), we obtain

$$P(x,\Delta_{\alpha,n,x})u(x)=\sum_{\gamma=0}^{N}a_\gamma(x)\mathcal{F}_\alpha^{-1}\big[\mathcal{F}_\alpha(\Delta^{\gamma}_{\alpha,n,x}u)(\xi)\big](x).$$

From (1.4.3), we find

$$P(x,\Delta_{\alpha,n,x})u(x)=\sum_{\gamma=0}^{N}a_\gamma(x)\mathcal{F}_\alpha^{-1}\big[(-1)^\gamma\|\xi\|^{2\gamma}(\mathcal{F}_\alpha u)(\xi)\big](x).$$

In view of (1.4.2), above expression becomes

$$\begin{aligned}P(x,\Delta_{\alpha,n,x})u(x)&=\sum_{\gamma=0}^{N}a_\gamma(x)(-1)^\gamma\Big(\int_{\mathbb{R}^{n+1}_+}W_\alpha(x,-\xi)\|\xi\|^{2\gamma}(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\Big)\\&=\int_{\mathbb{R}^{n+1}_+}W_\alpha(x,-\xi)\Big(\sum_{\gamma=0}^{N}a_\gamma(x)(-1)^\gamma\|\xi\|^{2\gamma}\Big)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi).\end{aligned}$$

From (2.2.2), we obtain

$$P(x,\Delta_{\alpha,n,x})u(x)=\int_{\mathbb{R}^{n+1}_+}W_\alpha(x,-\xi)P(x,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi),\tag{2.2.3}$$

where

$$W_\alpha(x,-\xi)=e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1}).$$

The expression (2.2.3) indicates that if we change the symbol $P(x,\xi)$ by more general symbol $\sigma(x,\xi)$ which is no longer polynomial in ξ. The operators so obtained are called pseudo-differential operators associated with the Weinstein transform.

Definition 2.2.1. Let $m \in \mathbb{R}$. Define S^m to be the set of all functions $\sigma(x,\xi)$ in $C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1})$ such that for all $k \in \mathbb{N}_0$ and $\beta, \gamma \in \mathbb{N}_0^{n+1}$, there exists a constant $C_{\beta,\gamma} > 0$ depending only on β and γ, for which

$$\left| D_\xi^\beta D_x^\gamma \sigma(x,\xi) \right| \leq C_{\beta,\gamma}(1+\|\xi\|^2)^{\frac{m-|\beta|}{2}}(1+\|x\|^2)^{-\frac{k}{2}}. \tag{2.2.4}$$

Definition 2.2.2. Let $\sigma(x,\xi)$ be a symbol. Then for $\phi \in S_*(\mathbb{R}_+^{n+1})$, the pseudo-differential operator T_σ associated to σ is defined by

$$\big(T_\sigma\phi\big)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)(\mathcal{F}_\alpha\phi)(\xi)d\mu_\alpha(\xi). \tag{2.2.5}$$

In view of (1.3.10), the last expression can be written as

$$\big(T_\sigma\phi\big)(x) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(x,-\xi)\sigma(x,\xi)(\mathcal{F}_\alpha\phi)(\xi)d\mu_\alpha(\xi). \tag{2.2.6}$$

Example 2.1. *Let $P(x,\Delta_{\alpha,n,x})$ be a linear partial differential operator on $\mathbb{R}_+^{n+1}$, such that*

$$P(x,\Delta_{\alpha,n,x}) = \sum_{\gamma=0}^{m/2} C_\gamma(x)\Delta_{\alpha,n,x}^\gamma, \tag{2.2.7}$$

where all the coefficients $C_\gamma(x)$ having bounded derivatives of all order and $\Delta_{\alpha,n,x}$ be the Weinstein operator. Then the polynomial $P(x,\xi)$ in ξ defined by

$$P(x,\xi) = \sum_{\gamma=0}^{m/2} C_\gamma(x)\ (-\|\xi\|^2)^\gamma, \tag{2.2.8}$$

is in S^m, $\frac{m}{2} \in \mathbb{N}$, *then* $P(x, \Delta_{\alpha,n,x})$ *is the pseudo-differential operator defined by*

$$\begin{aligned} &P(x, \Delta_{\alpha,n,x})\phi(x) \\ &\quad = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) P(x,\xi)(\mathcal{F}_\alpha \phi)(\xi) d\mu_\alpha(\xi), \ \phi \in S_*(\mathbb{R}_+^{n+1}). \end{aligned} \tag{2.2.9}$$

Proof. We need to show that for $\frac{m}{2} \in \mathbb{N}$ and $\beta, \delta \in \mathbb{N}_0^{n+1}$, there exists a constant $E_{m,\beta,\delta} > 0$, such that

$$\left|(D_\xi^\beta D_x^\delta P)(x,\xi)\right| \leq E_{m,\beta,\delta}(1 + \|\xi\|^2)^{\frac{m-|\beta|}{2}}.$$

For all $x, \xi \in \mathbb{R}_+^{n+1}$, we have

$$D_\xi^\beta D_x^\delta P(x,\xi) = \sum_{\gamma=0}^{m/2} D_x^\delta C_\gamma(x) D_\xi^\beta (-\|\xi\|^2)^\gamma.$$

Then

$$\begin{aligned} \left|(D_\xi^\beta D_x^\delta P)(x,\xi)\right| &\leq \sum_{\gamma=0}^{m/2} \left|D_x^\delta C_\gamma(x)\right| \left|D_\xi^\beta(-\|\xi\|^2)^\gamma\right| \\ &\leq \sum_{\gamma=0}^{m/2} C'_{\gamma,\delta} \left|D_\xi^\beta(\xi_1^2 + \ldots + \xi_n^2 + \xi_{n+1}^2)^\gamma\right|, \end{aligned} \tag{2.2.10}$$

where $C'_{\gamma,\delta} = \sup_{x \in \mathbb{R}_+^{n+1}} \left|D_x^\delta C_\gamma(x)\right|$.

Now, we use

$$(\xi_1^2 + \ldots + \xi_n^2 + \xi_{n+1}^2)^\gamma = \sum_{\gamma_1, \cdots, \gamma_n, \gamma_{n+1} \geq 0} \binom{\gamma}{\gamma_1, \cdots, \gamma_n, \gamma_{n+1}} \xi_1^{2\gamma_1} \xi_2^{2\gamma_2} \cdots \xi_n^{2\gamma_n} \xi_{n+1}^{2\gamma_{n+1}}.$$

Therefore,

$$\begin{aligned} D_\xi^\beta(\xi_1^2+\ldots+\xi_{n+1}^2)^\gamma &= \sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} \binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}} \frac{\partial^{|\beta|}}{\partial\xi_1^{\beta_1}\cdots\partial\xi_{n+1}^{\beta_{n+1}}}\xi_1^{2\gamma_1}\xi_2^{2\gamma_2}\cdots\xi_n^{2\gamma_n}\xi_{n+1}^{2\gamma_{n+1}} \\ &= \sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} \binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}} \prod_{j=1}^{n+1}\frac{\partial^{\beta_j}}{\partial\xi_j^{\beta_j}}\xi_j^{2\gamma_j}. \end{aligned} \tag{2.2.11}$$

Using the fact

$$\frac{\partial^{\beta_j}}{\partial\xi_j^{\beta_j}}\xi_j^{2\gamma_j} = \begin{cases} (\beta_j)!\binom{2\gamma_j}{\beta_j}\xi_j^{2\gamma_j-\beta_j}, & \beta_j\leq 2\gamma_j \\ \\ 0, & \text{otherwise} \end{cases} \tag{2.2.12}$$

in (2.2.11), we get

$$\begin{aligned} D_\xi^\beta(\xi_1^2+\ldots+\xi_{n+1}^2)^\gamma &= \sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} \binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}} \prod_{j=1}^{n+1}(\beta_j)!\binom{2\gamma_j}{\beta_j}\xi_j^{2\gamma_j-\beta_j} \\ &= \sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} \binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}} \beta!\binom{2\gamma}{\beta}\xi_1^{2\gamma_1-\beta_1}\cdots\xi_{n+1}^{2\gamma_{n+1}-\beta_{n+1}}. \end{aligned} \tag{2.2.13}$$

Using (2.2.13) in (2.2.10), we have

$$\begin{aligned} \left|(D_\xi^\beta D_x^\delta P)(x,\xi)\right| &\leq \sum_{\gamma=0}^{m/2}\sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} C'_{\gamma,s}\binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}}\beta!\binom{2\gamma}{\beta} \\ &\qquad \times|\xi_1|^{2\gamma_1-\beta_1}\cdots|\xi_{n+1}|^{2\gamma_{n+1}-\beta_{n+1}} \\ &\leq \sum_{\gamma=0}^{m/2}\sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} C'_{\gamma,s}\binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}}\beta!\binom{2\gamma}{\beta} \\ &\qquad \times\|\xi\|^{2(\gamma_1+\cdots+\gamma_{n+1})-(\beta_1+\cdots+\beta_{n+1})} \\ &\leq \sum_{\gamma=0}^{m/2}\sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} C'_{\gamma,s}\binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_n,\gamma_{n+1}}\beta!\binom{2\gamma}{\beta}\ \|\xi\|^{2|\gamma|-|\beta|}. \end{aligned}$$

Therefore,

$$\begin{aligned}\left|(D_\xi^\beta D_x^\delta P)(x,\xi)\right| &\leq E_{m,\beta,\delta}\ \|\xi\|^{m-|\beta|}\\ &\leq E_{m,\beta,\delta}\ (1+\|\xi\|^2)^{\frac{m-|\beta|}{2}},\end{aligned}$$

where $E_{m,\beta,\delta}=\sum_{\gamma=0}^{m/2}\sum_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}\geq 0} C'_{\gamma,s}\binom{\gamma}{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}}\beta!\binom{2\gamma}{\beta}$.

This shows that $P(x,\xi)$ given by (2.2.8) is a symbol which belongs to S^m and associated pseudo-differential operator is in the form of (2.2.5). □

Lemma 2.2.3. *Let $p\in\mathbb{N}_0$. Then for $x\in\mathbb{R}_+^{n+1}$, we prove the following relation*

$$(1+\|x\|^2)^p=\sum_{q=0}^{p}\sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0}\binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}x^{2\eta}, \tag{2.2.14}$$

where $\eta=(\eta_1,\cdots,\eta_n,\eta_{n+1})\in\mathbb{N}_0^{n+1}$ such that $|\eta|\leq q$.

Proof. Exploiting Binomial and multinomial theorems, we have

$$\begin{aligned}(1+\|x\|^2)^p&=\sum_{q=0}^{p}\binom{p}{q}(x_1^2+\cdots+x_n^2+x_{n+1}^2)^q\\ &=\sum_{q=0}^{p}\sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0}\binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}x_1^{2\eta_1}\cdots x_n^{2\eta_n}x_{n+1}^{2\eta_{n+1}}\\ &=\sum_{q=0}^{p}\sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0}\binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}x^{2\eta}.\end{aligned}$$

□

Lemma 2.2.4. *Let $\alpha>-\frac{1}{2}$ and $(x,\xi)\in\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{n+1}$. Then we show that*

1. For $\gamma\in\mathbb{N}_0^{n+1}$, we have

$$\begin{aligned}D_x^\gamma e^{i(x_1\xi_1+\cdots+x_n\xi_n)}\hat{J}_\alpha(x_{n+1}\xi_{n+1})=i^{|\gamma|}E_\alpha\xi^\gamma e^{i(x_1\xi_1+\cdots+x_n\xi_n)}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}t^{\gamma_{n+1}}\\ \times e^{i(x_{n+1}\xi_{n+1}t)}dt,\end{aligned} \tag{2.2.15}$$

where $E_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$.

2. Choose $\gamma_{n+1}, \eta_{n+1} \in \mathbb{N}_0$ *such that* γ_{n+1} *is even and* $\gamma_{n+1} \geq 2\eta_{n+1}$*, then we get*

$$x_{n+1}^{2\eta_{n+1}} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}} e^{i(x_{n+1}\xi_{n+1}t)} dt = i^{-2\eta_{n+1}} \frac{\partial^{2\eta_{n+1}}}{\partial \xi_{n+1}^{2\eta_{n+1}}} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}-2\eta_{n+1}} \times e^{i(x_{n+1}\xi_{n+1}t)} dt, \tag{2.2.16}$$

where the integral of right hand side of (2.2.16) *is bounded by the Beta function*

$$B\Big(\frac{\gamma_{n+1} - 2\eta_{n+1} + 1}{2}, \alpha + \frac{1}{2}\Big).$$

Proof. Using (1.3.12), we have

$$\begin{aligned} D_x^\gamma e^{i(x_1\xi_1+\cdots+x_n\xi_n)} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) &= E_\alpha \frac{\partial^{\gamma_1+\cdots+\gamma_n}}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} e^{i(x_1\xi_1+\cdots+x_n\xi_n)} \frac{\partial^{\gamma_{n+1}}}{\partial x_{n+1}^{\gamma_{n+1}}} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \\ &\quad \times e^{i(tx_{n+1}\xi_{n+1})} dt \\ &= i^{|\gamma|} E_\alpha \xi^\gamma e^{i(x_1\xi_1+\cdots+x_n\xi_n)} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \\ &\quad \times t^{\gamma_{n+1}} e^{i(tx_{n+1}\xi_{n+1})} dt. \end{aligned}$$

Now, using the relation

$$\frac{\partial^{2\eta_{n+1}}}{\partial \xi_{n+1}^{2\eta_{n+1}}} e^{i(tx_{n+1}\xi_{n+1})} = (itx_{n+1})^{2\eta_{n+1}} e^{i(tx_{n+1}\xi_{n+1})},$$

we have

$$\begin{aligned} x_{n+1}^{2\eta_{n+1}} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}} e^{i(x_{n+1}\xi_{n+1}t)} dt &= \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}} x_{n+1}^{2\eta_{n+1}} e^{i(x_{n+1}\xi_{n+1}t)} dt \\ &= i^{-2\eta_{n+1}} \frac{\partial^{2\eta_{n+1}}}{\partial \xi_{n+1}^{2\eta_{n+1}}} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}-2\eta_{n+1}} \\ &\quad \times e^{i(tx_{n+1}\xi_{n+1})} dt. \end{aligned}$$

Therefore,

$$\Big|\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}t^{\gamma_{n+1}-2\eta_{n+1}}e^{i(tx_{n+1}\xi_{n+1})}dt\Big| \leq 2\int_{0}^{1}(1-t^2)^{\alpha-\frac{1}{2}}t^{\gamma_{n+1}-2\eta_{n+1}}dt.$$

Now, from the properties of Beta function we have

$$\begin{aligned}\Big|\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}t^{\gamma_{n+1}-2\eta_{n+1}}e^{i(tx_{n+1}\xi_{n+1})}dt\Big| &\leq B\Big(\frac{\gamma_{n+1}-2\eta_{n+1}+1}{2},\alpha+\frac{1}{2}\Big)\\ &= \frac{\Gamma(\frac{\gamma_{n+1}-2\eta_{n+1}+1}{2})\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{\gamma_{n+1}-2\eta_{n+1}+2\alpha+2}{2})}. \qquad (2.2.17)\end{aligned}$$

This proves that the aforesaid integral is absolutely convergent for sufficiently large values of γ_{n+1}. □

Lemma 2.2.5. *Let* $\alpha > -\frac{1}{2}$ *and* $\gamma \in \mathbb{N}_0^{n+1}$. *Then for* $\phi \in S_*(\mathbb{R}_+^{n+1})$, $\xi^\gamma \xi_{n+1}^{2\alpha+1}(\mathcal{F}_\alpha\phi)(\xi) \in S_*(\mathbb{R}_+^{n+1})$.

Proof. Let $\psi(\xi) = \xi^\gamma \xi_{n+1}^{2\alpha+1}(\mathcal{F}_\alpha\phi)(\xi)$. Then for any $k \in \mathbb{N}$ and $\delta \in \mathbb{N}_0^{n+1}$, we need to show that

$$\sup_{\xi\in\mathbb{R}^{n+1}}\Big|(1+\|\xi\|^2)^k(D_\xi^\delta\psi)(\xi)\Big| < \infty,$$

where

$$(D_\xi^\delta\psi)(\xi) = D_\xi^\delta\big(\xi^\gamma\xi_{n+1}^{2\alpha+1}(\mathcal{F}_\alpha\phi)(\xi)\big). \qquad (2.2.18)$$

By the Leibniz formula (1.4.19) in (2.2.18), we have

$$\begin{aligned}(D_\xi^\delta\psi)(\xi) &= \sum_{\rho\leq\delta}\binom{\delta}{\rho}\frac{\partial^{|\rho|}}{\partial\xi_1^{\rho_1}\cdots\partial\xi_n^{\rho_n}\xi_{n+1}^{\rho_{n+1}}}\xi_1^{\gamma_1}\xi_2^{\gamma_2}\cdots\xi_n^{\gamma_n}\xi_{n+1}^{\gamma_{n+1}+2\alpha+1}D_\xi^{\delta-\rho}(\mathcal{F}_\alpha\phi)(\xi)\\ &= \sum_{\rho\leq\delta}\binom{\delta}{\rho}\prod_{j=1}^{n}\frac{\partial^{\rho_j}}{\partial\xi_j^{\rho_j}}\xi_j^{\gamma_j}\frac{\partial^{\rho_{n+1}}}{\partial\xi_{n+1}^{\rho_{n+1}}}\xi_{n+1}^{\gamma_{n+1}+2\alpha+1}D_\xi^{\delta-\rho}(\mathcal{F}_\alpha\phi)(\xi).\end{aligned}$$

Using (2.2.12), we get

$$\begin{aligned}(D_\xi^\delta \psi)(\xi) &= \sum_{\rho\le\delta}\binom{\delta}{\rho}\prod_{j=1}^{n}(\rho_j)!\binom{\gamma_j}{\rho_j}\xi_j^{\gamma_j-\rho_j}(\rho_{n+1})!\binom{\gamma_{n+1}+2\alpha+1}{\rho_{n+1}}\\ &\qquad\qquad \times \xi_{n+1}^{\gamma_{n+1}+2\alpha+1-\rho_{n+1}} D_\xi^{\delta-\rho}(\mathcal{F}_\alpha\phi)(\xi)\\ &= \sum_{\rho\le\delta}\binom{\delta}{\rho}\prod_{j=1}^{n}\binom{\gamma_j}{\rho_j}(\rho)!\binom{\gamma_{n+1}+2\alpha+1}{\rho_{n+1}}\xi^{\gamma-\rho}\xi_{n+1}^{2\alpha+1}D_\xi^{\delta-\rho}(\mathcal{F}_\alpha\phi)(\xi).\end{aligned} \tag{2.2.19}$$

The right hand side of (2.2.19) is bounded by

$$\sum_{\rho\le\delta}\binom{\delta}{\rho}\prod_{j=1}^{n}\binom{\gamma_j}{\rho_j}(\rho)!\binom{\gamma_{n+1}+2\alpha+1}{\rho_{n+1}}(1+\|\xi\|^2)^{|\gamma|-|\rho|+2\alpha+1}\big|D_\xi^{\delta-\rho}(\mathcal{F}_\alpha\phi)(\xi)\big|.$$

Let $l \geq |\gamma| + 2\alpha + 1$, then we have

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^{n+1}}\Big|(1+\|\xi\|^2)^k(D_\xi^\delta\psi)(\xi)\Big| &\le \sum_{\rho\le\delta}\binom{\delta}{\rho}\prod_{j=1}^{n}\binom{\gamma_j}{\rho_j}(\rho)!\binom{\gamma_{n+1}+2\alpha+1}{\rho_{n+1}}\\ &\qquad \times \sup_{\xi\in\mathbb{R}^{n+1}}\Big|(1+\|\xi\|^2)^{k+l}D_\xi^{\delta-\rho}(\mathcal{F}_\alpha\phi)(\xi)\Big|.\end{aligned}$$

Therefore, using (1.3.1) we get

$$\tau_{k,m}(\psi) \le C_{\alpha,\delta,\gamma}\tau_{k+l,p}(\mathcal{F}_\alpha\phi), \tag{2.2.20}$$

where $|\delta| \le m, |\delta-\rho| \le p$ and $C_{\alpha,\delta,\gamma}$ is a positive constant depends upon α,δ,γ only. Using the fact that $(\mathcal{F}_\alpha\phi) \in S_*(\mathbb{R}_+^{n+1})$, it follows that

$$\tau_{k,m}(\psi) = \sup_{\substack{|\delta|\le m\\ \xi\in\mathbb{R}^{n+1}}}\Big|(1+\|\xi\|^2)^k(D_\xi^\delta\psi)(\xi)\Big| < \infty.$$

This implies that $\xi^\gamma\xi_{n+1}^{2\alpha+1}(\mathcal{F}_\alpha\phi) \in S_*(\mathbb{R}_+^{n+1})$. □

Theorem 2.2.6. *Let $\sigma \in S^m, m \in \mathbb{R}$ be a symbol. Then the pseudo-differential operator T_σ associated to σ maps the Schwartz space $S_*(\mathbb{R}_+^{n+1})$ into itself.*

Proof. Let $\phi \in S_*(\mathbb{R}_+^{n+1})$. Then for $p \in \mathbb{N}_0$ and multi-index β, we need to show that

$$\sup_{x\in\mathbb{R}^{n+1}} \Big|(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x)\Big| < \infty.$$

Using Lemma 2.2.3 and (2.2.5), we have

$$(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x) = \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}} x^{2\eta}\int_{\mathbb{R}_+^{n+1}} D_x^\beta$$
$$\times \Big(e^{i(x_1\xi_1+\cdots+x_n\xi_n)}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)\Big)\big(\mathcal{F}_\alpha\phi\big)(\xi)d\mu_\alpha(\xi),$$

where p, q and η are independent from β.

Exploiting the Leibniz formula and Lemma 2.2.4, we get

$$\begin{aligned}
(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x) &= \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0}\sum_{\gamma\leq\beta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma} x^{2\eta}\int_{\mathbb{R}_+^{n+1}} D_x^\gamma \\
&\quad\times \Big(e^{i(x_1\xi_1+\cdots+x_n\xi_n)}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big)\Big(D_x^{\beta-\gamma}\sigma(x,\xi)\Big)\big(\mathcal{F}_\alpha\phi\big)(\xi)d\mu_\alpha(\xi) \\
&= \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0}\sum_{\gamma\leq\beta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma} i^{|\gamma|}E_\alpha x^{2\eta}\int_{\mathbb{R}_+^{n+1}} \\
&\quad\times \Big(\xi^\gamma e^{i(x_1\xi_1+\cdots+x_n\xi_n)}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}t^{\gamma_{n+1}}e^{i(tx_{n+1}\xi_{n+1})}dt\Big)\Big(D_x^{\beta-\gamma}\sigma(x,\xi)\Big) \\
&\quad\times \big(\mathcal{F}_\alpha\phi\big)(\xi)d\mu_\alpha(\xi) \\
&= \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0}\sum_{\gamma\leq\beta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma} i^{|\gamma|}E_\alpha \int_{\mathbb{R}_+^{n+1}} \\
&\quad\times \Big(x_1^{2\eta_1}\cdots x_n^{2\eta_n}e^{i(x_1\xi_1+\cdots+x_n\xi_n)}x_{n+1}^{2\eta_{n+1}}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}t^{\gamma_{n+1}}e^{i(tx_{n+1}\xi_{n+1})}dt\Big) \\
&\quad\times \Big(D_x^{\beta-\gamma}\sigma(x,\xi)\Big)\xi^\gamma\big(\mathcal{F}_\alpha\phi\big)(\xi)d\mu_\alpha(\xi).
\end{aligned}$$

Using (2.2.16), we get

$$\begin{aligned}(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x) &= \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma} i^{|\gamma|-2|\eta|} E_\alpha \\ &\quad\times \int_{\mathbb{R}_+^{n+1}} \Big(\frac{\partial^{2\eta_1+\cdots+2\eta_n}}{\partial\xi_1^{2\eta_1}\cdots\partial\xi_n^{2\eta_n}} e^{i(x_1\xi_1+\cdots+x_n\xi_n)} \frac{\partial^{2\eta_{n+1}}}{\partial\xi_{n+1}^{2\eta_{n+1}}} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}-2\eta_{n+1}} \\ &\quad\times e^{i(tx_{n+1}\xi_{n+1})} dt\Big)\Big(D_x^{\beta-\gamma}\sigma(x,\xi)\Big)\xi^\gamma \big(\mathcal{F}_\alpha\phi\big)(\xi) d\mu_\alpha(\xi) \\ &= \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma} i^{|\gamma|-2|\eta|} E'_\alpha \\ &\quad\times \int_{\mathbb{R}_+^{n+1}} \Big(D_\xi^{2\eta} e^{i(x_1\xi_1+\cdots+x_n\xi_n)} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}-2\eta_{n+1}} e^{i(tx_{n+1}\xi_{n+1})} dt\Big) \\ &\quad\times \Big(D_x^{\beta-\gamma}\sigma(x,\xi)\Big)\xi^\gamma \xi_{n+1}^{2\alpha+1} \big(\mathcal{F}_\alpha\phi\big)(\xi) d\xi,\end{aligned}$$

where $E'_\alpha = \frac{1}{(2\pi)^{\frac{n}{2}} 2^\alpha \Gamma(\alpha+1)} E_\alpha$.

Using integration by parts, we get

$$\begin{aligned}(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x) &= \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma} i^{|\gamma|-2|\eta|} E'_\alpha \\ &\quad\times \int_{\mathbb{R}_+^{n+1}} \Big(e^{i(x_1\xi_1+\cdots+x_n\xi_n)} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}-2\eta_{n+1}} e^{i(tx_{n+1}\xi_{n+1})} dt\Big) \\ &\quad\times D_\xi^{2\eta}\Big(D_x^{\beta-\gamma}\sigma(x,\xi)\Big)\xi^\gamma \xi_{n+1}^{2\alpha+1} \big(\mathcal{F}_\alpha\phi\big)(\xi) d\xi.\end{aligned}$$

Thus, the Leibniz formula gives

$$\begin{aligned}(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x) &= \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \sum_{\delta\leq 2\eta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma}\binom{2\eta}{\delta} i^{|\gamma|-2|\eta|} \\ &\quad\times E'_\alpha \int_{\mathbb{R}_+^{n+1}} \Big(e^{i(x_1\xi_1+\cdots+x_n\xi_n)} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} t^{\gamma_{n+1}-2\eta_{n+1}} e^{i(tx_{n+1}\xi_{n+1})} dt\Big) \\ &\quad\times \Big(D_\xi^{2\eta-\delta} D_x^{\beta-\gamma}\sigma(x,\xi)\Big) D_\xi^\delta\Big(\xi^\gamma \xi_{n+1}^{2\alpha+1} \big(\mathcal{F}_\alpha\phi\big)(\xi)\Big) d\xi.\end{aligned} \tag{2.2.21}$$

Choose $\gamma_{n+1}, \eta_{n+1} \in \mathbb{N}_0$ such that γ_{n+1} is even number and $\gamma_{n+1} \geq 2\eta_{n+1}$. Then from (2.2.17) and (2.2.21), we have

$$\begin{aligned}\Big|(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x)\Big| \leq \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \sum_{\delta\leq 2\eta} \binom{p}{q} \binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}} \binom{\beta}{\gamma} \binom{2\eta}{\delta} \\ \times E'_\alpha B\Big(\frac{\gamma_{n+1}-2\eta_{n+1}+1}{2}, \alpha+\frac{1}{2}\Big) \int_{\mathbb{R}_+^{n+1}} \Big|D_\xi^{2\eta-\delta} D_x^{\beta-\gamma}\sigma(x,\xi)\Big| \\ \times \Big|D_\xi^\delta \xi^\gamma \xi_{n+1}^{2\alpha+1} \big(\mathcal{F}_\alpha\phi\big)(\xi)\Big| d\xi.\end{aligned}$$

Using the fact that $\sigma \in S^m$, we can find a positive constant $C'_{\beta,\eta}$ depending on β, η only such that

$$\begin{aligned}\sup_{x\in\mathbb{R}^{n+1}} \Big|(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x)\Big| \leq \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \sum_{\delta\leq 2\eta} \binom{p}{q} \binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}} \binom{\beta}{\gamma} \\ \times \binom{2\eta}{\delta} E'_\alpha B\Big(\frac{\gamma_{n+1}-2\eta_{n+1}+1}{2}, \alpha+\frac{1}{2}\Big) C'_{\beta,\eta} \int_{\mathbb{R}_+^{n+1}} \\ \times (1+\|\xi\|^2)^{m-2|\eta|+|\delta|-k} \Big|(1+\|\xi\|^2)^k D_\xi^\delta \xi^\gamma \xi_{n+1}^{2\alpha+1} \big(\mathcal{F}_\alpha\phi\big)(\xi)\Big| d\xi.\end{aligned}$$

Choose $k > m + \frac{n}{2} - 2|\eta| + |\delta| + \frac{1}{2}$. Then $C''_{k,m,\eta} = \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{m-2|\eta|+|\delta|-k} d\xi$, be a finite positive constant depending on k, m, η, δ only such that

$$\begin{aligned}\sup_{x\in\mathbb{R}^{n+1}} \Big|(1+\|x\|^2)^p D_x^\beta (T_\sigma\phi)(x)\Big| \leq \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \sum_{\delta\leq 2\eta} \binom{p}{q} \binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}} \binom{\beta}{\gamma} \\ \times \binom{2\eta}{\delta} E'_\alpha B\Big(\frac{\gamma_{n+1}-2\eta_{n+1}+1}{2}, \alpha+\frac{1}{2}\Big) C'_{\beta,\eta} C''_{k,m,\eta} \\ \times \sup_{\xi\in\mathbb{R}^{n+1}} \Big|(1+\|\xi\|^2)^k D_\xi^\delta \xi^\gamma \xi_{n+1}^{2\alpha+1} \big(\mathcal{F}_\alpha\phi\big)(\xi)\Big|.\end{aligned} \tag{2.2.22}$$

Using Lemma 2.2.5 and (2.2.20) in (2.2.22), we get

$$\begin{aligned}\tau_{p,r}(T_\sigma\phi) \leq& \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \sum_{\delta\leq 2\eta} \binom{p}{q}\binom{q}{\eta_1,\cdots,\eta_n,\eta_{n+1}}\binom{\beta}{\gamma}\binom{2\eta}{\delta} \\ &\times E'_\alpha B\Big(\frac{\gamma_{n+1}-2\eta_{n+1}+1}{2},\alpha+\frac{1}{2}\Big)C'_{\beta,\eta}C''_{k,m,\eta}C_{\alpha,\delta,\zeta}\tau_{k+l,p}(\mathcal{F}_\alpha\phi) \\ \leq& \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \sum_{\delta\leq 2\eta} D_{\alpha,\beta,\eta,k,m,p}\tau_{k+l,p}(\mathcal{F}_\alpha\phi), \qquad (2.2.23)\end{aligned}$$

where $|\beta| \leq r$ and $D_{\alpha,\beta,\eta,k,m,p}$ is a positive constants, then (2.2.23) implies that $\tau_{p,r}(T_\sigma\phi) < \infty$. The continuity of T_σ follows from (2.2.23). □

Corollary 2.2.7. *The pseudo-differential operator T_σ associated to symbol σ maps the Schwartz space $S_*(\mathbb{R}_+^{n+1})$ continuously into itself. More precisely, if $\phi_s \to 0$ in $S_*(\mathbb{R}_+^{n+1})$, then $T_\sigma\phi_s \to 0$ in $S_*(\mathbb{R}_+^{n+1})$ as $s \to \infty$.*

Proof. From (2.2.23), we found that

$$\tau_{p,r}(T_\sigma\phi_s) \leq \sum_{q=0}^{p} \sum_{\eta_1,\cdots,\eta_n,\eta_{n+1}\geq 0} \sum_{\gamma\leq\beta} \sum_{\delta\leq 2\eta} D_{\alpha,\beta,\eta,k,m,p}\tau_{k+l,p}(\mathcal{F}_\alpha\phi_s). \qquad (2.2.24)$$

Since $\phi_s \to 0$ in $S_*(\mathbb{R}_+^{n+1})$ as $s \to \infty$, it follows from (2.2.24), $\tau_{p,r}(T_\sigma\phi_s) \to 0$ in $S_*(\mathbb{R}_+^{n+1})$. □

2.3 Boundedness of Pseudo-Differential Operators

In this section, an integral representation and various estimates are made by exploiting the theory of Weinstein transform. Further, boundedness of pseudo-differential operators are discussed.

An integral representation. The function $\sigma_x(\xi)$, associated with the symbol

$\sigma(x,\xi)$ and defined by

$$\sigma_x(\xi) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(\eta,\xi)\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big]d\mu_\alpha(\eta), \quad \xi \in \mathbb{R}_+^{n+1}, \tag{2.3.1}$$

will play a fundamental role in our investigation. An estimate for $\sigma_x(\xi)$ is given by the following lemma:

Lemma 2.3.1. *Let $\sigma \in S^m, m \in \mathbb{R}$. Then the following inequality holds*

$$\big|\sigma_x(\xi)\big| \leq E_{\alpha,m,p}(1+\|\xi\|^2)^{-p}(1+\|x\|^2)^{\frac{4p-k}{2}} \tag{2.3.2}$$

where $E_{\alpha,m,p}$ is a positive constant and $p,k \in \mathbb{N}_0$ such that $m < -2\alpha - n - p - 2$.

Proof. From (2.3.1) and Binomial theorem, we have

$$(1+\|\xi\|^2)^p\sigma_x(\xi) = \sum_{s=0}^{p}\binom{p}{s}\|\xi\|^{2s}\mathcal{F}_\alpha\Big(W_\alpha(x,-\eta)\sigma(x,\eta)\Big)(\xi).$$

First, we shall show that $x \to W_\alpha(x,-\eta)\sigma(x,\eta) \in L^1_\alpha(\mathbb{R}_+^{n+1})$, then for $\sigma \in S^m$, we get

$$\begin{aligned}\|W_\alpha(x,-\eta)\sigma(x,\eta)\|_{L^1_\alpha(\mathbb{R}_+^{n+1})} &= \int_{\mathbb{R}_+^{n+1}}|W_\alpha(x,-\eta)\sigma(x,\eta)|d\mu_\alpha(\eta)\\ &\leq C_{0,0}(1+\|x\|^2)^{\frac{-k}{2}}\int_{\mathbb{R}_+^{n+1}}(1+\|\eta\|^2)^{\frac{m}{2}}d\mu_\alpha(\eta)\\ &\leq C_{0,0}A_\alpha(1+\|x\|^2)^{\frac{-k}{2}}\int_{\mathbb{R}_+^{n+1}}(1+\|\eta\|^2)^{\frac{m+2\alpha+1}{2}}d\eta.\end{aligned}$$

Choose $m < -2\alpha - n - 2$, then there exists a finite constant $C_{\alpha,m}$ depends only on α, m such that $W_\alpha(x,-\eta)\sigma(x,\eta) \in L^1_\alpha(\mathbb{R}_+^{n+1})$.

Now, using (1.4.3) we get

$$\begin{aligned}(1+\|\xi\|^2)^p\sigma_x(\xi) &= \sum_{s=0}^{p}\binom{p}{s}(-1)^s\mathcal{F}_\alpha\Big(\Delta^s_{\alpha,n,\eta}\big[W_\alpha(x,-\eta)\sigma(x,\eta)\big]\Big)(\xi)\\ &= \sum_{s=0}^{p}\binom{p}{s}(-1)^s\int_{\mathbb{R}^{n+1}_+}W_\alpha(\eta,\xi)\Delta^s_{\alpha,n,\eta}\big[W_\alpha(x,-\eta)\sigma(x,\eta)\big]d\mu_\alpha(\eta).\end{aligned}$$

Therefore,

$$(1+\|\xi\|^2)^p\big|\sigma_x(\xi)\big| \leq \sum_{s=0}^{p}\binom{p}{s}\int_{\mathbb{R}^{n+1}_+}\Big|\Delta^s_{\alpha,n,\eta}\big[W_\alpha(x,-\eta)\sigma(x,\eta)\big]\Big|d\mu_\alpha(\eta). \tag{2.3.3}$$

Now, we find the estimate of $\Big|\Delta^s_{\alpha,n,\eta}\big[W_\alpha(x,-\eta)\sigma(x,\eta)\big]\Big|$ with the help of [19, p. 14], then there is a constant $E'_{\alpha,r}$ for $r\in\{0,1,\cdots,s\}$ depending only on α satisfying

$$\Delta^s_{\alpha,n,\eta}\big[W_\alpha(x,-\eta)\sigma(x,\eta)\big] = \sum_{j=0}^{s}\sum_{r=1}^{2j}\binom{s}{j}E'_{\alpha,r}\eta^{r-s}_{n+1}(\Delta_n)^{s-j}_{\eta'}\frac{\partial^r}{\partial\eta^r_{n+1}}\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big],$$

where $(\Delta_n)_{\eta'} = (\frac{\partial^2}{\partial\eta_1^2}+\cdots+\frac{\partial^2}{\partial\eta_n^2})$.

Using the Leibniz formula (1.4.19), we have

$$\begin{aligned}\Delta^s_{\alpha,n,\eta}\big[W_\alpha(x,-\eta)\sigma(x,\eta)\big] &= \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\binom{s}{j}\binom{r}{q}E'_{\alpha,r}\eta^{r-s}_{n+1}(\Delta_n)^{s-j}_{\eta'}\\ &\quad\times\Big[\frac{\partial^q}{\partial\eta^q_{n+1}}W_\alpha(x,-\eta)\frac{\partial^{r-q}}{\partial\eta^{r-q}_{n+1}}\sigma(x,\eta)\Big].\end{aligned} \tag{2.3.4}$$

Using (1.4.20) in the above expression, we get

$$\begin{aligned}\Delta^s_{\alpha,n,\eta}\big[W_\alpha(x,-\eta)\sigma(x,\eta)\big] &= \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\leq 2(s-j)}\binom{s}{j}\binom{r}{q}\frac{1}{\rho'!}E'_{\alpha,r}\eta^{r-s}_{n+1}\\ &\quad\times\left[\frac{\partial^{|\rho'|+q}}{\partial\eta_1^{\rho_1}\cdots\partial\eta_n^{\rho_n}\partial\eta^q_{n+1}}W_\alpha(x,-\eta)\right]\\ &\quad\times\left[\frac{\partial^{|\rho'|}}{\partial\eta_1^{\rho_1}\cdots\partial\eta_n^{\rho_n}}(\Delta_n)^{s-j}_{\eta'}\frac{\partial^{r-q}}{\partial\eta^{r-q}_{n+1}}\sigma(x,\eta)\right].\end{aligned} \tag{2.3.5}$$

For $s-j\in\mathbb{N}_0$, we have

$$(\Delta_n)_{\eta'}^{s-j}=\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s-j}{\delta_1,\cdots,\delta_n}\frac{\partial^{2\delta_1}}{\partial\eta_1^{2\delta_1}}\cdots\frac{\partial^{2\delta_n}}{\partial\eta_n^{2\delta_n}}. \tag{2.3.6}$$

From (2.3.5) and (2.3.6), we get

$$\begin{aligned}\Delta^s_{\alpha,n,\eta}\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big]&=\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\leq 2(s-j)}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{r}{q}\binom{s-j}{\delta_1,\cdots,\delta_n}\frac{1}{\rho'!}\\&\quad\times E'_{\alpha,r}\eta_{n+1}^{r-s}\left[\frac{\partial^{|\rho'|+q}}{\partial\eta_1^{\rho_1}\cdots\partial\eta_n^{\rho_n}\partial\eta_{n+1}^{q}}W_\alpha(x,-\eta)\right]\\&\quad\times\left[\frac{\partial^{|\rho'|+2|\delta'|+r-q}}{\partial\eta_1^{\rho_1+2\delta_1}\cdots\partial\eta_n^{\rho_n+2\delta_n}\partial\eta_{n+1}^{r-q}}\sigma(x,\eta)\right]\\&=\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\leq 2(s-j)}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{r}{q}\binom{s-j}{\delta_1,\cdots,\delta_n}\\&\quad\times\frac{1}{\rho'!}E'_{\alpha,r}\eta_{n+1}^{r-s}\Big[D_\eta^{\rho'+q}W_\alpha(x,-\eta)\Big]\Big[D_\eta^{\rho'+2\delta'+r-q}\sigma(x,\eta)\Big],\end{aligned} \tag{2.3.7}$$

where $\rho'+q=(\rho_1,\cdots,\rho_n,q)\in\mathbb{N}_0^{n+1}$, $\rho'+2\delta'+r-q=(\rho_1+2\delta_1,\cdots,\rho_n+2\delta_n,r-q)\in\mathbb{N}_0^{n+1}$ and $|\rho'|+q=\rho_1+\cdots+\rho_n+q$, $|\rho'|+2|\delta'|+r-q=\rho_1+2\delta_1+\cdots+\rho_n+2\delta_n+r-q$. Using the fact $\sigma\in S^m$ and Proposition 1.3.4, (2.3.7) becomes

$$\begin{aligned}\Big|\Delta^s_{\alpha,n,\eta}\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big]\Big|&\leq\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\leq 2(s-j)}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{r}{q}\binom{s-j}{\delta_1,\cdots,\delta_n}\frac{1}{\rho'!}\\&\quad\times E'_{\alpha,r}C_{\rho',\delta',q}|\eta_{n+1}|^{r-s}\|x\|^{|\rho'|+q}(1+\|\eta\|^2)^{\frac{m-(|\rho'|+2|\delta'|+r-q)}{2}}\\&\quad\times(1+\|x\|^2)^{-\frac{k}{2}}\\&\leq\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\leq 2(s-j)}\binom{s}{j}\binom{r}{q}\frac{1}{\rho'!}E'_{\alpha,r}C_{\rho',q}\\&\quad\times|\eta_{n+1}|^{r-s}(1+\|\eta\|^2)^{\frac{m-(|\rho'|+r-q)}{2}}(1+\|x\|^2)^{\frac{|\rho'|+q-k}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\Delta^s_{\alpha,n,\eta}\left[W_\alpha(x,-\eta)\sigma(x,\eta)\right]\right| &\leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\binom{s}{j}\binom{r}{q}E'_{\alpha,r}C_{2(s-j),q}|\eta_{n+1}|^{r-s}(1+\|\eta\|^2)^{\frac{m-r+q}{2}}\\ &\quad\times(1+\|x\|^2)^{\frac{2s-2j+q-k}{2}}\\ &\leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\binom{s}{j}E'_{\alpha,r}C_{2(s-j),r}|\eta_{n+1}|^{r-s}(1+\|\eta\|^2)^{\frac{m}{2}}\\ &\quad\times(1+\|x\|^2)^{\frac{2s-2j+r-k}{2}}\\ &\leq E''_{\alpha,s}(1+\|\eta\|^2)^{\frac{m+s}{2}}(1+\|x\|^2)^{\frac{4s-k}{2}}, \end{aligned} \tag{2.3.8}$$

where $E''_{\alpha,s}$ is a positive constant.

Now, using (2.3.8) in (2.3.3), we get

$$\begin{aligned}(1+\|\xi\|^2)^p\left|\sigma_x(\xi)\right| &\leq \sum_{s=0}^{p}\binom{p}{s}E''_{\alpha,s}A_\alpha(1+\|x\|^2)^{\frac{4s-k}{2}}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^{\frac{m+s}{2}}|\eta_{n+1}|^{2\alpha+1}d\eta\\ &\leq E'''_{\alpha,p}A_\alpha(1+\|x\|^2)^{\frac{4p-k}{2}}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^{\frac{m+p+2\alpha+1}{2}}d\eta.\end{aligned}$$

For $m<-2\alpha-n-p-2$, there exists a positive constant $E_{\alpha,m,p}$ such that

$$\left|\sigma_x(\xi)\right|\leq E_{\alpha,m,p}(1+\|\xi\|^2)^{-p}(1+\|x\|^2)^{\frac{4p-k}{2}}.$$

□

Theorem 2.3.2. *If $\sigma\in S^m, m\in\mathbb{R}$, then the pseudo-differential operator associated with σ can be represented in the following form*

$$\left(T_\sigma\phi\right)(x)=\int_{\mathbb{R}^{n+1}_+}\sigma_x(\xi)\phi(\xi)d\mu_\alpha(\xi),\quad \phi\in S_*(\mathbb{R}^{n+1}_+), \tag{2.3.9}$$

where $\sigma_x(\xi)$ is defined in (2.3.1) and involved estimate is convergent for $m<-2\alpha-n-p-2$ in Lemma 2.3.1.

Proof. Let $\sigma \in S^m$ be a symbol. Then from (2.2.5), we have

$$\big(T_\sigma\phi\big)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma(x,\eta)(\mathcal{F}_\alpha\phi)(\eta)d\mu_\alpha(\eta),$$

where

$$(\mathcal{F}_\alpha\phi)(\eta) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \eta',\xi'\rangle}\hat{J}_\alpha(\eta_{n+1}\xi_{n+1})\phi(\xi)d\mu_\alpha(\xi).$$

Therefore,

$$\begin{aligned}\big(T_\sigma\phi\big)(x) = \int_{\mathbb{R}_+^{n+1}} \Big(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle \eta',\xi'\rangle}\hat{J}_\alpha(\eta_{n+1}\xi_{n+1})\Big[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma(x,\eta)\Big]\\ \times d\mu_\alpha(\eta)\Big)\phi(\xi)d\mu_\alpha(\xi).\end{aligned}$$

Using Proposition 1.3.4, the last expression becomes

$$\begin{aligned}\big(T_\sigma\phi\big)(x) &= \int_{\mathbb{R}_+^{n+1}} \Big(\int_{\mathbb{R}_+^{n+1}} W_\alpha(\eta,\xi)\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big]d\mu_\alpha(\eta)\Big)\phi(\xi)d\mu_\alpha(\xi)\\ &= \int_{\mathbb{R}_+^{n+1}} \sigma_x(\xi)\phi(\xi)d\mu_\alpha(\xi).\end{aligned}$$

Next, we shall show that integral on right hand side of (2.3.9) is absolutely convergent. Since $\phi \in S_*(\mathbb{R}_+^{n+1})$, therefore

$$\big|\phi(\xi)\big| \leq C(1+\|\xi\|^2)^{-l}, \tag{2.3.10}$$

where C be a positive constant and $l \in \mathbb{N}$.

Taking absolute value of (2.3.9) and using (2.3.2), (2.3.10), we have

$$\begin{aligned}\big|\big(T_\sigma\phi\big)(x)\big| &\leq CE_{\alpha,m,p}\int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-l-p}(1+\|x\|^2)^{\frac{4p-k}{2}}d\mu_\alpha(\xi)\\ &\leq CE_{\alpha,m,p}A_\alpha(1+\|x\|^2)^{\frac{4p-k}{2}}\int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-l-p}\xi_{n+1}^{2\alpha+1}d\xi.\end{aligned}$$

For $l, p \in \mathbb{N}$ and $l > \frac{n}{2} + \alpha - p + 1$, there exists a positive constant $C_{\alpha,l,m,p}$ such that

$$\big|\big(T_\sigma\phi\big)(x)\big| \le C_{\alpha,l,m,p}(1+\|x\|^2)^{\frac{4p-k}{2}}, \quad \forall x \in \mathbb{R}_+^{n+1}.$$

□

Lemma 2.3.3. *Let $\sigma \in S^m, m \in \mathbb{R}$. Then*

$$\sigma_x(\xi) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(\eta,\xi)\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big]d\mu_\alpha(\eta),$$

can be expressed as

$$\int_{\mathbb{R}_+^{n+1}} W_\alpha(\eta,-\xi)\sigma_x(\xi)d\mu_\alpha(\xi) = W_\alpha(x,-\eta)\sigma(x,\eta). \tag{2.3.11}$$

Proof. Using Proposition 1.3.4 in (2.3.1), we get

$$\sigma_x(\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle\eta',\xi'\rangle}\hat{J}_\alpha(\eta_{n+1}\xi_{n+1})\Big[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma(x,\eta)\Big]d\mu_\alpha(\eta).$$

In view of (1.4.1), above expression yields

$$\sigma_x(\xi) = \mathcal{F}_\alpha\Big[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma(x,\eta)\Big](\xi).$$

From (1.4.2), the last expression becomes

$$\int_{\mathbb{R}_+^{n+1}} e^{i\langle\eta',\xi'\rangle}\hat{J}_\alpha(\eta_{n+1}\xi_{n+1})\sigma_x(\xi)d\mu_\alpha(\xi) = e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma(x,\eta).$$

In view of Proposition 1.3.4, the last expression yields

$$\int_{\mathbb{R}_+^{n+1}} W_\alpha(\eta,-\xi)\sigma_x(\xi)d\mu_\alpha(\xi) = W_\alpha(x,-\eta)\sigma(x,\eta).$$

□

Theorem 2.3.4. *Let* $u \in S_*(\mathbb{R}_+^{n+1})$*. Then the pseudo-differential operator* T_σ *associated with symbol* $\sigma \in S^m, m \in \mathbb{R}$ *can be represented as*

$$(T_\sigma u)(x) = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} e^{i\langle \eta', \xi' \rangle} \hat{J}_\alpha(\eta_{n+1}\xi_{n+1}) \sigma_x(\xi) (\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta) d\mu_\alpha(\xi), \tag{2.3.12}$$

where $\sigma_x(\xi)$ *is defined in* (2.3.1) *and all involved integrals are convergent for* $k, p \in \mathbb{N}_0$*,* $l \in \mathbb{N}$ *such that* $p > \alpha + \frac{n}{2} + 1$ *and* $l > \alpha + \frac{n}{2} + 1$*.*

Proof. Using (2.2.6) and Lemma 2.3.3, we have

$$(T_\sigma u)(x) = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} W_\alpha(\eta, -\xi) \sigma_x(\xi) (\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta) d\mu_\alpha(\xi). \tag{2.3.13}$$

From Proposition 1.3.4, the last expression yields

$$(T_\sigma u)(x) = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} e^{i\langle \eta', \xi' \rangle} \hat{J}_\alpha(\eta_{n+1}\xi_{n+1}) \sigma_x(\xi) (\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta) d\mu_\alpha(\xi).$$

Next, we shall show that all involved integrals in (2.3.13) are absolutely convergent. Since $(\mathcal{F}_\alpha u) \in S_*(\mathbb{R}_+^{n+1})$, therefore

$$\big|(\mathcal{F}_\alpha u)(\eta)\big| \leq C(1 + \|\eta\|^2)^{-l}, \tag{2.3.14}$$

where C be a positive constant and $l \in \mathbb{N}$.

Therefore, we take

$$\big|(T_\sigma u)(x)\big| \leq \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} \big|W_\alpha(x, -\eta)\big| \big|\sigma_x(\xi)\big| \big|(\mathcal{F}_\alpha u)(\eta)\big| d\mu_\alpha(\eta) d\mu_\alpha(\xi).$$

Using (2.3.2) and (2.3.14), we have

$$\big|(T_\sigma u)(x)\big| \leq CE_{\alpha,m,p}A_\alpha^2(1+\|x\|^2)^{\frac{4p-k}{2}}\int_{\mathbb{R}_+^{n+1}}(1+\|\xi\|^2)^{\frac{2\alpha-2p+1}{2}}d\xi\int_{\mathbb{R}_+^{n+1}}(1+\|\eta\|^2)^{\frac{2\alpha-2l+1}{2}}d\eta.$$

Since the last integrals are convergent for $p > \alpha+\frac{n}{2}+1$ and $l > \alpha+\frac{n}{2}+1$.

Hence,

$$\big|(T_\sigma u)(x)\big| \leq E'_{\alpha,l,m,p}(1+\|x\|^2)^{\frac{4p-k}{2}},$$

where $E'_{\alpha,l,m,p}$ is a positive constant given by

$$E'_{\alpha,l,m,p} = CE_{\alpha,m,p}A_\alpha^2\int_{\mathbb{R}_+^{n+1}}(1+\|\xi\|^2)^{\frac{2\alpha-2p+1}{2}}d\xi\int_{\mathbb{R}_+^{n+1}}(1+\|\eta\|^2)^{\frac{2\alpha-2l+1}{2}}d\eta.$$

□

The Translation operator associated with the Weinstein transform

We define the basic function for the Weinstein transform by

$$\mathcal{D}_\alpha(x,y,z) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(x,t)W_\alpha(-y,t)W_\alpha(z,t)d\mu_\alpha(t). \tag{2.3.15}$$

Next, we shall show that $\mathcal{D}_\alpha(x,y,z)$ is well defined function for each $x,y,z\in\mathbb{R}_+^{n+1}$. In view of (1.3.10), above expression becomes

$$\begin{aligned}\mathcal{D}_\alpha(x,y,z) = &\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',t'\rangle}\hat{J}_\alpha(x_{n+1}t_{n+1})e^{i\langle y',t'\rangle}\hat{J}_\alpha(y_{n+1}t_{n+1})\\ &\times e^{-i\langle z',t'\rangle}\hat{J}_\alpha(z_{n+1}t_{n+1})d\mu_\alpha(t) \qquad (2.3.16)\\ = &\int_0^\infty \hat{J}_\alpha(x_{n+1}t_{n+1})\hat{J}_\alpha(y_{n+1}t_{n+1})\hat{J}_\alpha(z_{n+1}t_{n+1})t_{n+1}^{2\alpha+1}\frac{dt_{n+1}}{2^\alpha\Gamma(\alpha+1)}\\ &\times\int_{\mathbb{R}^n} e^{-i\langle x'-y'+z',t'\rangle}\frac{dt'}{(2\pi)^{\frac{n}{2}}}. \qquad (2.3.17)\end{aligned}$$

From [1, p. 81], we obtain

$$\mathcal{D}_\alpha(x,y,z) = (2\pi)^{\frac{n}{2}} D(x_{n+1}, y_{n+1}, z_{n+1})\delta(x' - y' + z'), \tag{2.3.18}$$

where

$$\begin{aligned} &D(x_{n+1}, y_{n+1}, z_{n+1}) \\ &\quad = \int_0^\infty \hat{J}_\alpha(x_{n+1}t_{n+1})\hat{J}_\alpha(y_{n+1}t_{n+1})\hat{J}_\alpha(z_{n+1}t_{n+1}) t_{n+1}^{2\alpha+1} \frac{dt_{n+1}}{2^\alpha \Gamma(\alpha+1)}, \end{aligned} \tag{2.3.19}$$

and $\delta(\cdot)$ is the Dirac delta function.

Using [1, p. 81], we have

$$\begin{aligned} \int_{\mathbb{R}_+^{n+1}} \mathcal{D}_\alpha(x,y,z) d\mu_\alpha z &= \int_0^\infty D(x_{n+1}, y_{n+1}, z_{n+1}) z_{n+1}^{2\alpha+1} \frac{dz_{n+1}}{2^\alpha \Gamma(\alpha+1)} \\ &\quad \times \int_{\mathbb{R}^n} \delta(x' - y' + z') dz' \\ &= 1. \end{aligned} \tag{2.3.20}$$

Lemma 2.3.5. *For all $x, y, t \in \mathbb{R}_+^{n+1}$, the following formula holds*

$$W_\alpha(x,t)W_\alpha(-y,t) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(-z,t)\mathcal{D}_\alpha(x,y,z) d\mu_\alpha(z). \tag{2.3.21}$$

Proof. From (2.3.16), we have

$$\mathcal{D}_\alpha(x,y,z) = \mathcal{F}_\alpha\big[e^{-i\langle x',t'\rangle}\hat{J}_\alpha(x_{n+1}t_{n+1})e^{i\langle y',t'\rangle}\hat{J}_\alpha(y_{n+1}t_{n+1})\big](z).$$

By invoking (1.4.2), we get

$$\begin{aligned} &e^{-i\langle x',t'\rangle}\hat{J}_\alpha(x_{n+1}t_{n+1})e^{i\langle y',t'\rangle}\hat{J}_\alpha(y_{n+1}t_{n+1}) \\ &\quad = \int_{\mathbb{R}_+^{n+1}} e^{i\langle z',t'\rangle}\hat{J}_\alpha(z_{n+1}t_{n+1})\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z). \end{aligned}$$

Using (1.3.10), we get

$$W_\alpha(x,t)W_\alpha(-y,t) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(-z,t)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z).$$

□

Theorem 2.3.6. *If* $\phi \in S_*(\mathbb{R}_+^{n+1})$, *then the translation of* ϕ *associated with the Weinstein transform can be expressed as*

$$(\tau_x^\alpha\phi)(y) = \int_{\mathbb{R}_+^{n+1}} \phi(z)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z). \tag{2.3.22}$$

Proof. From [42, p. 596], we have

$$\mathcal{F}_\alpha(\tau_x^\alpha\phi)(\xi) = W_\alpha(x,\xi)(\mathcal{F}_\alpha\phi)(\xi).$$

Exploiting (1.4.2), we get

$$(\tau_x^\alpha\phi)(y) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(x,\xi)W_\alpha(-y,\xi)(\mathcal{F}_\alpha\phi)(\xi)d\mu_\alpha(\xi). \tag{2.3.23}$$

By invoking (1.4.1), we obtain

$$(\tau_x^\alpha\phi)(y) = \int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}} W_\alpha(x,\xi)W_\alpha(-y,\xi)W_\alpha(z,\xi)\phi(z)d\mu_\alpha(\xi)d\mu_\alpha(z).$$

Therefore, using (2.3.15) we get

$$(\tau_x^\alpha \phi)(y) = \int_{\mathbb{R}_+^{n+1}} \phi(z)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z).$$

□

2.4 $L^p_{\alpha,s}(\mathbb{R}_+^{n+1})$- boundedness of pseudo-differential operators

In this section, $L^p_{\alpha,s}$- boundedness of pseudo-differential operators associated with the Weinstein transform is shown and its various properties studied.

Definition 2.4.1. Let $\sigma \in S^m, m \in \mathbb{R}$ then for each fixed $x \in \mathbb{R}_+^{n+1}$, $K_x(z)$ is defined by

$$K_x(z) = \int_{\mathbb{R}_+^{n+1}} W_\alpha(-z,\eta)\sigma(x,\eta)d\mu_\alpha(\eta), \quad z \in \mathbb{R}_+^{n+1}. \tag{2.4.1}$$

The convergent of (2.4.1) is shown by Lemma 2.4.2 which is useful in the proof of boundedness for T_σ.

Lemma 2.4.2. *Let $\sigma \in S^m, m \in \mathbb{R}$. Then the following inequality holds*

$$\left|K_x(z)\right| \leq C_{\alpha,m,p}(1+\|z\|^2)^{-p}(1+\|x\|^2)^{-\frac{k}{2}} \tag{2.4.2}$$

where $C_{\alpha,m,p}$ is a positive constant and $k,p,s \in \mathbb{N}_0$ such that $m < -2\alpha - n - s - 1$.

Proof. Using Binomial theorem, we get

$$(1+\|z\|^2)^p K_x(z) = \sum_{s=0}^{p} \binom{p}{s} \|z\|^{2s} K_x(z).$$

From (2.4.1), above expression becomes

$$(1+\|z\|^2)^p K_x(z) = \sum_{s=0}^{p} \binom{p}{s} \|z\|^{2s} \int_{\mathbb{R}_+^{n+1}} W_\alpha(-z,\eta)\sigma(x,\eta)d\mu_\alpha(\eta).$$

In view of (1.3.10), we get

$$(1+\|z\|^2)^p K_x(z) = \sum_{s=0}^{p} \binom{p}{s} \int_{\mathbb{R}_+^{n+1}} \|z\|^{2s} e^{i\langle z',\eta'\rangle} \hat{J}_\alpha(z_{n+1}\eta_{n+1})\sigma(x,\eta)d\mu_\alpha(\eta).$$

Using (1.3.9), we obtain

$$\begin{aligned}(1+\|z\|^2)^p K_x(z) &= \sum_{s=0}^{p} \binom{p}{s} \int_{\mathbb{R}_+^{n+1}} (-1)^s \Delta_{\alpha,n,\eta}^s e^{i\langle z',\eta'\rangle} \hat{J}_\alpha(z_{n+1}\eta_{n+1})\sigma(x,\eta)d\mu_\alpha(\eta)\\ &= \sum_{s=0}^{p} \binom{p}{s} (-1)^s \int_{\mathbb{R}_+^{n+1}} e^{i\langle z',\eta'\rangle} \hat{J}_\alpha(z_{n+1}\eta_{n+1}) \Delta_{\alpha,n,\eta}^s \sigma(x,\eta)d\mu_\alpha(\eta).\end{aligned}$$

Therefore,

$$(1+\|z\|^2)^p \big|K_x(z)\big| \leq \sum_{s=0}^{p} \binom{p}{s} \int_{\mathbb{R}_+^{n+1}} \Big|\Delta_{\alpha,n,\eta}^s \sigma(x,\eta)\Big| d\mu_\alpha(\eta). \tag{2.4.3}$$

From [19, p. 14], we find

$$\begin{aligned}(1+\|z\|^2)^p \big|K_x(z)\big| \leq& \sum_{s=0}^{p}\sum_{j=0}^{s}\sum_{r=1}^{2j} \binom{p}{s}\binom{s}{j} E'_{\alpha,r}\\ &\times \int_{\mathbb{R}_+^{n+1}} \Big|\eta_{n+1}^{r-s}(\Delta_n)_{\eta'}^{s-j} \frac{\partial^r}{\partial\eta_{n+1}^r}\sigma(x,\eta)\Big| d\mu_\alpha(\eta).\end{aligned} \tag{2.4.4}$$

From (2.3.6) and (2.4.4), we get

$$(1+\|z\|^2)^p\big|K_x(z)\big| \leq \sum_{s=0}^{p}\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{p}{s}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r} \times \int_{\mathbb{R}^{n+1}_+}\Big|\eta_{n+1}^{r-s}D_\eta^{2\delta'+r}\sigma(x,\eta)\Big|d\mu_\alpha(\eta),$$

where $2\delta'+r=(2\delta_1,\cdots,2\delta_n,r)\in\mathbb{N}_0^{n+1}$ and $2|\delta'|+r=2\delta_1+\cdots+2\delta_n+r$.

Since $\sigma\in S^m$, then from (2.2.4) we have

$$\begin{aligned}(1+\|z\|^2)^p\big|K_x(z)\big| &\leq \sum_{s=0}^{p}\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{p}{s}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}C_{\delta',r} \\ &\quad\times(1+\|x\|^2)^{-\frac{k}{2}}\int_{\mathbb{R}^{n+1}_+}|\eta_{n+1}|^{r-s}(1+\|\eta\|^2)^{\frac{m-(2|\delta'|+r)}{2}}d\mu_\alpha(\eta) \\ &\leq \sum_{s=0}^{p}\sum_{j=0}^{s}\binom{p}{s}\binom{s}{j}E'_{\alpha,2s}C_{2s}(1+\|x\|^2)^{-\frac{k}{2}}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^{\frac{m+s-1}{2}}d\mu_\alpha(\eta) \\ &\leq \sum_{s=0}^{p}\binom{p}{s}C''_{\alpha,s}(1+\|x\|^2)^{-\frac{k}{2}}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^{\frac{m+s-1}{2}}d\mu_\alpha(\eta),\end{aligned}$$

where $C''_{\alpha,s}$ is a positive constant.

Therefore,

$$\begin{aligned}(1+\|z\|^2)^p\big|K_x(z)\big| &\leq \sum_{s=0}^{p}\binom{p}{s}C''_{\alpha,s}A_\alpha(1+\|x\|^2)^{-\frac{k}{2}}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^{\frac{m+s-1}{2}}\eta_{n+1}^{2\alpha+1}d\eta \\ &\leq \sum_{s=0}^{p}\binom{p}{s}C''_{\alpha,s}A_\alpha(1+\|x\|^2)^{-\frac{k}{2}}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^{\frac{m+s+2\alpha}{2}}d\eta.\end{aligned}$$

The last integral is convergent for $m<-2\alpha-n-s-1$, therefore there exists a positive constant $C_{\alpha,m,p}$ such that

$$\big|K_x(z)\big|\leq C_{\alpha,m,p}(1+\|z\|^2)^{-p}(1+\|x\|^2)^{\frac{-k}{2}}.$$

□

Definition 2.4.3. Let $s \in \mathbb{R}$. Denote $L^p_{\alpha,s}(\mathbb{R}^{n+1}_+)$, $1 \leq p \leq \infty$ by the space of all functions $\phi \in S'_*(\mathbb{R}^{n+1}_+)$ which satisfy

$$\|\phi\|_{L^p_{\alpha,s}(\mathbb{R}^{n+1}_+)} = \left\|(1+\|\xi\|^2)^{\frac{s}{2}}\phi(\xi)\right\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{2.4.5}$$

From (2.4.5), we have

$$\|\phi\|_{L^p_{\alpha,0}(\mathbb{R}^{n+1}_+)} = \|\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{2.4.6}$$

Theorem 2.4.4. *Let $\alpha > -\frac{1}{2}$ and $p', q', r' \in [1,\infty]$ such that $\frac{1}{p'} + \frac{1}{q'} - \frac{1}{r'} = 1$. Then for $u \in S_*(\mathbb{R}^{n+1}_+)$, the pseudo-differential operator T_σ associated with symbol $\sigma \in S^m, m \in \mathbb{R}$ defined by (2.2.5) maps continuously $L^{p'}_\alpha(\mathbb{R}^{n+1}_+)$ to $L^{r'}_{\alpha,k}(\mathbb{R}^{n+1}_+)$. Moreover,*

$$\|T_\sigma u\|_{L^{r'}_{\alpha,k}(\mathbb{R}^{n+1}_+)} \leq C_{\alpha,k,m,p}\|(1+\|z\|^2)^{-p}\|_{L^{q'}_\alpha(\mathbb{R}^{n+1}_+)}\|u\|_{L^{p'}_\alpha(\mathbb{R}^{n+1}_+)}, \tag{2.4.7}$$

where $C_{\alpha,k,m,p}$ is a positive constant for $p \in \mathbb{N}$ and $q' > \frac{2\alpha+n+2}{2p}$.

Proof. From (2.3.9) and (2.3.1), we have

$$\big(T_\sigma u\big)(x) = \int_{\mathbb{R}^{n+1}_+} \sigma_x(\xi)u(\xi)d\mu_\alpha(\xi),$$

where

$$\sigma_x(\xi) = \int_{\mathbb{R}^{n+1}_+} W_\alpha(\eta,\xi)\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big]d\mu_\alpha(\eta).$$

Therefore,

$$\big(T_\sigma u\big)(x) = \int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+} W_\alpha(\eta,\xi)\Big[W_\alpha(x,-\eta)\sigma(x,\eta)\Big]u(\xi)d\mu_\alpha(\eta)d\mu_\alpha(\xi).$$

Using (2.3.21), above expression becomes

$$\begin{aligned}\big(T_\sigma u\big)(x) &= \int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}}\Big(\int_{\mathbb{R}_+^{n+1}} W_\alpha(-z,\eta)\mathcal{D}_\alpha(x,\xi,z)d\mu_\alpha(z)\Big)\sigma(x,\eta)\\ &\qquad\qquad\times u(\xi)d\mu_\alpha(\eta)d\mu_\alpha(\xi)\\ &= \int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}}\Big(\int_{\mathbb{R}_+^{n+1}} W_\alpha(-z,\eta)\sigma(x,\eta)d\mu_\alpha(\eta)\Big)\mathcal{D}_\alpha(x,\xi,z)\\ &\qquad\qquad\times u(\xi)d\mu_\alpha(z)d\mu_\alpha(\xi).\end{aligned}$$

From the relation (2.4.1), the last expression yields

$$\big(T_\sigma u\big)(x) = \int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}} K_x(z)\mathcal{D}_\alpha(x,\xi,z)u(\xi)d\mu_\alpha(\xi)d\mu_\alpha(z). \tag{2.4.8}$$

In view of an inequality (2.4.2), (2.4.8) finds

$$\begin{aligned}\big|\big(T_\sigma u\big)(x)\big| \leq C_{\alpha,m,p}\int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}} &(1+\|z\|^2)^{-p}(1+\|x\|^2)^{-\frac{k}{2}}\mathcal{D}_\alpha(x,\xi,z)\\ &\times |u(\xi)|d\mu_\alpha(\xi)d\mu_\alpha(z).\end{aligned}$$

If $F(z) = (1+\|z\|^2)^{-p}$, then from (1.4.11) above expression becomes

$$\Big|(1+\|x\|^2)^{\frac{k}{2}}\big(T_\sigma u\big)(x)\Big| \leq C_{\alpha,m,p}(F *_W |u|)(x).$$

Therefore,

$$\Big(\int_{\mathbb{R}_+^{n+1}}\Big|(1+\|x\|^2)^{\frac{k}{2}}\big(T_\sigma u\big)(x)\Big|^{r'}d\mu_\alpha(x)\Big)^{\frac{1}{r'}} \leq C_{\alpha,m,p}\Big(\int_{\mathbb{R}_+^{n+1}}(F *_W |u|)^{r'}(x)d\mu_\alpha(x)\Big)^{\frac{1}{r'}}.$$

Using (1.4.17) for $F \in L_\alpha^{q'}(\mathbb{R}_+^{n+1})$ and $|u| \in L_\alpha^{p'}(\mathbb{R}_+^{n+1})$, we obtain

$$\begin{aligned}\|(1+\|x\|^2)^{\frac{k}{2}}\big(T_\sigma u\big)(x)\|_{L_\alpha^{r'}(\mathbb{R}_+^{n+1})} &\leq C_{\alpha,k,m,p}\|F *_W |u|\|_{L_\alpha^{r'}(\mathbb{R}_+^{n+1})} \\ &\leq C_{\alpha,k,m,p}\|F\|_{L_\alpha^{q'}(\mathbb{R}_+^{n+1})}\|u\|_{L_\alpha^{p'}(\mathbb{R}_+^{n+1})}, \end{aligned} \tag{2.4.9}$$

where $1 \leq p' \leq \infty$ such that $\frac{1}{p'} + \frac{1}{q'} - \frac{1}{r'} = 1$ and $q' > \frac{2\alpha+n+2}{2p}$.

Therefore, from (2.4.5), we obtain

$$\|T_\sigma u\|_{L_{\alpha,k}^{r'}(\mathbb{R}_+^{n+1})} \leq C_{\alpha,k,m,p}\|(1+\|z\|^2)^{-p}\|_{L_\alpha^{q'}(\mathbb{R}_+^{n+1})}\|u\|_{L_\alpha^{p'}(\mathbb{R}_+^{n+1})}.$$

□

2.5 Pseudo-differential Operators associated with Heat equation

In this section, pseudo-differential operators associated with heat equation involving Weinstein transform is studied.

Proposition 2.5.1. *Let $f \in S_*(\mathbb{R}_+^{n+1})$. The pseudo-differential operator $u(x,t) = (T_{\sigma_t}f)(x)$, solves the Weinstein heat equation*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta_{\alpha,n,x}u(x,t), & (x,t) \in \mathbb{R}_+^{n+1} \times (0,\infty) \\ \\ u(x,0) = f(x), & \forall x \in \mathbb{R}_+^{n+1} \end{cases} \tag{2.5.1}$$

where

$$(T_{\sigma_t} f)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x^{'},\xi^{'}\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma_t(\xi)(\mathcal{F}_\alpha f)(\xi)d\mu_\alpha(\xi), \tag{2.5.2}$$

and $\sigma_t(\xi) = e^{-t\|\xi\|^2}, \ \forall t > 0.$

Proof Using the Weinstein transform and their inversion on $S_*(\mathbb{R}_+^{n+1})$, we have

$$u(x,t) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x^{'},\xi^{'}\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})(\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi).$$

Therefore,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x^{'},\xi^{'}\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\frac{\partial}{\partial t}\Big((\mathcal{F}_\alpha u)(\xi,t)\Big)d\mu_\alpha(\xi), \tag{2.5.3}$$

and

$$\begin{aligned}\Delta_{\alpha,n,x} u(x,t) &= \int_{\mathbb{R}_+^{n+1}} \Big(\Delta_{\alpha,n,x} e^{i\langle x^{'},\xi^{'}\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big)(\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}_+^{n+1}} \Big(\Big[\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}\Big] e^{i\sum_{k=1}^{n} x_k\xi_k} \\ &\qquad \times \hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big)(\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi).\end{aligned}$$

Thus,

$$\begin{aligned}\Delta_{\alpha,n,x} u(x,t) = \int_{\mathbb{R}_+^{n+1}} \Big(&\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} e^{i\sum_{k=1}^{n} x_k\xi_k} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) + e^{i\sum_{k=1}^{n} x_k\xi_k} \\ &\times \Big[\frac{\partial^2}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}\Big]\hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big) \\ &\times (\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi).\end{aligned}$$

From (1.3.9), we get

$$\begin{aligned}\Delta_{\alpha,n,x}u(x,t) &= \int_{\mathbb{R}_+^{n+1}} \Big(\sum_{j=1}^{n}(i\xi_j)^2 e^{i\sum_{k=1}^{n}x_k\xi_k}\hat{J}_\alpha(x_{n+1}\xi_{n+1}) + e^{i\langle x',\xi'\rangle} \\ &\qquad \times (-\xi_{n+1}^2)\hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big)(\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}_+^{n+1}} (-\|\xi\|^2)e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi). \qquad (2.5.4)\end{aligned}$$

Using (2.5.3) and (2.5.4) in (2.5.1), we have

$$\begin{aligned}&\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\frac{\partial}{\partial t}\Big((\mathcal{F}_\alpha u)(\xi,t)\Big)d\mu_\alpha(\xi) \\ &\qquad = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(-\|\xi\|^2)(\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi),\end{aligned}$$

$$\begin{aligned}&\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\frac{\partial}{\partial t}\Big((\mathcal{F}_\alpha u)(\xi,t)\Big)d\mu_\alpha(\xi) \\ &\qquad + \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(\|\xi\|^2)(\mathcal{F}_\alpha u)(\xi,t)d\mu_\alpha(\xi) = 0.\end{aligned}$$

It gives

$$\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big(\frac{\partial}{\partial t}(\mathcal{F}_\alpha u)(\xi,t) + \|\xi\|^2(\mathcal{F}_\alpha u)(\xi,t)\Big)d\mu_\alpha(\xi) = 0.$$

Therefore,

$$\Big(\frac{\partial}{\partial t} + \|\xi\|^2\Big)(\mathcal{F}_\alpha u)(\xi,t)) = 0. \qquad (2.5.5)$$

For fixed value of ξ, the solution of (2.5.5)

$$(\mathcal{F}_\alpha u)(\xi,t) = C(\xi)e^{-t\|\xi\|^2}. \qquad (2.5.6)$$

From (1.4.1), the initial condition gives

$$\big(\mathcal{F}_\alpha u\big)(\xi,0)=\big(\mathcal{F}_\alpha f\big)(\xi). \tag{2.5.7}$$

Using (2.5.7) in (2.5.6), we find out $C(\xi)=\big(\mathcal{F}_\alpha f\big)(\xi)$.

Therefore, (2.5.6) becomes

$$(\mathcal{F}_\alpha u)(\xi,t)=\big(\mathcal{F}_\alpha f\big)(\xi)e^{-t\|\xi\|^2}. \tag{2.5.8}$$

Since $f\in S_*(\mathbb{R}^{n+1}_+)$, then from (1.4.6), we have $(\mathcal{F}_\alpha f)\in S_*(\mathbb{R}^{n+1}_+)$.

Taking inverse Weinstein transform in (2.5.8), we get

$$\begin{aligned} u(x,t)&=\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})e^{-t\|\xi\|^2}(\mathcal{F}_\alpha f)(\xi)d\mu_\alpha(\xi)\\ &=\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma_t(\xi)(\mathcal{F}_\alpha f)(\xi)d\mu_\alpha(\xi). \end{aligned}$$

This shows that, $u(x,t)=(T_{\sigma_t}f)(x)$ is the solution of heat equation (6.4.1) for any initial function f.

Conclusion.

The Weinstein transform contains strong calculus and it is used in many problems of mathematics, particularly in wavelets, partial-differential equations and distributions. Taking the theory of aforesaid transform, many authors found useful observations in harmonic analysis and other areas. In their research works, Chettaoui-Trimèche [10], Hleili [24], Hleili-Hleili [25], Saudi [66] expounded different problems in their papers. In the present chapter, authors found the theory of pseudo-differential operators associated with the Weinstein transform and studied its properties. Utilizing the aforesaid results, the Sobolev type of spaces and associated results were discussed in this chapter. The solution of the heat equation is obtained and it is expressed in the form of pseudo-differential operators by using the Weinstein transform technique.

***