APPROXIMATION TECHNIQUES AND ALGORITHMS FOR THE FRACTIONAL ORDER MATHEMATICAL MODELS

Thesis submitted in partial fulfillment for the

Award of Degree

DOCTOR OF PHILOSOPHY

By

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Dedicated to

My Parents & Brothers

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It is certified that the work contained in this thesis titled **"Approximation Techniques and Algorithms for the Fractional Order Mathematical Models"** by **Mr. Nikhil Srivastava** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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DECLARATION BY THE CANDIDATE

I, **Nikhil Srivastava**, certify that the work embodied in this thesis is my own bona fide work and carried out by me under the supervision of **Dr. Vineet Kumar Singh** from **July 2018** to **March 2023** at the Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not wilfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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It is certified that the above statement made by the candidate is correct to the best of my knowledge.

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LIST OF SYMBOLS

$\mathbb{Z}^+$	Set of non-negative integers
$\otimes$	Kronecker product
Γ	Gamma function
$(-\Delta)^\alpha$	Fractional Laplacian operator
${}_0^{RZ}D_t^\alpha u(t)$	Riesz fractional derivative of the function $u(t)$ of order α.
${}_0^{C}D_t^\alpha u(t)$	Caputo fractional derivative of the function $u(t)$ of order α.
$\|.\|_2$	L_2 norm
$\|.\|_\infty$	L_∞ norm
h	Step size in spatial direction
τ	Step size in time direction
Ψ	Basis w.r.t. shifted Legendre polynomial
Φ	Basis w.r.t. shifted Chebyshev polynomial of second kind.

PREFACE

The thesis consists of six chapters. **Chapter 1**, provides a brief introduction of the thesis. Basic definition of fractional integral and derivatives including the literature survey of various fractional partial differential equations are provided. The challenges and motivation behind choosing the topic and problem statement of the thesis are explained.

Chapter 2 presents numerical algorithms for the Riesz-space fractional partial differential equations based on the combination of finite difference and operational matrix technique. In this chapter, we construct two efficient numerical schemes by combining the finite difference method and operational matrix method (OMM) to solve Riesz-space fractional diffusion equation (RFDE) and Riesz-space fractional advection-dispersion equation (RFADE) with initial and Dirichlet boundary conditions. We applied matrix transform method (MTM) for discretization of Riesz-space fractional derivative and OMM based on shifted Legendre polynomials (SLP) and shifted Chebyshev polynomial (SCP) of second kind for approximating the time derivatives. The proposed schemes transform the RFDE and RFADE into the system of linear algebraic equations. For a better understanding of the methods, numerical algorithms are also provided for the considered problems. Furthermore, optimal error bound for the numerical approximation is derived. The stability of the schemes is also established numerically. The schemes are observed to be of second-order accurate in space. The effectiveness and accuracy of the schemes are tested by taking two numerical examples of RFDE and RFADE and found to be in good agreement with the exact solutions. It is observed that the numerical schemes are simple, easy to implement, yield high accurate results with both the basis functions. Moreover, the CPU time taken by the schemes with SLP basis is very less as compared to schemes with SCP basis. The content of this chapter is published in **"Applied Numerical Mathematics, 161 (2021) 244-274"**.

Chapter 3 is based on a computational algorithm for financial mathematical model based on the European option. In this chapter, the computational approach is based on the combination of finite difference with the operational matrix approach for solving the time-fractional Black-Scholes model (TFBSM) arising in the financial market. In order to discretized the Caputo time fractional derivative, the L1-2 approximation is used. To approximate the space derivatives, the OMM based on SLP and SCP is used. The proposed algorithm transforms the TFBSM into a system of algebraic equations, which can be solved easily to get the numerical solution. Furthermore, theoretical unconditional stability and convergence of the numerical scheme are established for $0 < \alpha \leq \bar{\alpha}$ and the stability of the proposed algorithm is also verified numerically. Various numerical problems, including the European double barrier call option and the European call and put option is provided to validate our numerical scheme. The effect of different parameters like volatility, interest rate, fractional order, etc., on the option pricing, is also investigated. A comparative study of the numerical results by the proposed algorithm with the existing schemes given in [1] and [2] is also provided to show its effectiveness and accuracy. The content of this chapter is published in **"Mathematical Sciences"** DOI: https://doi.org/10.1007/s40096-022-00474-0.

Chapter 4 presents L3 approximation of the Caputo derivative and its application to time-fractional wave equation-(I). In this chapter, we have developed two new approximations of the Caputo derivatives of order $\alpha \in (1, 2)$ (called L3 and modified L3) for the Caputo fractional derivative of order $\alpha \in (1, 2)$. We have used the cubic interpolating polynomial on uniform grid points $[(\mathfrak{t}_{j-2}, \mathfrak{U}_{j-2})$,$(\mathfrak{t}_{j-1}, \mathfrak{U}_{j-1})$,$(\mathfrak{t}_j, \mathfrak{U}_j)$,

$(\mathfrak{t}_{j+1}, \mathfrak{U}_{j+1})]$ for $2 \le j \le \mathsf{k} - 1$, while the quadratic interpolating polynomial is applied on the first interval $[\mathfrak{t}_0, \mathfrak{t}_1]$. We have modified the L3 approximation by using cubic Hermite interpolation in the sub-interval $[\mathfrak{t}_0, \mathfrak{t}_2]$ and name it as "Modified L3" (ML3) approximation. The novel L3 and ML3, both approximation are second order accurate for all α. Both approximations are tested on various examples and gives highly accurate results. Later, using this L3 approximation, a difference scheme is proposed to solve the time-fractional wave equation (TFWE). The proposed difference scheme is second order accurate in space and time for all α. The scheme is again tested on three numerical problems of TFWE, and the comparative study of the numerical results by the proposed scheme with some existing schemes is also provided to show the effectiveness and accuracy of our scheme. The content of this chapter is published in **"Mathematics and Computers in Simulation, 205 (2023) 532-557"**.

Chapter 5 presents a new difference scheme for time-fractional telegraph equation. We have applied L3 approximation and L123 approximation to approximate the Caputo derivative of order $\alpha \in (1, 2)$ and $\beta \in (0, 1)$ respectively. To discretize the spatial derivative, central difference scheme is used. The proposed difference scheme is of second order in space and time direction for all α. Four different test problems of TFTE are given to validate the efficacy and accuracy of the difference scheme. To demonstrate the effectiveness of scheme, a comparative study is also provided with the existing schemes [3]. The content of this chapter is communicated in a reputed international journal.