Chapter 5

Construction of Quantum and LCD Codes from Skew Constacyclic Codes over a Class of Non-chain Rings

After thoroughly examining skew constacyclic codes over $\mathcal{T}$ in Chapter 4, the focus in Chapter 5 shifts to exploring their applications in constructing quantum and LCD codes. This chapter is divided into three main sections. In Sections 5.1 and 5.2, we investigate the Euclidean and Hermitian duals of skew constacyclic codes over $\mathcal{T}$, respectively. We establish dual-containing criteria for these codes and, using these criteria, develop methods to construct quantum codes. As a result, we present numerous new quantum codes, including several MDS codes and others that outperform existing codes. Additionally, we demonstrate the construction of quantum codes over nine different rings that belong to the general class of rings $\mathcal{T}$.

In Section 5.3, we explore skew constacyclic LCD codes over $\mathcal{T}$. We provide methods to construct both Euclidean and Hermitian LCD codes and prove that the Gray image of LCD codes over $\mathcal{T}$ is also an LCD code over $\mathbb{F}_q$. Consequently, we find numerous LCD codes, derived as Gray images of skew constacyclic LCD codes over $\mathcal{T}$, including some MDS, AMDS, and BKLCs. These codes are particularly notable because they are not only not only LCD but also skew quasi-$(\alpha_{11\ldots1}, \ldots, \alpha_{l_1 l_2 \ldots l_r})$-twisted codes, adding further interest to their structure and potential applications.

5.1 Euclidean dual of skew (Θ, α)-constacyclic codes over $\mathcal{T}$ and construction of Quantum codes

This section is primarily concerned with the construction of quantum codes from Euclidean dual-containing skew constacyclic codes over the ring $\mathcal{T}$. Initially, we establish some important results on Euclidean dual of skew constacyclic codes over $\mathcal{T}$. Further, we characterize Euclidean dual-containing skew constacyclic codes over $\mathcal{T}$. We then present a method to generate quantum codes from dual-containing skew constacyclic codes over $\mathcal{T}$, and we finally employ this method to produce some novel quantum codes, some of which are MDS and some outperform the existing ones.

Lemma 5.1.1. *Let $\Theta \in Aut(\mathcal{T})$, n be a multiple of $o(\Theta)$ and $\alpha \in \mathcal{U}(\mathcal{T})$ be such that $\Theta(\alpha) = \alpha$. Then the Euclidean dual of a skew (Θ, α)-constacyclic code $\mathcal{C}$ of length n over $\mathcal{T}$, denoted as $\mathcal{C}^{\perp_E}$ is a skew (Θ, α^{-1})-constacyclic code of length n over $\mathcal{T}$.*

Proof. Let $\mathbf{v} = (\mathsf{v}^0, \mathsf{v}^1, \ldots, \mathsf{v}^{n-1}) \in \mathcal{C}$ and $\mathbf{w} = (\mathsf{w}^0, \mathsf{w}^1, \ldots, \mathsf{w}^{n-1}) \in \mathcal{C}^{\perp_E}$ be arbitrary elements. Since $\mathcal{C}$ is given to be a skew (Θ, α)-constacyclic code, $\sigma^{n-1}_{(\Theta,\ \alpha)}(\mathbf{v}) =$

$(\alpha\Theta^{n-1}(\mathsf{v}^1), \alpha\Theta^{n-1}(\mathsf{v}^2), \dots, \alpha\Theta^{n-1}(\mathsf{v}_{n-1}), \Theta^{n-1}(\mathsf{v}^0)) \in \mathcal{C}$. Therefore, we have

$$\begin{aligned}
0 &= \left\langle \sigma^{n-1}_{(\Theta,\ \alpha)}(\mathbf{v}), \mathbf{w} \right\rangle_E \\
&= \left\langle \left(\alpha\Theta^{n-1}(\mathsf{v}^1), \alpha\Theta^{n-1}(\mathsf{v}^2), \dots, \alpha\Theta^{n-1}(\mathsf{v}_{n-1}), \Theta^{n-1}(\mathsf{v}^0)\right), \left(\mathsf{w}^0, \mathsf{w}^1, \dots, \mathsf{w}^{\mathsf{n-1}}\right) \right\rangle_E \\
&= \alpha \left(\sum_{k=1}^{n-1} \Theta^{n-1}(\mathsf{v}_k)\mathsf{w}_{k-1} + \alpha^{-1}\Theta^{n-1}(\mathsf{v}_0)\mathsf{w}_{n-1} \right)
\end{aligned}$$

Now since, $o(\Theta)|n$ and Θ fixes α, applying Θ on both sides, we obtain

$$\begin{aligned}
0 &= \alpha\Theta \left(\sum_{k=1}^{n-1} \Theta^{n-1}(\mathsf{v}_k)\mathsf{w}_{k-1} + \alpha^{-1}\Theta^{n-1}(\mathsf{v}_0)\mathsf{w}_{n-1} \right) \\
0 &= \alpha \left(\sum_{k=1}^{n-1} \mathsf{v}_k\Theta(\mathsf{w}_{k-1}) + \alpha^{-1}\mathsf{v}_0\Theta(\mathsf{w}_{n-1}) \right) \\
0 &= \sum_{k=1}^{n-1} \mathsf{v}_k\Theta(\mathsf{w}_{k-1}) + \alpha^{-1}\mathsf{v}_0\Theta(\mathsf{w}_{n-1}) \text{ (because } \alpha \text{ is a unit.)} \\
0 &= \left\langle (\mathsf{v}_0, \mathsf{v}_1, \dots, \mathsf{v}_{n-1}), \left(\alpha^{-1}\Theta(\mathsf{w}_{n-1}), \Theta(\mathsf{w}_0), \dots, \Theta(\mathsf{w}_{n-2})\right) \right\rangle_E \\
0 &= \left\langle \mathbf{v}, \sigma_{(\Theta,\ \alpha^{-1})}(\mathbf{w}) \right\rangle_E
\end{aligned}$$

Hence, $\forall\ \mathbf{w} \in \mathcal{C}^{\perp_E}$, $\sigma_{(\Theta,\ \alpha^{-1})}(\mathbf{w}) \in \mathcal{C}^{\perp_E}$. This proves that $\mathcal{C}^{\perp_E}$ is a skew (Θ, α^{-1})-constacyclic code of length n over $\mathcal{T}$. □

Remark 5.1.2. In particular, if $\alpha^2 = 1$ then $\mathcal{C}^{\perp_E}$ is also skew (Θ, α)-constacyclic.

Lemma 5.1.3. [19] Let C be skew (θ_t, β)-constacyclic code of length n over $\mathbb{F}_q$ generated by $f(y) \in \mathbb{F}_q[y; \theta_t]/\langle y^n - \beta\rangle$ such that $\theta_t(\beta) = \beta$, $\beta^2 = 1$, $o(\theta_t) = t$ divides n and $f(y) = \sum_{i=0}^{k} f_i y^i$. Then the Euclidean dual of C, denoted as $C^{\perp_E}$, is generated by the monic skew reciprocal polynomial of $g(y)$ defined as $g^{\dagger}(y) := \frac{1}{\theta_t^k(g_0)} \sum_{i=0}^{k} \theta_t^i(g_{k-i}) y^i$, where $g(y)f(y) = y^n - \beta$.

Theorem 5.1.4. Let $\Theta_t \in \mathrm{Aut}(\mathcal{T})$, n be a multiple of $o(\Theta_t)$, and $\alpha \in \mathcal{U}(\mathcal{T})$ be such that $\Theta_t(\alpha) = \alpha$ and $\alpha = \sum_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \alpha_{i_1 i_2 \dots i_r}$. Furthermore, let $\mathcal{C} =$

$\bigoplus_{i_1i_2\ldots i_r}\eta_{i_1i_2\ldots i_r}\mathcal{C}_{i_1i_2\ldots i_r}$ be a skew (Θ_t,α)-constacyclic code of length n over $\mathcal{T}$ such that $\mathcal{C}_{i_1i_2\ldots i_r}=\langle f_{i_1i_2\ldots i_r}(y)\rangle$ and $f_{i_1i_2\ldots i_r}(y)g_{i_1i_2\ldots i_r}(y)=y^n-\alpha_{i_1i_2\ldots i_r}$, for $i_j\in\{1,\ 2,\ldots,\ l_j\}$, $j=1,2,\ldots,r$. Then

(i) $\mathcal{C}^{\perp_E}=\langle g^{\dagger}(y)\rangle$, where $g^{\dagger}(y)=\sum_{i_1i_2\ldots i_r}\eta_{i_1i_2\ldots i_r}g^{\dagger}_{i_1i_2\ldots i_r}(y)$;

(ii) $|\mathcal{C}^{\perp_E}|=q^L$, where $L=\sum_{i_1,i_2,\ldots,i_r}deg(f_{i_1i_2\ldots i_r}(y))$.

Proof. The proof follows from Theorems 4.2.4, 4.2.5, Lemma 5.1.1 and Lemma 5.1.3. □

Lemma 5.1.5. ([31]) Let C be a skew (θ_t,β)-constacyclic code of length n over $\mathbb{F}_q$ such that $ord(\theta_t)|n$ and $\beta\in\mathbb{F}_q^*$ is such that $\beta^2=1$. If $f(y)$ is the generator polynomial of C such that $g(y)f(y)=y^n-\beta$. Then C contains its Euclidean dual if and only if $g^{\dagger}(y)g(y)$ is divisible by $y^n-\alpha$ from the right, where $g^{\dagger}(y)$ denotes the left monic skew reciprocal polynomial of $g(y)$.

Theorem 5.1.6. Let $\Theta_t\in\mathrm{Aut}(\mathcal{T})$, n be a multiple of $o(\Theta_t)$, and $\alpha\in\mathcal{U}(\mathcal{T})$ be such that $\alpha=\sum_{i_1i_2\ldots i_r}\eta_{i_1i_2\ldots i_r}\alpha_{i_1i_2\ldots i_r}$ and $\alpha^2=1$. Furthermore, let $\mathcal{C}=\bigoplus_{i_1i_2\ldots i_r}\eta_{i_1i_2\ldots i_r}\mathcal{C}_{i_1i_2\ldots i_r}$ be a skew (Θ_t,α)-constacyclic code of length n over $\mathcal{T}$ such that $\mathcal{C}_{i_1i_2\ldots i_r}=\langle f_{i_1i_2\ldots i_r}(y)\rangle$ and $f_{i_1i_2\ldots i_r}(y)g_{i_1i_2\ldots i_r}(y)=y^n-\alpha_{i_1i_2\ldots i_r}$, $i_j\in\{1,2,\ldots,l_j\}$, $j=1,2,\ldots,r$. Then

(i) $\mathcal{C}$ contains its Euclidean dual if and only if $g^{\dagger}_{i_1i_2\ldots i_r}(y)g_{i_1i_2\ldots i_r}(y)$ is divisible by $y^n-\alpha_{i_1i_2\ldots i_r}$ from right, for all $i_j\in\{1,2,\ldots,l_j\},j=1,2,\ldots,r$.

(ii) If $g^{\dagger}_{i_1i_2\ldots i_r}(y)g_{i_1i_2\ldots i_r}(y)$ is divisible by $y^n-\alpha_{i_1i_2\ldots i_r}$ from right, for all $i_j\in\{1,2,\ldots,l_j\},j=1,2,\ldots,r$ then there exists an $[[N,K,D]]_q$ quantum code, where $N=l_1l_2\ldots l_rn$, $K=l_1l_2\ldots l_rn-2\sum_{i_1i_2\ldots i_r}deg(f_{i_1i_2\ldots i_r}(y))$ and $D\geq d_L$, the Lee distance of $\mathcal{C}$.

Proof. (i) Let us suppose that $\mathcal{C}^{\perp_E} \subseteq \mathcal{C}$. Then by Theorem 4.1.9(iii), we get

$$\bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}^{\perp_E}_{i_1 i_2 \dots i_r} \subseteq \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}.$$

Now taking modulo $\eta_{i_1 i_2 \dots i_r}$ both sides, we get $\mathcal{C}^{\perp_E}_{i_1 i_2 \dots i_r} \subseteq \mathcal{C}_{i_1 i_2 \dots i_r}$, for all $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$. Then by Lemma 5.1.5, $g^{\dagger}_{i_1 i_2 \dots i_r}(y) g_{i_1 i_2 \dots i_r}(y)$ is divisible by $y^n - \alpha_{i_1 i_2 \dots i_r}$ from right, $\forall\ i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$. Conversely, if $g^{\dagger}_{i_1 i_2 \dots i_r}(y) g_{i_1 i_2 \dots i_r}(y)$ is divisible by $y^n - \alpha_{i_1 i_2 \dots i_r}$ from right, $\forall\ i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$ then again by Lemma 5.1.5, $\mathcal{C}^{\perp_E}_{i_1 i_2 \dots i_r} \subseteq \mathcal{C}_{i_1 i_2 \dots i_r}$, for all $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$. Thus, $\bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}^{\perp_E}_{i_1 i_2 \dots i_r} \subseteq \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$ and hence by Theorem 4.1.9(iii), $\mathcal{C}^{\perp_E} \subseteq \mathcal{C}$.

(ii) If $g^{\dagger}_{i_1 i_2 \dots i_r}(y) g_{i_1 i_2 \dots i_r}(y)$ is divisible by $y^n - \alpha_{i_1 i_2 \dots i_r}$ from right, $\forall\ i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$ then by part (i), $\mathcal{C}^{\perp_E} \subseteq \mathcal{C}$. Therefore by Theorem 4.1.9(v), we have $\Phi(\mathcal{C})^{\perp_E} = \Phi(\mathcal{C}^{\perp_E}) \subseteq \Phi(\mathcal{C})$. Then by Theorems 4.1.9(i) and 4.2.8, $\Phi(\mathcal{C})$ is an $[l_1 l_2 \dots l_r n,\ l_1 l_2 \dots l_r n - \sum_{i_1 i_2 \dots i_r} deg(f_{i_1 i_2 \dots i_r}(y)),\ d_L]$ dual-containing skew quasi-$(\alpha_{11\dots1}, \dots, \alpha_{l_1 l_2 \dots l_r})$-twisted code over $\mathbb{F}_q$. Hence, by Lemma 1.3.6, there exists a quantum code with parameters $[[N, K, D]]_q$, where $N = l_1 l_2 \dots l_r n$, $K = l_1 l_2 \dots l_r n - 2 \sum_{i_1 i_2 \dots i_r} deg(f_{i_1 i_2 \dots i_r}(y))$ and $D \geq d_L$, the Lee distance of $\mathcal{C}$.

□

Now we present an example to justify and explain the construction of a new quantum code utilizing the technique discussed in this section.

Example 5.1.7. For $q = 27$, consider $\mathbb{F}_q = \mathbb{F}_{27} = \mathbb{F}_3[X]/\langle X^3 + 2X + 1 \rangle = \mathbb{F}_3(w)$, where $w^3 + 2w + 1$. Further, let $\mathcal{T} = T_3 = \mathbb{F}_{27}[u_1, u_2, u_3]/\langle u_1^3 - u_1, u_2^2 - u_2, u_3^2 - u_3, u_1 u_2 - u_2 u_1, u_1 u_3 - u_3 u_1, u_2 u_3 - u_3 u_2 \rangle$. Then $\eta_{111} = (1 - u_1^2) u_2 u_3$, $\eta_{112} = (1 - u_1^2) u_2 (1 - u_3)$, $\eta_{121} = (1 - u_1^2)(1 - u_2) u_3$, $\eta_{122} = (1 - u_1^2)(1 - u_2)(1 - u_3)$, $\eta_{211} =$

$(\frac{u_1^2+u_1}{2})u_2u_3$, $\eta_{212} = (\frac{u_1^2+u_1}{2})u_2(1-u_3)$, $\eta_{221} = (\frac{u_1^2+u_1}{2})(1-u_2)u_3$, $\eta_{222} = (\frac{u_1^2+u_1}{2})(1-u_2)(1-u_3)$, $\eta_{311} = (\frac{u_1^2-u_1}{2})u_2u_3$, $\eta_{312} = (\frac{u_1^2-u_1}{2})u_2(1-u_3)$, $\eta_{321} = (\frac{u_1^2-u_1}{2})(1-u_2)u_3$ and $\eta_{322} = (\frac{u_1^2-u_1}{2})(1-u_2)(1-u_3)$ are the primitive orthogonal idempotents. Thus every element $\mathsf{v} \in \mathcal{T}$ can be uniquely written as $\mathsf{v} = \sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\eta_{i_1i_2i_3}v_{i_1i_2i_3}$. Let us define an automorphism $\Theta_3 : \mathcal{T} \to \mathcal{T}$ as

$$\sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\eta_{i_1i_2i_3}v_{i_1i_2i_3} \longmapsto \sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\eta_{i_1i_2i_3}v_{i_1i_2i_3}^{3}.$$

For $n = 6$, consider a factorisation of $y^n - 1 = y^6 - 1 \in \mathbb{F}_{27}[y;\theta_3]$ given as

$$y^6 - 1 = (y + w^2 + 1)(y + w)(y + 2w^2 + w)(y + 2w)(y + 2w + 1)(y + 2w + 2).$$

Let $u_1(y) = 1$ and $v_1(y)u_1(y) = y^6 - 1$. Then $v_1(y) = y^6 - 1$ and $v_1^\dagger(y) = y^6 - 1$ and so $v_1^\dagger(y)v_1(y)$ is divisible by $y^6 - 1$ from right. Let $u_2(y) = y + 2w + 2 = y + w^{22}$ and $v_2(y)u_2(y) = y^6 - 1$, then $v_2(y) = (y + w^2 + 1)(y + w)(y + 2w^2 + w)(y + 2w)(y + 2w + 1) = y^5 + (w + 2)y^4 + (w^2 + 2w)y^3 + 2y^2 + (2w + 1)y + 2w^2 + w$ and $v_2^\dagger(y) = y^5 + (2w^2 + 1)y^4 + (2w + 1)y^3 + 2y^2 + (w^2 + 2)y + w + 2$. Then $v_2^\dagger(y)v_2(y) = (y^4 + (2w^2 + w + 1)y^3 + 2y + w^2 + 2w + 2)(y^6 - 1)$ i.e. $v_2^\dagger(y)v_2(y)$ divisible by $y^6 - 1$ from right.

Next, let us consider one of the factorization of $y^6 + 1 \in \mathbb{F}_{27}[y;\theta_3]$ given as $y^6 + 1 = (y^2 + 2w^2y + w^2)(y^2 + w^2)(y^2 + (w^2 + 2w + 1)y + 2w + 2)$. Let $u_3(y) = y^2 + (w^2 + 2w + 1)y + 2w + 2 = y^2 + w^{18}y + w^{22}$ and $v_3(y)u_3(y) = y^6 + 1$. Then $v_3(y) = (y^2 + 2w^2y + w^2)(y^2 + w^2) = y^4 + 2w^2y^3 + (2w^2 + 2w + 1)y^2 + (w^2 + 1)y + w^2 + 2w$ and $v_3^\dagger(y) = y^4 + (w + 1)y^3 + (2w + 2)y^2 + (w^2 + 2w + 2)y + 2w$ so $v_3^\dagger(y)u_3(y) = (y^2 + 2w^2y + w^2 + 2w + 1)(y^6 + 1)$ i.e. $v_3^\dagger(y)v_3(y)$ is divisible by $y^6 + 1$ from the right. Let us take $f_{111}(y) = f_{122}(y) = f_{222}(y) = u_2(y) \in \mathbb{F}_{27}[y;\theta_3]/\langle y^6 - 1\rangle$, $f_{112}(y) = f_{121}(y) = f_{211}(y) = f_{212}(y) = f_{221}(y) = f_{311}(y) = f_{312}(y) = f_{321}(y) =$

$u_1(y) \in \mathbb{F}_{27}[y;\theta_3]/\langle y^6-1\rangle$, and $f_{322}(y) = u_3(y) \in \mathbb{F}_{27}[y;\theta_3]/\langle y^6+1\rangle$. Then $\mathcal{C} = \langle \sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2} \eta_{i_1i_2i_3} f_{i_1i_2i_3}(y)\rangle$ is a skew (Θ,α)-constacyclic code of length 6 over $\mathcal{T}$, where $\alpha = \eta_{111}+\eta_{112}+\eta_{121}+\eta_{122}+\eta_{211}+\eta_{212}+\eta_{221}+\eta_{222}+\eta_{311}+\eta_{312}+\eta_{321}-\eta_{322} = 1-2\eta_{322} \in \mathcal{U}(\mathcal{T})$. Let $M = M_{27,3}\otimes M_{27,2}\otimes M_{27,2} \in GL_{12}(\mathbb{F}_{27})$ be the matrix used in the Gray map $\phi : \mathcal{T} \rightarrow \mathbb{F}_{25}^{12}$, where

$$M_{27,2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, M_{27,3} = \begin{bmatrix} 2w^2+w & w+1 & 2 \\ w & 2w & w+1 \\ w^2+w+2 & w & 2w^2+w \end{bmatrix}$$

and $MM^T = (2w+1)I_{12}$. Then $\Phi(\mathcal{C})$ is a skew-θ_t quasi-$(1,1,1,1,1,1,1,1,1,1,1,-1)$-twisted Euclidean dual-containing $[72,67,4]$ code over $\mathbb{F}_{27}$. Hence, by Theorem 5.1.6, there exists a $[[72,62,4]]_{27}$ quantum code which is a new code as per database [9].

Now, we conclude this section by enlisting the quantum codes constructed using Theorem 5.1.6 in Tables 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8 and 5.9.

TABLE 5.1: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1]/\langle u_1^2 - u_1\rangle$, $\alpha = \eta_1\alpha_1 + \eta_2\alpha_2$

q	n	(α_1, α_2)	$f_1(y)$	$f_2(y)$	$\Phi(\mathcal{C})$	$[[n, k, d]]_q$	Remark
25	8	$(1, -1)$	$2w^{17}w^{23}1$	$w^4w^{11}w^{23}w^{23}1$	$[16, 9, 7]$	$[[16, 2, 7]]_{25}$	new
25	10	$(1, -1)$	w^4w^31	$2w^3w^{17}w^{11}w^91$	$[20, 13, 6]$	$[[20, 6, 6]]_{25}$	new
25	10	$(1, -1)$	w^4w^31	$w^4w^9w^5w^{13}1$	$[20, 14, 5]$	$[[20, 8, 5]]_{25}$	new
27	6	$(1, -1)$	$w^{22}1$	$w^{22}w^{18}1$	$[12, 9, 4]$	$[[12, 6, 4]]_{27}$	MDS
27	9	$(1, -1)$	$w^{22}2w^91$	$w^{10}1$	$[[18, 13, 3]]$	$[[18, 8, 3]]_{27}$	new
27	9	$(1, -1)$	ww^801	$w^42w^3w^{19}1$	$[18, 11, 6]$	$[[18, 4, 6]]_{27}$	new
27	9	$(1, -1)$	$w^2w^{22}1$	$w^42w^3w^{19}1$	$[18, 12, 5]$	$[[18, 6, 5]]_{27}$	new
27	12	$(1, -1)$	$w0w^{11}w^{15}1$	$w^2w^3w^{25}01$	$[24, 16, 6]$	$[[24, 8, 6]]_{27}$	new
27	15	$(1, -1)$	$w^{10}w^{19}w^401$	$w^{24}w^{24}w^{15}w^{20}w^31$	$[30, 21, 7]$	$[[30, 12, 7]]_{27}$	new

TABLE 5.2: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1]/\langle u_1^3 - u_1\rangle$, $\alpha = \eta_1\alpha_1 + \eta_2\alpha_2 + \eta_3\alpha_3$

q	n	$(\alpha_1, \alpha_2, \alpha_3)$	$f_1(y)$	$f_2(y)$	$f_3(y)$	$\Phi(\mathcal{C})$	$[[n, k, d]]_q$	Remark
27	9	$(1, -1, -1)$	ww^801	w^81	$w^42w^3w^{19}1$	$[27, 19, 6]$	$[[27, 11, 6]]_{27}$	new

TABLE 5.3: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2]/\langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1\rangle$, $\alpha = \eta_{11}\alpha_{11} + \eta_{12}\alpha_{12} + \eta_{21}\alpha_{21} + \eta_{22}\alpha_{22}$

q	n	$(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$	$f_{11}(y)$	$f_{12}(y)$	$f_{21}(y)$	$f_{22}(y)$	$\Phi(\mathcal{C})$	$[[n, k, d]]_q$	Remark
27	6	$(1, -1, 1, 1)$	$w^{22}1$	$w^{22}w^{18}1$	1	$w^{22}1$	$[24, 20, 4]$	$[[24, 16, 4]]_{27}$	new
27	9	$(1, -1, 1, -1)$	$w^{22}20w^91$	$w^{10}1$	1	1001	$[36, 28, 3]$	$[[36, 20, 3]]_{27}$	new

TABLE 5.4: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2]/\langle u_1^2-u_1, u_2^3-u_2, u_1u_2-u_2u_1\rangle$, $\alpha=\sum_{i_1=1}^{2}\sum_{i_2=1}^{3}\eta_{i_1i_2}\alpha_{i_1i_2}$

q	n	$(\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{21},\alpha_{22},\alpha_{23})$	$f_{11}(y)$	$f_{12}(y)$	$f_{13}(y)$	$f_{21}(y)$	$f_{22}(y)$	$f_{23}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
25	4	$(1,1,-1,1,-1,-1)$	21	31	301	1	$w^{11}1$	$1w^{21}1$	$[24,17,5]$	$[[24,10,5]]_{25}$	new
27	6	$(1,1,1,1,1,-1)$	$w^{22}1$	1	1	1	1	$w^{22}w^{18}1$	$[36,33,3]$	$[[36,30,3]]_{27}$	new
27	6	$(1,1,1,1,1,1)$	$w^{16}1$	$w^{9}1$	$w1$	21	$w^{3}1$	$w^{14}1$	$[36,30,4]$	$[[36,24,4]]_{27}$	new
27	6	$(1,1,1,1,-1,1)$	$w^{16}1$	$w^{9}1$	$w^{9}w^{20}1$	1	$w^{10}1$	$w^{14}1$	$[36,29,4]$	$[[36,22,5]]_{27}$	new
27	9	$(1,1,1,-1,1,-1)$	$w^{22}20w^{9}1$	1	$w^{23}1$	$w^{10}1$	1	1001	$[54,45,3]$	$[[54,36,3]]_{27}$	new

TABLE 5.5: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2,u_3]/\langle u_1^2-u_1, u_2^2-u_2, u_3^2-1, u_iu_j-u_iu_i\rangle$, $\alpha=\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\eta_{i_1i_2i_3}\alpha_{i_1i_2i_3}$, $\beta=(\alpha_{111},\alpha_{112},\ldots,\alpha_{222})$

q	n	β	$f_{111}(y)$	$f_{112}(y)$	$f_{121}(y)$	$f_{122}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{221}(y)$	$f_{222}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
27	6	$(1,1,1,1,1,1,1,-1)$	$w^{22}1$	1	1	$w^{22}1$	1	$w^{22}1$	1	$w^{22}w^{18}1$	$[48,43,4]$	$[[48,38,4]]_{27}$	new
27	9	$(1,1,1,-1,1,1,-1,-1)$	1	$w^{23}1$	$w^{22}20w^{9}1$	$w^{10}1$	1	$w^{23}1$	$w^{10}1$	$w^{6}w^{11}1$	$[72,62,4]$	$[[72,52,4]]_{27}$	new

TABLE 5.6: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2]/\langle u_1^3-u_1, u_2^3-u_2, u_1u_2-u_2u_1\rangle$, $\alpha=\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\eta_{i_1i_2}\alpha_{i_1i_2}$, $\beta=(\alpha_{11},\alpha_{12},\ldots,\alpha_{33})$

q	n	β	$f_{11}(y)$	$f_{12}(y)$	$f_{13}(y)$	$f_{21}(y)$	$f_{22}(y)$	$f_{23}(y)$	$f_{31}(y)$	$f_{32}(y)$	$f_{33}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
9	6	$(1,1,1,-1,-1,1,1,1,-1)$	201	21	1	$w1$	$w1$	1	21	1	$w1$	$[54,47,4]$	$[[54,40,4]]_9$	new
25	10	$(-1,1,1,-1,1,1,1,1,-1)$	31	21	1	1	311231	1	1	$w^{20}1$	31	$[90,81,4]$	$[[90,72,4]]_{25}$	new
27	6	$(1,1,1,1,1,1,1,1,-1)$	$w^{22}1$	21	1	1	$w^{22}1$	1	$w^{22}1$	1	$w^{22}w^{18}1$	$[54,48,4]$	$[[54,42,4]]_{27}$	new
27	9	$(1,1,1,-1,1,1,-1,-1)$	1	$w^{23}1$	21	$w^{22}20w^{9}1$	$w^{10}1$	1	$w^{23}1$	$w^{10}1$	$w^{6}w^{11}1$	$[81,70,4]$	$[[81,61,4]]_{27}$	new

TABLE 5.7: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2, u_3]/\langle u_1^3 - u_1, u_2^2 - u_2, u_3^2 - 1, u_iu_j - u_ju_i\rangle$, $\alpha = \sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\eta_{i_1i_2i_3}\alpha_{i_1i_2i_3}$, $\beta = (\alpha_{111}, \alpha_{112}, \ldots, \alpha_{322})$

q	n	β	$f_{111}(y)$	$f_{112}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{211}(y)$	$f_{212}(y)$
25	10	$(-1,1,1,1,-1,1,-1,1,1,-1,1,-1)$	31	1	$w^{20}1$	1	311231	1
27	6	$(1,1,1,1,1,1,1,1,1,1,1,-1)$	$w^{22}1$	1	1	$w^{22}1$	1	1
27	6	$(1,1,1,1,1,1,1,1,1,1,1,-1)$	$w^{22}1$	1	1	1	1	$w^{22}1$
27	9	$(1,1,1,1,1,-1,1,1,1,-1,1,-1)$	1	$w^{23}1$	1	$w^{22}20w^{9}1$	1	$w^{10}1$

Table 5.7 continued

$f_{221}(y)$	$f_{222}(y)$	$f_{311}(y)$	$f_{312}(y)$	$f_{321}(y)$	$f_{322}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
31	$w^{20}1$	1	31	1	101	$[120, 108, 4]$	$[[120, 96, 4]]_{25}$	new
1	$w^{22}1$	1	1	1	$w^{22}w^{18}1$	$[72, 67, 4]$	$[[72, 62, 4]]_{27}$	new
1	1	1	1	1	$w^{22}w^{18}1$	$[72, 68, 4]$	$[[72, 64, 3]]_{27}$	new
1	$w^{23}1$	1	$w^{10}1$	1	$w^{6}w^{11}1$	$[108, 98, 4]$	$[[108, 88, 4]]_{27}$	new

TABLE 5.8: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2,u_3,u_4]/\langle u_1^2-u_1,u_2^2-u_2,u_3^2-1,u_4^2-1u_iu_j-u_ju_i\rangle$, $\alpha=\sum_{i_1,i_2,i_3,i_4=1}^{2}\eta_{i_1i_2i_3i_4}\alpha_{i_1i_2i_3i_4}$, $\beta=(\alpha_{1111},\alpha_{1112},\ldots,\alpha_{2222})$

q	n	β	$f_{1111}(y)$	$f_{1112}(y)$	$f_{1121}(y)$	$f_{1122}(y)$	$f_{1211}(y)$	$f_{1212}(y)$	$f_{1221}(y)$
25	10	$(-1,1,1,1,-1,1,-1,1,1,1,-1,1,1,-1,1,1)$	31	1	$w^{20}1$	1	311231	1	31
27	6	$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1)$	$w^{22}1$	1	1	$w^{22}1$	1	1	1
27	9	$(1,1,1,1,1,1,1,-1,1,1,1,1,1,-1,1,-1)$	1	$w^{23}1$	1	1	$w^{22}20w^{9}1$	1	1

Table 5.8 continued

$f_{1222}(y)$	$f_{2111}(y)$	$f_{2112}(y)$	$f_{2121}(y)$	$f_{2122}(y)$	$f_{2211}(y)$	$f_{2212}(y)$	$f_{2221}(y)$	$f_{2222}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
1	$w^{20}1$	1	31	1	$w^{20}1$	31	1	$w^{20}1$	$[160,147,4]$	$[[160,134,4]]_{25}$	new
$w^{22}1$	1	1	$w^{22}1$	1	1	1	$w^{22}1$	$w^{22}w^{18}1$	$[96,89,4]$	$[[96,82,4]]_{27}$	new
$w^{10}1$	1	1	$w^{23}1$	1	1	$w^{10}1$	1	$w^{6}w^{11}1$	$[144,134,4]$	$[[144,124,4]]_{27}$	new

TABLE 5.9: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2,u_3]/\langle u_1^3-u_1, u_2^2-u_2, u_3^3-1, u_iu_j-u_ju_i\rangle$, $\alpha=\sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{3}\eta_{i_1i_2i_3}\alpha_{i_1i_2i_3}$, $\beta=(\alpha_{111},\alpha_{112},\ldots,\alpha_{323})$

q	n	β	$f_{111}(y)$	$f_{112}(y)$	$f_{113}(y)$	$f_{121}(y)$	$f_{122}(y)$	$f_{123}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{213}(y)$
9	6	$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1)$	1	1	1	1	1	1	1	1	1
9	6	$(1,1,1,1,-1,1,1,1,1,-1,1,1,1,1-1,1,1,1)$	21	1	w^62w^21	1	$w1$	1	21	21	1
25	8	$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	$w^{23}1$	1	1	1	1	1	1	1	1
25	10	$(-1,1,1,1,-1,1,-1,1,1,1,-1,1,1,-1,1,1,1,-1)$	31	1	$w^{20}1$	1	311231	1	31	1	$w^{20}1$
27	6	$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1)$	$w^{22}1$	1	1	$w^{22}1$	1	1	1	$w^{22}1$	1
27	9	$(-1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	$w^{10}1$	1	1	1	1	1	1	1	1
27	9	$(1,1,1,1,1,1,1,1,1,-1,1,1,1,1,1,1,-1,1,-1)$	1	1	$w^{23}1$	1	1	$w^{22}20w^91$	1	1	$w^{10}1$

Table 5.9 continued

$f_{221}(y)$	$f_{222}(y)$	$f_{223}(y)$	$f_{311}(y)$	$f_{312}(y)$	$f_{313}(y)$	$f_{321}(y)$	$f_{322}(y)$	$f_{323}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
1	1	1	1	1	1	1	1	$w1$	$[108,107,2]$	$[[108,106,2]]_9$	MDS, new
$w1$	1	21	1	1	101	1	1	21	$[108,96,4]$	$[[108,84,4]]_9$	new
1	1	1	1	1	1	1	1	$w1$	$[144,143,2]$	$[[144,142,2]]_{25}$	MDS, new
1	31	1	$w^{20}1$	31	1	$w^{20}1$	1	31	$[180,162,4]$	$[[180,154,4]]_{25}$	new
1	$w^{22}1$	1	1	1	$w^{22}1$	1	1	$w^{22}w^{18}1$	$[108,101,4]$	$[[108,94,4]]_{27}$	new
1	1	1	1	1	1	1	1	$w1$	$[162,161,2]$	$[[162,160,2]]_{27}$	MDS, new
1	1	1	$w^{23}1$	1	1	$w^{10}1$	1	$w^6w^{11}1$	$[162,152,3]$	$[[162,142,3]]_{27}$	new

5.2 Hermitian Dual of Skew (Θ, α)-Constacyclic Codes over $\mathcal{T}$ and Construction of Quantum Codes

This section deals with the construction of quantum codes from Hermitian dual-containing skew constacyclic codes over the ring $\mathcal{T}$. We start the section with some important results on Hermitian dual of skew (Θ_t, α)-constacyclic codes over $\mathcal{T}$ and then characterize Hermitian dual-containing skew constacyclic codes over $\mathcal{T}$. These findings are then employed to outline the procedure for constructing quantum codes. We wrap up this section by presenting examples of numerous new quantum codes generated through this technique.

Lemma 5.2.1. Let $\Theta \in Aut(\mathcal{T})$, n be a multiple of $o(\Theta)$ and $\alpha \in \mathcal{U}(\mathcal{T})$ be such that $\Theta(\alpha) = \alpha$. Then the Hermitian dual of a skew (Θ, α)-constacyclic code $\mathcal{C}$ of length n over $\mathcal{T}$, denoted as $\mathcal{C}^{\perp_H}$, is a skew $(\Theta, \overline{\alpha}^{-1})$-constacyclic code of length n over $\mathcal{T}$.

Proof. Let $\mathbf{v} = (\mathsf{v}^0, \mathsf{v}^1, \dots, \mathsf{v}^{\mathsf{n}-1}) \in \mathcal{C}$ and $\mathbf{w} = (\mathsf{w}^0, \mathsf{w}^1, \dots, \mathsf{w}^{\mathsf{n}-1}) \in \mathcal{C}^{\perp_H}$ be arbitrary elements. Since $\mathcal{C}$ is given to be a skew (Θ, α)-constacyclic code, we have $\sigma^{n-1}_{(\Theta,\, \overline{\alpha})}(\mathbf{v}) = (\alpha\Theta^{n-1}(\mathsf{v}^1), \alpha\Theta^{n-1}(\mathsf{v}^2), \dots, \alpha\Theta^{n-1}(\mathsf{v}_{n-1}), \Theta^{n-1}(\mathsf{v}^0)) \in \mathcal{C}$. Therefore, we have

$$
\begin{aligned}
0 &= \left\langle \sigma^{n-1}_{(\Theta,\, \overline{\alpha})}(\mathbf{v}), \mathbf{w} \right\rangle_H \\
&= \big\langle \big(\alpha\Theta^{n-1}(\mathsf{v}^1), \alpha\Theta^{n-1}(\mathsf{v}^2), \dots, \alpha\Theta^{n-1}(\mathsf{v}_{n-1}), \Theta^{n-1}(\mathsf{v}^0)\big), \big(\mathsf{w}^0, \mathsf{w}^1, \dots, \mathsf{w}^{\mathsf{n}-1}\big) \big\rangle_H \\
&= \alpha \left(\sum_{l=1}^{n-1} \Theta^{n-1}(\mathsf{v}_l)\overline{\mathsf{w}}_{l-1} + \alpha^{-1}\Theta^{n-1}(\mathsf{v}_0)\overline{\mathsf{w}}_{n-1} \right)
\end{aligned}
$$

Now since, $o(\Theta)|n$ and Θ fixes α, applying Θ on both sides, we obtain

$$\begin{aligned}
0 &= \alpha\Theta\left(\sum_{l=1}^{n-1}\Theta^{n-1}(\mathsf{v}_l)\overline{\mathsf{w}}_{l-1} + \alpha^{-1}\Theta^{n-1}(\mathsf{v}_0)\overline{\mathsf{w}}_{n-1}\right)\\
0 &= \alpha\left(\sum_{l=1}^{n-1}\mathsf{v}_l\Theta(\overline{\mathsf{w}}_{l-1}) + \alpha^{-1}\mathsf{v}_0\Theta(\overline{\mathsf{w}}_{n-1})\right)\\
0 &= \alpha\left(\sum_{l=1}^{n-1}\mathsf{v}_l\overline{\Theta(\mathsf{w}_{l-1})} + \alpha^{-1}\mathsf{v}_0\overline{\Theta(\mathsf{w}_{n-1})}\right)\\
0 &= \sum_{l=1}^{n-1}\mathsf{v}_l\overline{\Theta(\mathsf{w}_{l-1})} + \mathsf{v}_0\overline{\overline{\alpha}^{-1}\Theta(\mathsf{w}_{n-1})}\\
0 &= \big\langle(\mathsf{v}_0,\mathsf{v}_1,\ldots,\mathsf{v}_{n-1})\,,\big(\overline{\alpha}^{-1}\Theta(\mathsf{w}_{n-1}),\Theta(\mathsf{w}_0),\ldots,\Theta(\mathsf{w}_{n-2})\big)\big\rangle_H\\
0 &= \big\langle\mathbf{v},\sigma_{(\Theta,\ \overline{\alpha}^{-1})}(\mathbf{w})\big\rangle_H
\end{aligned}$$

Hence, $\forall\ \mathbf{w}\in\mathcal{C}^{\perp_H}$, $\sigma_{(\Theta,\ \overline{\alpha}^{-1})}(\mathbf{w})\in\mathcal{C}^{\perp_H}$. This proves that $\mathcal{C}^{\perp_H}$ is a skew $(\Theta,\ \overline{\alpha}^{-1})$-constacyclic code over $\mathcal{T}$. □

Lemma 5.2.2. ([19, Theorem 1]) Let $\gamma\in\mathbb{F}_q^*$ and $\theta_t\in Aut(\mathbb{F}_q)$ which fixes γ. Further suppose that C is a skew (θ,γ)-constacyclic code of length n over $\mathbb{F}_q$ generated by $f(y)$ such that $o(\theta)|n$. Then, there exists a polynomial $g(y)\in\mathbb{F}_q[y;\theta]$ such that $y^n-\gamma=g(y)f(y)$ and the Hermitian dual $C^{\perp_H}$ is generated by the skew Hermitian reciprocal polynomial $\overline{g}^{\dagger}(y)$ of $g(y)$, where $\overline{g}^{\dagger}(y)$ is obtained by taking conjugates of the coefficients of $g^{\dagger}(y)$.

Theorem 5.2.3. Let $\Theta_t\in\mathrm{Aut}(\mathcal{T})$ and $\alpha=\sum_{i_1i_2\ldots i_r}\eta_{i_1i_2\ldots i_r}\alpha_{i_1i_2\ldots i_r}\in\mathcal{U}(\mathcal{T})$ be such that $\Theta_t(\alpha)=\alpha$ and $o(\Theta_t)$ divides n. Furthermore, let $\mathcal{C}=\bigoplus_{i_1i_2\ldots i_r}\eta_{i_1i_2\ldots i_r}\mathcal{C}_{i_1i_2\ldots i_r}$ be a skew (Θ_t,α)-constacyclic code of length n over $\mathcal{T}$ such that $\mathcal{C}_{i_1i_2\ldots i_r}=\langle f_{i_1i_2\ldots i_r}(y)\rangle$ and $f_{i_1i_2\ldots i_r}(y)g_{i_1i_2\ldots i_r}(y)=y^n-\alpha_{i_1i_2\ldots i_r}$, for $i_j\in\{1,2,\ldots l_j\}$, $j=1,2,\ldots,r$. Then

(i) $\mathcal{C}^{\perp_H}=\langle\overline{g}^{\dagger}(y)\rangle$, where $\overline{g}^{\dagger}(y)=\sum_{i_1i_2\ldots i_r}\eta_{i_1i_2\ldots i_r}\overline{g}^{\dagger}(y)_{i_1i_2\ldots i_r}$;

(ii) $|\mathcal{C}^{\perp_H}|=q^L$, where $L=\sum_{i_1,i_2,\ldots,i_r}deg(f_{i_1i_2\ldots i_r}(y))$.

Proof. The proof easily follows by combining Theorems 4.2.4, 4.2.5, Lemma 5.2.1 and 5.2.2. ☐

Lemma 5.2.4. ([75, Lemma 9]) Let $C = \langle u(y) \rangle$ be a skew (θ, γ)-constacyclic code of length n over $\mathbb{F}_{q^2}$, where $\gamma \in \mathbb{F}_{q^2}^*$ such that $\gamma^2 = 1$. Then, C is Hermitian dual-containing if and only if $y^n - \gamma$ divides $\overline{v}^{\dagger}(y)v(y)$ from right, where $y^n - \gamma = v(y)u(y)$ and $\overline{v}^{\dagger}(y)$ denotes the skew Hermitian reciprocal polynomial of $v(y)$.

Now, we present the main result of this section which provides the construction of quantum codes from Hermitian dual-containing skew constacyclic codes over $\mathcal{T} = \mathbb{F}_{q^2}[u_1, u_2, \dots, u_r]/\langle f_i(u_i), u_iu_j - u_ju_i \rangle$.

Theorem 5.2.5. Let $\alpha = \sum_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \alpha_{i_1 i_2 \dots i_r} \in \mathcal{U}(\mathcal{T})$ be such that $\alpha^2 = 1$, $\Theta_t \in \text{Aut}(\mathcal{T})$ and n be a multiple of $o(\Theta_t)$. Furthermore, let $\mathcal{C} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$ be a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$ such that $\mathcal{C}_{i_1 i_2 \dots i_r} = \langle f_{i_1 i_2 \dots i_r}(y) \rangle$ and $g_{i_1 i_2 \dots i_r}(y) \in \mathbb{F}_{q^2}[y; \theta_t]$ be such that $g_{i_1 i_2 \dots i_r}(y) f_{i_1 i_2 \dots i_r}(y) = y^n - \alpha_{i_1 i_2 \dots i_r}$, $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$.

(i) $\mathcal{C}$ contains its Hermitian dual if and only if $\overline{g}^{\dagger}_{i_1 i_2 \dots i_r}(y) g_{i_1 i_2 \dots i_r}(y)$ is divisible by $y^n - \alpha_{i_1 i_2 \dots i_r}$ from right, $\forall\ i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$.

(ii) If $\overline{g}^{\dagger}_{i_1 i_2 \dots i_r}(y) g_{i_1 i_2 \dots i_r}(y)$ is divisible by $y^n - \alpha_{i_1 i_2 \dots i_r}$ from right, $\forall\ i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2, \dots, r$ then there exists an $[[N, K, D]]_q$ quantum code, where $N = l_1 l_2 \dots l_r n$, $K = l_1 l_2 \dots l_r n - 2 \sum_{i_1 i_2 \dots i_r} deg(f_{i_1 i_2 \dots i_r}(y))$ and $D \geq d_L$, the Lee distance of $\mathcal{C}$.

Proof. (i) Let $\mathcal{C} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$ be a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$ such that $\mathcal{C}^{\perp_H} \subseteq \mathcal{C}$. Then by Theorem (iii), we get $\bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}^{\perp_H}_{i_1 i_2 \dots i_r} \subseteq \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$. Since $o(\Theta_t) \mid n$ and $\alpha^2 = 1$, by taking modulo $\eta_{i_1 i_2 \dots i_r}$ in above equation, we get $\mathcal{C}^{\perp_H}_{i_1 i_2 \dots i_r} \subseteq \mathcal{C}_{i_1 i_2 \dots i_r}$ for all $i_j \in$

$\{1, 2, \ldots, l_j\}$, $j = 1, 2, \ldots, r$. Therefore, by Lemma 5.2.4, $\overline{g}^{\dagger}(y)_{i_1i_2\ldots i_r}g_{i_1i_2\ldots i_r}(y)$ is right divisible by $y^n - \alpha_{i_1i_2\ldots i_r}$, where $y^n - \alpha_{i_1i_2\ldots i_r} = g_{i_1i_2\ldots i_r}(y)f_{i_1i_2\ldots i_r}(y)$ and $\overline{g}^{\dagger}(y)_{i_1i_2\ldots i_r}$ is the skew Hermitian reciprocal of $g_{i_1i_2\ldots i_r}(y)$.

Conversely, let $\overline{g}^{\dagger}(y)_{i_1i_2\ldots i_r}g_{i_1i_2\ldots i_r}(y)$ be divisible by $y^n - \alpha_{i_1i_2\ldots i_r}$ from right for all $i_j \in \{1, 2, \ldots, l_j\}$, $j = 1, 2, \ldots, r$. Then by Lemma 5.2.4, we get $\mathcal{C}_{i_1i_2\ldots i_r}^{\perp_H} \subseteq \mathcal{C}_{i_1i_2\ldots i_r}$ for all $i_j \in \{1, 2, \ldots, l_j\}$, $j = 1, 2, \ldots, r$. This implies that $\bigoplus_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r}\mathcal{C}_{i_1i_2\ldots i_r}^{\perp_H} \subseteq \bigoplus_{i_1i_2\ldots i_r} \eta_{i_1i_2\ldots i_r}\mathcal{C}_{i_1i_2\ldots i_r}$ and hence, $\mathcal{C}^{\perp_H} \subseteq \mathcal{C}$.

(ii) If $g_{i_1i_2\ldots i_r}^{\dagger}(y)g_{i_1i_2\ldots i_r}(y)$ is divisible by $y^n - \alpha_{i_1i_2\ldots i_r}$ from right, $\forall\, i_j \in \{1, 2, \ldots, l_j\}$, $j = 1, 2, \ldots, r$, then by part(i), $\mathcal{C}^{\perp_H} \subseteq \mathcal{C}$. Since, the Gray map Φ is bijective, linear, distance preserving (by Theorem 4.1.2), and it preserves Hermitian orthogonality (by Theorem 4.1.6), we have, $\Phi(\mathcal{C})$ is a $[l_1l_2\ldots l_rn, \sum_{i_1i_2\ldots i_r}(n - deg(f_{i_1i_2\ldots i_r}(y)), d_L]$ linear code over $\mathbb{F}_{q^2}$ such that $\Phi(\mathcal{C})^{\perp_H} \subseteq \Phi(\mathcal{C})$. (In fact, by Theorem 4.2.8, it is a skew-θ_t quasi $(\alpha_{11\ldots1}, \ldots \alpha_{l_1l_2\ldots l_r}))$-twisted code.) Then, $\Phi(\mathcal{C})^{\perp_H}$ is a self-orthogonal code of length $l_1l_2\ldots l_rn = N$ (say) and dimension $l_1l_2\ldots l_rn - \sum_{i_1i_2\ldots i_r}(n - deg(f_{i_1i_2\ldots i_r}(y)) = \sum_{i_1i_2\ldots i_r}(deg(f_{i_1i_2\ldots i_r}(y)) = \frac{N-K}{2}$ (say). Hence, by Lemma 1.3.7, there exists an $[[N, K, D]]_q$ QECC, where where $N = l_1l_2\ldots l_rn$, $K = l_1l_2\ldots l_rn - 2\sum_{i_1i_2\ldots i_r} deg(f_{i_1i_2\ldots i_r}(y))$ and $D = \min\{\mathrm{wt}(\mathbf{c}) : \mathbf{c} \in (\Phi(\mathcal{C})^{\perp_H})^{\perp_H} \setminus \Phi(\mathcal{C})^{\perp_H} = \Phi(\mathcal{C}) \setminus \Phi(\mathcal{C})^{\perp_H}\} \geq d_L$.

□

Now, we present an example to justify and explain the construction of a new quantum code utilizing the technique discussed in this section. Finally, we'll conclude this section by enlisting the quantum codes constructed using Theorem 5.2.5 in Tables 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, and 5.9.

Example 5.2.6. For $q = 5$, $\mathbb{F}_{q^2} = \mathbb{F}_{25} = \mathbb{F}_5[X]/\langle X^2 + 4X + 2\rangle = \mathbb{F}_5(w)$, where $w^2 + 4w + 2$. Further, let $\mathcal{T} = T_3 = \mathbb{F}_{25}[u_1, u_2, u_3]/\langle u_1^3 - u_1, u_2^2 - u_2, u_3^3 - u_3, u_1u_2 -$

$u_2u_1, u_1u_3 - u_3u_1, u_2u_3 - u_3u_2\rangle$. Then $\eta_{111} = (1-u_1^2)u_2(1-u_3^2)$, $\eta_{112} = (1-u_1^2)u_2(\frac{u_3^2+u_3}{2})$, $\eta_{113} = (1-u_1^2)u_2(\frac{u_3^2-u_3}{2})$, $\eta_{121} = (1-u_1^2)(1-u_2)(1-u_3^2)$, $\eta_{122} = (1-u_1^2)(1-u_2)(\frac{u_3^2+u_3}{2})$, $\eta_{123} = (1-u_1^2)(1-u_2)(\frac{u_3^2-u_3}{2})$, $\eta_{211} = (\frac{u_1^2+u_1}{2})u_2(1-u_3^2)$, $\eta_{212} = (\frac{u_1^2+u_1}{2})u_2(\frac{u_3^2+u_3}{2})$, $\eta_{213} = (\frac{u_1^2+u_1}{2})u_2(\frac{u_3^2-u_3}{2})$, $\eta_{221} = (\frac{u_1^2+u_1}{2})(1-u_2)(1-u_3^2)$, $\eta_{222} = (\frac{u_1^2+u_1}{2}))(1-u_2)(\frac{u_3^2+u_3}{2})$, $\eta_{223} = (\frac{u_1^2+u_1}{2})(1-u_2)(\frac{u_3^2-u_3}{2})$, $\eta_{311} = (\frac{u_1^2-u_1}{2})u_2(1-u_3^2)$, $\eta_{312} = (\frac{u_1^2-u_1}{2})u_2(\frac{u_3^2+u_3}{2})$, $\eta_{313} = (\frac{u_1^2-u_1}{2})u_2(\frac{u_3^2-u_3}{2})$, $\eta_{321} = (\frac{u_1^2-u_1}{2})(1-u_2)(1-u_3^2)$, $\eta_{322} = (\frac{u_1^2-u_1}{2})(1-u_2)(\frac{u_3^2+u_3}{2})$ and $\eta_{323} = (\frac{u_1^2-u_1}{2})(1-u_2)(\frac{u_3^2-u_3}{2})$ are the primitive orthogonal idempotents. Thus, every element $\mathbf{v} \in \mathcal{T}$ can be uniquely written as $\mathbf{v} = \sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{3}\eta_{i_1i_2i_3}v_{i_1i_2i_3}$. Let us define an automorphism $\Theta_1 : \mathcal{T} \to \mathcal{T}$ as

$$\sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{3}\eta_{i_1i_2i_3}v_{i_1i_2i_3} \longmapsto \sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{3}\eta_{i_1i_2i_3}v_{i_1i_2i_3}^5.$$

For $n = 10$, consider a factorisation of $y^n - 1 = y^{10} - 1 \in \mathbb{F}_{25}[y;\theta_2]$ as $y^{10} - 1 = (y+3w+3)(y+3w+4)(y+3w+3)(y+3w+4)(y+3w+3)(y+3w+4)(y+3w+3)(y+3w+4)(y+3w+3)(y+3w+4)$. Let $u_1(y) = 1$ and $v_1(y)u_1(y) = y^{10}-1$. Then $v_1(y) = y^{10}-1$ and $\overline{v}_1^{\dagger}(y) = y^{10}-1$ so $\overline{v}_1^{\dagger}(y)v_1(y)$ is divisible by $y^{10}-1$ from right. Let $u_2(y) = y+3w+4 = y+w^{20}$ and $v_2(y)u_2(y) = y^{10}-1$. Then $v_2(y) = (y+3w+3)(y+3w+4)(y+3w+3)(y+3w+4)(y+3w+3)(y+3w+4)(y+3w+3)(y+3w+4)(y+3w+3) = y^9+(3w+3)y^8+y^7+(3w+3)y^6+y^5+(3w+3)y^4+y^3+(3w+3)y^2+y+3w+3$ and $\overline{v}_2^{\dagger}(y) = y^9+(2w+1)y^8+y^7+(2w+1)y^6+y^5+(2w+1)y^4+y^3+(2w+1)y^2+y+2w+1$. Then $\overline{v}_2^{\dagger}(y)v_2(y) = (y^8+(4w+2)y^7+3y^6+(3w+4)y^5+(2w+1)y^3+2y^2+(w+3)y+4)(y^{10}-1)$ i.e. $\overline{v}_2^{\dagger}(y)v_2(y)$ divisible by $y^{10}-1$ from right.

Next let us consider one of the factorizations of $y^{10}+1 \in \mathbb{F}_{25}[y;\theta_2]$ given as $y^{10}+1 = (y+w+3)(y+w+1)(y+w+3)(y+w+1)(y+w+3)(y+w+1)(y+w+3)(y+w+1)(y+w+3)(y+w+1)$. Let $u_3(y) = y+w+1 = y+w^{22}$ and $v_3(y)u_3(y) = y^{10}+1$. Then $v_3(y) = (y+w+3)(y+w+1)(y+w+3)(y+w+1)(y+w+3)(y+w+1)(y+w+3)(y+w+1)(y+w+3) = y^9+(w+3)y^8+4y^7+(4w+2)y^6+y^5+(w+3)y^4+4y^3+(4w+2)y^2+y+w+3$

and $\overline{v}_3^{\dagger}(y) = y^9+(w+1)y^8+4y^7+(4w+4)y^6+y^5+(w+1)y^4+4y^3+(4w+4)y^2+y+w+1$ so $v_3^{\dagger}(y)u_3(y) = (y^8+4y^6+y^4+4y^2+1)(y^{10}+1)$ i.e. $\overline{v}_3^{\dagger}(y)v_3(y)$ divisible by $y^{10}+1$ from right. Let $u_4(y) = (y+w+1)(y+w+3)(y+w+1)(y+w+3)(y+w+1) = y^5+(w+1)y^4+2y^3+(2w+2)y^2+y+w+1 = y^5+w^{22}y^4+2y^3+w^4y^2+y+w^{22}$ and $v_4(y)u_4(y) = y^{10}+1$. Then $v_4(y) = y^5+(w+1)y^4+2y^3+(2w+2)y^2+y+w+1$ and $\overline{v}_4^{\dagger}(y) = y^5+(w+3)y^4+2y^3+(2w+1)y^2+y+w+3$. Then $\overline{v}_4^{\dagger}(y)v_4(y) = y^{10}+1$ i.e. $\overline{v}_4^{\dagger}(y)v_4(y)$ divisible by $y^{10}+1$ from right.

Let us take $f_{111}(y) = f_{211}(y) = f_{322}(y) = u_3(y) \in \mathbb{F}_{5^2}[y;\theta_2]/\langle y^{10}+1\rangle$, $f_{112}(y) = f_{113}(y) = f_{121}(y) = f_{123}(y) = f_{212}(y) = f_{221}(y) = f_{222}(y) = f_{223}(y) = f_{312}(y) = f_{313}(y) = f_{321}(y) = f_{323}(y) = u_1(y) \in \mathbb{F}_{5^2}[y;\theta_2]/\langle y^{10}-1\rangle$, $f_{122}(y) = f_{311}(y) = u_2(y) \in \mathbb{F}_{5^2}[y;\theta_2]/\langle y^{10}-1\rangle$ and $f_{213}(y) = u_4(y) \in \mathbb{F}_{5^2}[y;\theta_2]/\langle y^{10}+1\rangle$. Then $\mathcal{C} = \langle \sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{3}\eta_{i_1i_2i_3}f_{i_1i_2i_3}(y)\rangle$ is a skew (Θ,α)-constacyclic code of 10 over $\mathcal{T}$, where $\alpha = -\eta_{111}+\eta_{112}+\eta_{113}+\eta_{121}+\eta_{122}+\eta_{123}-\eta_{211}+\eta_{212}-\eta_{213}+\eta_{221}+\eta_{222}+\eta_{223}+\eta_{311}+\eta_{312}+\eta_{313}+\eta_{321}-\eta_{322}+\eta_{323} = 1-2u_2-2u_1u_2+u_1u_3+2u_1u_2u_3+u_1u_3^2-u_1^2u_3-u_1^2u_3^2+2u_2u_3^2+2u_1u_2u_3^2-2u_1^2u_2u_3 \in \mathcal{U}(\mathcal{T})$. Let $M = M_{25,2}\otimes M_{25,2}\otimes M_{25,3}$ be the matrix used in Gray map $\phi : \mathcal{T} \to \mathbb{F}_{25}^{18}$, where

$$M_{25,2} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, M_{25,3} = \begin{bmatrix} 4w+1 & 3w+2 & w+3 \\ 3w+2 & 3w+1 & w \\ 4w+2 & 4w & 3w \end{bmatrix} \text{ and } MM^T = (4w+2)I_{18}.$$

Then $\Phi(\mathcal{C})$ is a skew quasi $(-1,1,1,1,1,1,-1,1,-1,1,1,1,1,1,1,1,-1,1)$-twisted Hermitian dual-containing $[180,170,4]$ code over $\mathbb{F}_{5^2}$. Hence by Theorem 5.2.5, there exists a $[[180,160,\geq 4]]_5$ quantum code which is a new code as per database [9].

Table 5.10: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1]/\langle u_1^2 - u_1\rangle,\ \alpha = \eta_1\alpha_1 + \eta_2\alpha_2$

q	n	(α_1, α_2)	$f_1(y)$	$f_2(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
9	6	$(-1,-1)$	$w^2 11$	$w^5 w 11$	$[12,7,5]$	$[[12,2,\geq 5]]_3$	new
9	8	$(-1,1)$	$w w^5 1$	$w^7 w^5 w^6 1$	$[16,11,5]$	$[[16,6,\geq 5]]_3$	new
9	8	$(-1,-1)$	$11w21$	$w^7 21$	$[16,10,6]$	$[[16,4,\geq 6]]_3$	new
9	10	$(1,-1)$	$1ww^6 w^5 21$	$w^7 1 w^6 w^5 w^7 1$	$[20,11,6]$	$[[20,2,\geq 6]]_3$	new
9	10	$(1,-1)$	$1w^5 w^6 w1$	11021	$[20,12,5]$	$[[20,4,\geq 5]]_3$	new
25	6	$(1,-1)$	$4w^{21} 1$	$w^{22} w^7 w^9 1$	$[12,7,6]$	$[[12,2,6]]_5$	MDS, new
25	12	$(1,-1)$	$w^2 w w^{15} w^{17} w^7 1$	$w w^3 w^2 w^{16} 11$	$[24,14,9]$	$[[24,4,\geq 9]]_5$	new

Table 5.11: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1]/\langle u_1^3 - u_1\rangle,\ \alpha = \eta_1\alpha_1 + \eta_2\alpha_2 + \eta_3\alpha_3$

q	n	$(\alpha_1, \alpha_2, \alpha_3)$	$f_1(y)$	$f_2(y)$	$f_3(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
25	6	$(1,-1,-1)$	$4w^{21} 1$	$w^{22} w^7 w^9 1$	$4w^{14} 1$	$[18,11,6]$	$[[18,4,\geq 6]]_5$	new
25	6	$(-1,-1,-1)$	$w^{22} 1$	$w^{22} w^7 w^9 1$	$4w^{14} 1$	$[18,12,5]$	$[[18,6,\geq 5]]_5$	$[[18,2,5]]_5$ [50]
25	8	$(-1,1,-1)$	$1w^9 w^{10} w^{21} 1$	$w^{23} w^{22} 1$	$1w^{17} w^2 w^5 1$	$[24,14,7]$	$[[24,4,\geq 7]]$	new

Table 5.12: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2]/\langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1\rangle,\ \alpha = \eta_{11}\alpha_{11} + \eta_{12}\alpha_{12} + \eta_{21}\alpha_{21} + \eta_{22}\alpha_{22}$

q	n	$(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$	$f_{11}(y)$	$f_{12}(y)$	$f_{21}(y)$	$f_{22}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
9	6	$(-1,-1,-1,-1)$	$w^3 1 w^3 1$	$w^5 1$	$w^2 11$	$w^5 w 11$	$[24,15,6]$	$[[24,6,\geq 6]]_3$	new
25	6	$(1,-1,-1,1)$	$4w^{17} 1$	$w^{22} w^7 w^9 1$	$w^{22} 1$	$4w^9 1$	$[24,16,6]$	$[[24,8,\geq 6]]_5$	new

TABLE 5.13: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2]/\langle u_1^2 - u_1, u_2^3 - u_2, u_1u_2 - u_2u_1\rangle$, $\alpha = \sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\eta_{i_1i_2}\alpha_{i_1i_2}$

q	n	$(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23})$	$f_{11}(y)$	$f_{12}(y)$	$f_{13}(y)$	$f_{21}(y)$	$f_{22}(y)$	$f_{23}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
9	6	$(-1, 1, -1, -1, -1, 1)$	w^31w^31	11	w^211	w^51	w^5w11	21	$[36, 25, 6]$	$[[36, 14, \geq 6]]_3$	new
25	6	$(1, 1, -1, 1, -1, 1)$	$4w^{17}1$	1	$w^{22}w^7w^91$	1	$w^{22}1$	$4w^91$	$[36, 28, 5]$	$[[36, 20, \geq 5]]_5$	$[[36, 18, 5]]_5$ [67]
25	6	$(1, -1, -1, 1, -1, 1)$	$4w^{17}1$	w^21	$w^{22}w^7w^91$	1	$w^{22}1$	$4w^91$	$[36, 27, 6]$	$[[36, 18, \geq 6]]_5$	$[[36, 18, 5]]_5$ [67]

TABLE 5.14: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2, u_3]/\langle u_1^2 - u_1, u_2^2 - u_2, u_3^2 - 1, u_iu_j - u_iu_i\rangle$, $\alpha = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\eta_{i_1i_2i_3}\alpha_{i_1i_2i_3}$, $\beta = (\alpha_{111}, \alpha_{112}, \ldots, \alpha_{222})$

q	n	β	$f_{111}(y)$	$f_{112}(y)$	$f_{121}(y)$	$f_{122}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{221}(y)$	$f_{222}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
9	12	$(1, 1, 1, 1, 1, -1, 1, 1)$	$w1w1$	$w1$	1	$w1$	1	$1w^71$	1	$w1$	$[96, 88, 4]$	$[[96, 80, \geq 4]]_3$	new
25	8	$(1, 1, 1, 1, -1, 1, 1, 1)$	31	1	31	1	$w^{17}w^21$	1	1	31	$[64, 59, 3]$	$[[64, 54, \geq 3]]_5$	new
25	10	$(1, -1, 1, -1, 1, 1, 1, 1)$	$w^{20}1$	$w^{22}1$	1	$w^{22}1w^42w^{22}1$	1	1	$w^{20}1$	1	$[80, 72, 4]$	$[[80, 64, \geq 4]]_5$	new

TABLE 5.15: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2]/\langle u_1^3 - u_1, u_2^3 - u_2, u_1u_2 - u_2u_1\rangle$, $\alpha = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\eta_{i_1i_2}\alpha_{i_1i_2}$, $\beta = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{33})$

q	n	β	$f_{11}(y)$	$f_{12}(y)$	$f_{13}(y)$	$f_{21}(y)$	$f_{22}(y)$	$f_{23}(y)$	$f_{31}(y)$	$f_{32}(y)$	$f_{33}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
9	6	$(1, 1, -1, 1, -1, -1, -1, -1, 1)$	w^61	1	w^31w^31	11	w^71	w^211	w^51	w^5w11	21	$[54, 41, 6]$	$[[54, 28, \geq 6]]_3$	new
9	12	$(1, 1, 1, 1, 1, 1, -1, 1, 1)$	1	$w1w1$	$w1$	1	$w1$	1	$1w^71$	1	$w1$	$[108, 100, 4]$	$[[108, 92, \geq 4]]_3$	new
25	6	$(-1, -1, -1, -1, -1, 1, -1, 1, -1)$	21	w^21	$w^{14}1$	$4w^21$	$w^{22}w^7w^91$	1	$w^{22}1$	$4w^91$	$w^{18}1$	$[54, 42, 6]$	$[[54, 30, \geq 6]]_5$	new
25	10	$(1, 1, -1, 1, -1, 1, 1, 1, 1)$	$w^{20}1$	1	$w^{22}1$	1	$w^{22}1w^42w^{22}1$	1	1	$w^{20}1$	1	$[90, 82, 4]$	$[[90, 74, \geq 4]]_5$	new

TABLE 5.16: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2,u_3]/\langle u_1^3-u_1,u_2^2-u_2,u_3^2-1,u_iu_j-u_ju_i\rangle$, $\alpha=\sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\eta_{i_1i_2i_3}\alpha_{i_1i_2i_3}$, $\beta=(\alpha_{111},\alpha_{112},\ldots,\alpha_{322})$

q	n	β	$f_{111}(y)$	$f_{112}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{211}(y)$	$f_{212}(y)$
9	6	$(1,-1,1,1,1,-1,1,-1,-1,1,-1,1)$	$w^6 1$	$w1$	1	$w^6 1$	1	$w^3 1 w^3 1$
25	10	$(-1,1,1,-1,1,-1,1,1,1,1,-1,1)$	$w^{22}1$	1	$w^{20}1$	$w^{22}1$	1	$w^{22}1w^4 2w^{22}1$

Table 5.16 continued..

$f_{221}(y)$	$f_{222}(y)$	$f_{311}(y)$	$f_{312}(y)$	$f_{321}(y)$	$f_{322}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
1	$w^7 1$	$w^2 11$	1	$w^5 w11$	1	$[72,60,6]$	$[[72,48,\geq 6]]_3$	$[[72,48,4]]_3$ [67]
1	1	$w^{20}1$	1	$w^{22}1$	1	$[120,110,4]$	$[[120,100,\geq 4]]_5$	$[[120,88,3]]_5$[2]

TABLE 5.17: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2,u_3,u_4]/\langle u_1^2-u_1,u_2^2-u_2,u_3^2-1,u_4^2-1u_iu_j-u_ju_i\rangle$, $\alpha=\sum_{i_1,i_2,i_3,i_4=1}^{2}\eta_{i_1i_2i_3i_4}\alpha_{i_1i_2i_3i_4}$, $\beta=(\alpha_{1111},\alpha_{1112},\ldots,\alpha_{2222})$

q	n	β	$f_{1111}(y)$	$f_{1112}(y)$	$f_{1121}(y)$	$f_{1122}(y)$	$f_{1211}(y)$	$f_{1212}(y)$	$f_{1221}(y)$	$f_{1222}(y)$
9	12	$(1,1,1,1,1,1,1,1,1,1,-1,1,1,1,1,1)$	1	$w1$	1	$w1$	1	$w1w1$	$w1$	1
25	8	$(1,1,1,1,-1,1,1,1,1,1,1,1,1,1,1,1)$	31	1	1	1	$w^{27}w^2 1$	1	1	1
25	10	$(-1,1,1,1,1,-1,1,-1,1,1,1,1,1,1,-1,1)$	$w^{22}1$	1	1	1	$w^{20}1$	$w^{22}1$	1	$w^{22}1w^4 2w^{22}1$

Table 5.17 continued...

$f_{2111}(y)$	$f_{2112}(y)$	$f_{2121}(y)$	$f_{2122}(y)$	$f_{2211}(y)$	$f_{2212}(y)$	$f_{2221}(y)$	$f_{2222}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
$w1$	1	$1w^7 1$	1	$w1$	1	$w1$	1	$[192,181,4]$	$[[192,170,\geq 4]]_3$	new
31	1	31	1	1	1	1	31	$[128,122,3]$	$[[128,116,\geq 3]]_5$	new
1	1	$w^{20}1$	1	1	1	$w^{22}1$	1	$[160,150,4]$	$[[160,140,\geq 4]]_5$	new

TABLE 5.18: Quantum Codes from Skew constacyclic codes over $\mathbb{F}_q[u_1,u_2,u_3]/\langle u_1^3-u_1, u_2^2-u_2, u_3^3-1, u_iu_j-u_ju_i\rangle$, $\alpha=\sum_{i_1=1}^{3}\sum_{i_2=1}^{2}\sum_{i_3=1}^{3}\eta_{i_1i_2i_3}\alpha_{i_1i_2i_3}$, $\beta=(\alpha_{111},\alpha_{112},\dots,\alpha_{323})$

q	n	β	$f_{111}(y)$	$f_{112}(y)$	$f_{113}(y)$	$f_{121}(y)$	$f_{122}(y)$	$f_{123}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{213}(y)$
9	12	$(1,1,1,1,1,1,1,1,1,1,-1,1,1,1,1,1)$	1	$w1$	1	$w1$	1	$w1w1$	$w1$	1	$w1$
25	10	$(-1,1,1,1,1,1,-1,1,-1,1,1,1,1,1,1,1,-1,1)$	$w^{22}1$	1	1	1	$w^{20}1$	1	$w^{22}1$	1	$w^{22}1w^{4}2w^{22}1$

Table 5.18 continued...

$f_{221}(y)$	$f_{222}(y)$	$f_{223}(y)$	$f_{311}(y)$	$f_{312}(y)$	$f_{313}(y)$	$f_{321}(y)$	$f_{322}(y)$	$f_{323}(y)$	$\Phi(\mathcal{C})$	$[[n,k,d]]_q$	Remark
1	$1w^{7}1$	1	$w1$	1	$w1$	1	$w1$	1	$[216,204,4]$	$[[216,192,\geq 4]]_3$	new
1	1	1	$w^{20}1$	1	1	1	$w^{22}1$	1	$[180,170,4]$	$[[180,160,\geq 4]]_5$	new

5.3 Euclidean and Hermitian LCD Skew (Θ, α)-Constacyclic Codes over $\mathcal{T}$

This section deals with the construction of Euclidean and Hermitian LCD codes over $\mathcal{T}$ from skew constacyclic codes. We begin by reviewing the definition of LCD codes and an essential criterion established by Boulanouar et al. [20] that identifies when a skew constacyclic code over a finite field is LCD. Further, we present a technique for deriving LCD codes from skew constacyclic codes over $\mathcal{T}$, based on these criteria and a decomposition method for skew constacyclic codes. Finally, we conclude this section with illustrative examples.

Definition 5.3.1. A linear code $\mathcal{C}$ over $\mathcal{T}$ whose Hull is trivial, is called a Linear Complementary Dual (LCD) code.

$Hull_E(C) = \mathcal{C} \cap \mathcal{C}^{\perp_E}$, $Hull_H(C) = \mathcal{C} \cap \mathcal{C}^{\perp_H}$ are called Euclidean and Hermitian hulls of $\mathcal{C}$ respectively and $\mathcal{C}$ is Euclidean (Hermitian) LCD if $Hull_E(C) = 0$ ($Hull_H(C) = 0$).

Now, we recall a characterization of Euclidean and Hermitian skew constacyclic codes over a finite field to be LCD, provided by Boulanour et al. [20].

Lemma 5.3.2. ([20, Theorem 2]) Let $\theta_t \in \mathrm{Aut}(\mathbb{F}_q)$ and $\gamma \in \mathbb{F}_q^*$ be such that $\gamma^2 = 1$. Furthermore, let $C = \langle f(y) \rangle$ be a skew (θ_t, γ)-constacyclic code of length n over $\mathbb{F}_q$ such that $t \mid n$, and let $g(y) \in \mathbb{F}_q[y; \theta_t]$ be such that $g(y)f(y) = f(y)g(y) = y^n - \gamma$.

- (i) C is Euclidean LCD if and only if $GCRD(f, g^\dagger) = 1$.
- (ii) If q is an even power of a prime number i.e., $q = p^{2e}$, L is Hermitian LCD if and only if $GCRD(f, \overline{g}^\dagger) = 1$.

Theorem 5.3.3. A linear code $\mathcal{C} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$ of length n over $\mathcal{T}$ is an LCD code if and only if $\mathcal{C}_{i_1 i_2 \dots i_r}$ is an LCD code of length n over $\mathbb{F}_q$, for all $i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r$.

Proof. Since $\mathcal{C}^{\perp} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}^{\perp}_{i_1 i_2 \dots i_r}$, we have

$$\begin{aligned}
\mathcal{C} \cap \mathcal{C}^{\perp} &= \Big(\bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r} \Big) \cap \Big(\bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r} \Big)^{\perp} \\
&= \Big(\bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r} \Big) \cap \Big(\bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}^{\perp}_{i_1 i_2 \dots i_r} \Big) \\
&= \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} (\mathcal{C}_{i_1 i_2 \dots i_r} \cap \mathcal{C}^{\perp}_{i_1 i_2 \dots i_r})
\end{aligned}$$

Therefore, $Hull(\mathcal{C}) = \{0\}$ if and only if $Hull(\mathcal{C}_{i_1 i_2 \dots i_r}) = \{0\}$, for all $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2 \dots r$. Hence, the result follows. ☐

Theorem 5.3.4. Let $\alpha = \sum_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \alpha_{i_1 i_2 \dots i_r} \in \mathcal{U}(\mathcal{T})$ be such that $\alpha^2 = 1$, $\Theta_t \in \mathrm{Aut}(\mathcal{T})$ and n be a multiple of $o(\Theta_t)$. Furthermore, let $\mathcal{C} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$ be a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$ such that $\mathcal{C}_{i_1 i_2 \dots i_r} = \langle f_{i_1 i_2 \dots i_r}(y) \rangle$ and $g_{i_1 i_2 \dots i_r}(y) \in \mathbb{F}_q[y; \theta_t]$ be such that $g_{i_1 i_2 \dots i_r}(y) f_{i_1 i_2 \dots i_r}(y) = y^n - \alpha_{i_1 i_2 \dots i_r}$. Then $\mathcal{C}$ is Euclidean LCD if and only if $\mathrm{GCRD}(f_{i_1 i_2 \dots i_r}, g^{\dagger}_{i_1 i_2 \dots i_r}) = 1$ for all $i_j \in \{1, 2, \dots, l_j\}$, $j = 1, 2 \dots r$.

Proof. Let $\mathcal{C} = \bigoplus_{i_1 i_2 \dots i_r} \eta_{i_1 i_2 \dots i_r} \mathcal{C}_{i_1 i_2 \dots i_r}$ be a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$. If $\mathcal{C}$ is Euclidean LCD i.e. $\mathcal{C} \cap \mathcal{C}^{\perp_E} = \{0\}$, then by Theorem 5.3.3, we have $\mathcal{C}_{i_1 i_2 \dots i_r} \cap \mathcal{C}^{\perp_E}_{i_1 i_2 \dots i_r} = \{0\}$ for all $i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots r$. As n is a multiple of $o(\Theta_t)$ and $\alpha^2 = 1$, we have n is a multiple of $o(\theta_t)$ and $\alpha^2_{i_1 i_2 \dots i_r} = 1$ for all $i_j \in \{1, 2, \dots, l_j\}, j = 1, 2 \dots r$. Therefore by Lemma 5.3.2, $GCRD(f_{i_1 i_2 \dots i_r}, g^{\dagger}_{i_1 i_2 \dots i_r}) = 1$, for all $i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r$ where $y^n - \alpha_{i_1 i_2 \dots i_r} = g_{i_1 i_2 \dots i_r}(y) f_{i_1 i_2 \dots i_r}(y)$ and $g^{\dagger}_{i_1 i_2 \dots i_r}$ is the left monic skew reciprocal polynomial of $g_{i_1 i_2 \dots i_r}$. Conversely, let $GCRD(f_{i_1i_2\dots i_r}, g^{\dagger}_{i_1i_2\dots i_r}) = 1$ for all $i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r$. Then by Lemma 5.3.2, $\mathcal{C}_{i_1i_2\dots i_r} \cap \mathcal{C}^{\perp_E}_{i_1i_2\dots i_r} = \{0\}$ for all $i_j \in \{1, 2, \dots, l_j\},\ j = 1, 2, \dots, r$. Hence, by Theorem 5.3.3, $\mathcal{C}$ is a Euclidean LCD code. □

Theorem 5.3.5. Let q be an even power of a prime number, say $q = p^{2e}$, $\alpha = \sum_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\alpha_{i_1i_2\dots i_r} \in \mathcal{U}(\mathcal{T})$ be such that $\alpha^2 = 1$, $\Theta_t \in \mathrm{Aut}(\mathcal{T})$ and n be a multiple of $o(\Theta_t)$. Furthermore, let $\mathcal{C} = \bigoplus_{i_1i_2\dots i_r} \eta_{i_1i_2\dots i_r}\mathcal{C}_{i_1i_2\dots i_r}$ be a skew (Θ_t, α)-constacyclic code of length n over $\mathcal{T}$ such that $\mathcal{C}_{i_1i_2\dots i_r} = \langle f_{i_1i_2\dots i_r}(y)\rangle$ and $g_{i_1i_2\dots i_r}(y) \in \mathbb{F}_q[y;\theta_t]$ be such that $g_{i_1i_2\dots i_r}(y)f_{i_1i_2\dots i_r}(y) = y^n - \alpha_{i_1i_2\dots i_r}$. Then $\mathcal{C}$ is Hermitian LCD if and only if $\mathrm{GCRD}(f_{i_1i_2\dots i_r}, \overline{g}^{\dagger}_{i_1i_2\dots i_r}) = 1$ for all $i_j \in \{1, 2, \dots, l_j\}, j = 1, 2 \dots r$.

Proof. The proof is similar to that of Theorem 5.3.4. □

Lemma 5.3.6. For a linear code $\mathcal{C}$ of length n over $\mathcal{T}$, $\Phi(Hull(\mathcal{C})) = Hull(\Phi(\mathcal{C}))$.

Proof. Let $\mathbf{w} \in \Phi(Hull(\mathcal{C}))$. Since Φ is onto, $\exists\ \mathbf{v} \in Hull(\mathcal{C})$ such that $\Phi(\mathbf{v}) = \mathbf{w}$. As $\mathbf{v} \in Hull(\mathcal{C})$, $\mathbf{v} \in \mathcal{C}$ and $\mathbf{v} \in \mathcal{C}^{\perp}$. Therefore, $\mathbf{w} \in \Phi(\mathcal{C})$ and $\mathbf{w} \in \Phi(\mathcal{C}^{\perp})$ and so $\mathbf{w} \in \Phi(\mathcal{C}) \cap \Phi(\mathcal{C}^{\perp}) = \Phi(\mathcal{C}) \cap \Phi(\mathcal{C})^{\perp} = Hull(\Phi(\mathcal{C}))$. Since, $\mathbf{w} \in \Phi(Hull(\mathcal{C}))$ is arbitrary, we have $\Phi(Hull(\mathcal{C})) \subseteq Hull(\Phi(\mathcal{C}))$.

Conversely, let $\mathbf{w} \in Hull(\Phi(\mathcal{C}))$, i.e. $\mathbf{w} \in \Phi(\mathcal{C})$ and $\mathbf{w} \in \Phi(\mathcal{C})^{\perp} = \Phi(\mathcal{C}^{\perp})$. Then, $\exists\ \mathbf{u} \in \mathcal{C}$ and $\exists\ \mathbf{v} \in \mathcal{C}^{\perp}$ such that $\Phi(\mathbf{u}) = \mathbf{w}$ and $\Phi(\mathbf{v}) = \mathbf{w}$. Since Φ is one-one as well, we have $\mathbf{u} = \mathbf{v}$ and so $\mathbf{u}(= \mathbf{v}) \in \mathcal{C} \cap \mathcal{C}^{\perp}$. Therefore, $\Phi(\mathbf{u}) = \mathbf{w} \in \Phi(\mathcal{C} \cap \mathcal{C}^{\perp}) = \Phi(Hull(\mathcal{C}))$. Since $\mathbf{w} \in Hull(\Phi(\mathcal{C}))$ is arbitrary, we have $Hull(\Phi(\mathcal{C})) \subseteq \Phi(Hull(\mathcal{C}))$. Hence, $\Phi(Hull(\mathcal{C})) = Hull(\Phi(\mathcal{C}))$. □

Theorem 5.3.7. A linear code of length n over $\mathcal{T}$ is LCD code if and only if its Gray image is a q-ary LCD code of length $l_1l_2\dots l_rn$.

Proof. Let $\mathcal{C}$ be an LCD code of length n over the ring $\mathcal{T}$. Then by definition, $Hull(\mathcal{C}) = \{0\}$. By Lemma 5.3.6, we get $Hull(\Phi(\mathcal{C})) = \Phi(Hull(\mathcal{C})) = \Phi(\{0\}) = \{0\}$

which concludes that $\Phi(\mathcal{C})$ is an LCD code of length $l_1 l_2 \dots l_r n$ over $\mathcal{T}$. Conversely, if $\Phi(\mathcal{C})$ is an LCD of length $l_1 l_2 \dots l_r n$ over $\mathbb{F}_q$ then $Hull(\Phi(\mathcal{C})) = \{0\}$. Therefore, by Lemma 5.3.6, we have $\Phi(Hull(\mathcal{C})) = Hull(\Phi(\mathcal{C})) = \{0\}$ which implies that $Hull(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^{\perp} = \{0\}$, as Φ is one-one. Hence, $\mathcal{C}$ is an LCD code of length n over $\mathcal{T}$. □

Now, we utilize the results obtained in this section to provide some examples of LCD codes over $\mathcal{T}$. For computation purposes, SageMath [90] and MAGMA [17, 23] software are used.

Example 5.3.8. Consider $\mathbb{F}_9 = \mathbb{F}_3[X]/\langle X^2 + 2X + 2\rangle = \mathbb{F}_3(w)$, where $w^2 = w + 1$. Let $\theta_1 : b \mapsto b^3$ be the Frobenius automorphism. For $n = 4$, consider a factorization of $y^n - 1$ as

$$y^4 - 1 = (y + 2)(y + 1)(y + w + 2)(y + w).$$

If we take $u_1(y) = 1$ and $v_1(y)u_1(y) = y^4 - 1$, then $v_1(y) = y^4 - 1$ and $v_1^{\dagger}(y) = y^4 - 1$. Clearly, $GCRD(u_1(y), v_1^{\dagger}(y)) = 1$. Next, let $u_2(y) = y + w$ and $v_2(y)u_2(y) = y^4 - 1$. Then, $v_2(y) = (y + 2)(y + 1)(y + w + 2) = y^3 + (w + 2)y^2 + 2y + 2w + 1$ and $v_2^{\dagger}(y) = y^3 + (2w + 1)y^2 + 2y + w + 2$. Thus, $GCRD(u_2(y), v_2^{\dagger}(y)) = 1$.

Consider another factorization of $y^4 - 1$ as

$$y^4 - 1 = (y + 2)(y + 2w)(y + 2)(y + w + 2).$$

Let $u_3(y) = (y + 2)(y + w + 2) = y^2 + (2w + 2)y + 2w + 1 = y^2 + w^6 y + w^3$ and $v_3(y)u_3(y) = y^4 - 1$. Then, $v_3(y) = (y + 2)(y + 2w) = y^2 + (w + 1)y + w$, $v_3^{\dagger}(y) = y^2 + 2wy + w + 2$, and $GCRD(u_3(y), v_3^{\dagger}(y)) = 1$.

Further, let us consider a factorization of y^4+1 as

$$y^4+1=(y^2+(2w+2)y+2)(y^2+(w+1)y+2).$$

If we take $u_4(y)=y^2+(w+1)y+2$ and $v_4(y)u_4(y)=y^4+1$, then $v_4(y)=y^2+(2w+2)y+2$ and $v_4(y)^{\dagger}=y^2+(2w+2)y+2$. This gives $GCRD(u_4(y),v_4^{\dagger}(y))=1$.

Now, let $T_1=\mathbb{F}_9[u_1]/\langle u_1^2-w^2\rangle$. Then, $\eta_1=\frac{w+u_1}{2w}$ and $\eta_2=\frac{w-u_1}{2w}$ are primitive orthogonal idempotent elements. So, every element v of T_1 can be uniquely written as $\mathsf{v}=\eta_1 v_1+\eta_2 v_2$, for some $v_1,v_2\in\mathbb{F}_9$. Let us define $\Theta_1:T_1\to T_1$ as $\eta_1 v_1+\eta_2 v_2\mapsto\eta_1 v_1^3+\eta_2 v_2^3$, which is an automorphism. Let $f_1(y)=u_4(y)$ and $f_2(y)=u_2(y)$. Then, by Theorem 5.3.4, $\mathcal{C}=\langle\eta_1 f_1+\eta_2 f_2\rangle$ is a skew (Θ_1,w^3u_1)-constacyclic Euclidean LCD code of length 4, size 9^5 and minimum Lee distance 4 over T_1. Let $\phi:T_1\to\mathbb{F}_q^2$ be a Gray map defined as $\mathbf{v}\longmapsto(v_1+v_2,v_1-2v_2)$ and Φ be the extension of ϕ over T_1^4. Hence, $\Phi(\mathcal{C})$ is an $[8,5,4]_9$ skew quasi-$(-1,1)$-twisted Euclidean LCD code, which is a maximum distance separable **(MDS)** code.

Furthermore, let $T_3=\mathbb{F}_9[u_1,u_2,u_3]/\langle u_i^2-u_i,u_iu_j-u_ju_i\rangle_{i,j=1}^3$. Let $\kappa_{j,1}=u_j$ and $\kappa_{j,2}=1-u_j$. Then, $\{\eta_{i_1i_2i_3}=\kappa_{1,i_1}\kappa_{2,i_2}\kappa_{3,i_3}:i_j=1,2\}$ is set of primitive orthogonal idempotent elements. So, every element v of T_3 can be uniquely written as $\mathbf{v}=\sum_{i_1=1}^2\sum_{i_2=1}^2\sum_{i_3=1}^2\eta_{i_1i_2i_3}v_{i_1i_2i_3}$. Let us define an automorphism $\Theta_1:T_3\to T_3$ as

$$\sum_{i_1=1}^2\sum_{i_2=1}^2\sum_{i_3=1}^2\eta_{i_1i_2i_3}v_{i_1i_2i_3}\longmapsto\sum_{i_1=1}^2\sum_{i_2=1}^2\sum_{i_3=1}^2\eta_{i_1i_2i_3}v_{i_1i_2i_3}^3.$$

Let us take $f_{111}(y)=f_{121}(y)=f_{222}(y)=u_2(y)\in\mathbb{F}_9[y;\theta_2]/\langle y^4-1\rangle$, $f_{112}(y)=f_{122}(y)=f_{212}(y)=f_{221}(y)=u_1(y)\in\mathbb{F}_9[y;\theta_2]/\langle y^4-1\rangle$ and $f_{211}(y)=u_3(y)\in\mathbb{F}_9[y;\theta_1]/\langle y^4-1\rangle$. From the above discussion, it is clear that $GCRD(f_{i_1i_2i_3},g_{i_1i_2i_3}^{\dagger})=1$, for all $i_j=1,2$, $j=1,2,3$, where $g_{i_1i_2i_3}f_{i_1i_2i_3}(y)(y)=y^4-1$. Then, $\mathcal{C}=\langle\sum_{i_1=1}^2\sum_{i_2=1}^2\sum_{i_3=1}^2\eta_{i_1i_2i_3}f_{i_1i_2i_3}\rangle$ is a skew Θ_1-cyclic LCD code of length 4, size 9^{27}

and minimum Lee distance 4 over T_3. The matrix M used in Gray map $\phi : T_3 \to \mathbb{F}_9^8$ is given as $M = M_{9,2} \otimes M_{9,2} \otimes M_{9,2}$, where $M_{9,2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Hence, $\Phi(\mathcal{C})$ is a $[32, 27, 4]_9$ skew 8-quasi cyclic LCD code which is a best known linear code **(BKLC)**.

Example 5.3.9. For $q = 25$, consider $\mathbb{F}_q = \mathbb{F}_{25} = \mathbb{F}_5[X]/\langle X^2 + 4X + 2\rangle = \mathbb{F}_5(w)$, where $w^2 + 4w + 2 = 0$. Let $\theta_1 : b \mapsto b^5$ be the Frobenius automorphism on $\mathbb{F}_{25}$. For $n = 6$, consider a factorization of $y^n - 1$ as

$$y^6 - 1 = (y^2 + (2w + 4)y + 4)(y^2 + (3w + 1)y + 4)(y + 1)(y + 4).$$

If we take $u_1(y) = 1$ and $v_1(y)u_1(y) = y^6 - 1$. Then, $v_1(y) = y^6 - 1$ and $v_1^\dagger(y) = y^6 - 1$. Clearly, $GCRD(u_1(y), \overline{v_1}^\dagger(y)) = 1$. Next, let $u_2(y) = y + 4$ and $v_1(y)u_1(y) = y^6 - 1$. Then, $v_2(y) = (y^2 + (2w+4)y+4)(y^2 + (3w+1)y+4)(y+1) = y^5 + y^4 + y^3 + y^2 + y + 1$ and $\overline{v_2}^\dagger(y) = y^5 + y^4 + y^3 + y^2 + y + 1$. We have, $GCRD(u_2(y), \overline{v_2}^\dagger(y)) = 1$. Consider another factorization of $y^6 - 1$ as

$$y^6 - 1 = (y^2 + (3w + 3)y + 1)(y^2 + (2w + 2)y + 1)(y + 3w + 3)(y + 3w + 4).$$

Let $u_3(y) = y + 3w + 4$ and $v_3(y)u_3(y) = y^6 - 1$. Then, $v_3(y) = (y^2 + (3w + 3)y + 1)(y^2 + (2w + 2)y + 1)(y + 3w + 3) = y^5 + (3w + 3)y^4 + y^3 + (3w + 3)y^2 + y + 3w + 3$ and $\overline{v_3}^\dagger(y) = y^5 + (2w + 1)y^4 + y^3 + (2w + 1)y^2 + y + 2w + 1$. Again, we get, $GCRD(u_3(y), \overline{v_3}^\dagger(y)) = 1$. Furthermore, let us consider a factorization of $y^6 + 1$ as

$$y^6 + 1 = (y^2 + (2w + 4)y + 2w + 2)(y + 2)(y + 3)(y^2 + (3w + 1)y + 3w + 4).$$

If we take $u_4(y) = y^2 + (3w + 1)y + 3w + 4$ and $v_4(y)u_4(y) = y^4 + 1$, then $v_4(y) = (y^2+(2w+4)y+2w+2)(y+2)(y+3) = y^4+(2w+4)y^3+(2w+3)y^2+(2w+4)y+2w+2$ and $\overline{v_4}(y)^\dagger = y^4+wy^3+(2w+3)y^2+wy+2w+2$. This gives $GCRD(u_4(y), \overline{v_4}^\dagger(y)) = 1$.

Now, let $\mathcal{T} = T_2 = \mathbb{F}_{25}[u_1, u_2]/\langle u_i^2 - u_i, u_iu_j - u_ju_i\rangle_{i,j=1}^2$. Further, let $\kappa_{j,1} = u_j$ and $\kappa_{j,2} = 1 - u_j,\ j = 1, 2$. Then, $\{\eta_{i_1i_2} = \kappa_{1,i_1}\kappa_{2,i_2} : i_j = 1, 2\}$ is set of primitive orthogonal elements. So, every v element of T_2 can be uniquely written as $\mathsf{v} = \sum_{i_1=1}^2 \sum_{i_2=1}^2 \eta_{i_1i_2}v_{i_1i_2}$. Let us define an automorphism $\theta_1 : \mathcal{T} \to \mathcal{T}$ as

$$\sum_{i_1=1}^{2}\sum_{i_2=1}^{2} \eta_{i_1i_2}v_{i_1i_2} \longmapsto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2} \eta_{i_1i_2}v_{i_1i_2}^5.$$

If we take $f_{11}(y) = u_2(y) \in \mathbb{F}_{25}[y;\theta_1]/\langle y^6 - 1\rangle$, $f_{12}(y) = u_4(y) \in \mathbb{F}_{25}[y;\theta_1]/\langle y^6 + 1\rangle$, $f_{21}(y) = u_1(y) \in \mathbb{F}_{25}[y;\theta_1]/\langle y^6 - 1\rangle$ and $f_{22}(y) = u_3(y) \in \mathbb{F}_{25}[y;\theta_1]/\langle y^6 - 1\rangle$. From the above discussion, it is clear that $GCRD(f_{i_1i_2}(y), \overline{g}^{\dagger}_{i_1i_2}(y)) = 1$, for all $i_j = 1, 2$, $j = 1, 2$, where $f_{i_1i_2}(y)f_{i_1i_2}(y) = y^6 - \alpha_{ij}$ with $\alpha_{11} = \alpha_{21} = \alpha_{22} = 1$ and $\alpha_{12} = -1$. Then, $\mathcal{C} = \langle \sum_{i_1=1}^2 \sum_{i_2=1}^2 \eta_{i_1i_2}f_{i_1i_2}\rangle$ is a skew $(\theta_1, 1 - 2u_2 + 2u_1u_2)$ constacyclic Hermitian LCD code of length 6, size 25^{20}, and minimum Lee distance 4 over $\mathcal{T} = T_2$. The matrix M used in Gray map $\phi : \mathcal{T} \to \mathbb{F}_{25}^4$ is given as $M = M_{25,2} \otimes M_{25,2}$, where $M_{25,2} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$. Hence, $\Phi(\mathcal{C})$ is a $[24, 20, 4]_{25}$ skew quasi-(1,-1,1,1)-twisted Hermitian LCD code which is **almost MDS**.

Now, we conclude this section by enlisting Euclidean and Hermitian LCD codes over $\mathcal{T}$ and their Gray images.

TABLE 5.19: Euclidean LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1]/\langle u_1^2 - u_1 \rangle$ and their Gray images

q	n	(α_1, α_2)	$f_1(y)$	$f_2(y)$	$\Phi(\mathcal{C})$	Remark
3	4	$(1, 1)$	1	1111	$[8, 4, 4]$	AMDS
3	20	$(1, 1)$	1111	11111	$[40, 33, 4]$	BKLC
9	4	$(-1, 1)$	$2w^2 1$	$w1$	$[8, 5, 4]$	MDS

TABLE 5.20: Euclidean LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2]/\langle u_1^2 - 1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$ and their Gray images

q	n	$(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$	$f_{11}(y)$	$f_{12}(y)$	$f_{21}(y)$	$f_{22}(y)$	$\Phi(\mathcal{C})$	Remark
3	20	$(1, 1, 1, 1)$	11	1	1111	11111	$[80, 72, 4]$	BKLC
9	4	$(-1, 1, 1, 1)$	$2w^2 1$	$w1$	1	$w1$	$[16, 12, 4]$	AMDS, BKLC

TABLE 5.21: Euclidean LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2]/\langle u_1^2 - u_1, u_2^3 - u_2, u_1u_2 - u_2u_1 \rangle$

q	n	$(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23})$	$f_{11}(y)$	$f_{12}(y)$	$f_{13}(y)$	$f_{21}(y)$	$f_{22}(y)$	$f_{23}(y)$	$\Phi(\mathcal{C})$	Remark
9	4	$(1, 1, 1, 1, 1, 1)$	$w1$	1	$w1$	1	$w^3 w^6 1$	$w1$	$[24, 19, 4]$	BKLC

TABLE 5.22: Euclidean LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2, u_3]/\langle u_1^2 - u_1, u_2^2 - u_2, u_3^2 - 1, u_iu_j - u_iu_i\rangle$

q	n	$(\alpha_{111}, \alpha_{112}, \alpha_{121}, \alpha_{122}, \alpha_{211}, \alpha_{212}, \alpha_{221}, \alpha_{222}$	$f_{111}(y)$	$f_{112}(y)$	$f_{121}(y)$	$f_{122}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{221}(y)$	$f_{222}(y)$	$\Phi(\mathcal{C})$	Remark
9	4	$(1, 1, 1, 1, 1, 1, 1, 1)$	$w1$	1	$w1$	1	w^3w^61	1	1	$w1$	$[32, 27, 4]$	BKLC

TABLE 5.23: Hermitian LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1]/\langle u_1^2 - u_1\rangle$ and their Gray images

q	n	(α_1, α_2)	$f_1(y)$	$f_2(y)$	$\Phi(\mathcal{C})$	Remark
25	6	$(1, -1)$	$w^{20}1$	$w^{20}w^5w^{19}w^51$	$[12, 7, 4]$	

TABLE 5.24: Hermitian LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2]/\langle u_1^2 - 1, u_2^2 - u_2, u_1u_2 - u_2u_1\rangle$ and their Gray images

q	n	$(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$	$f_{11}(y)$	$f_{12}(y)$	$f_{21}(y)$	$f_{22}(y)$	$\Phi(\mathcal{C})$	Remark
25	6	$(1, -1, 1, 1)$	41	$w^{20}w^91$	1	$w^{20}1$	$[24, 20, 4]$	AMDS

TABLE 5.25: Hermitian LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2]/\langle u_1^2 - u_1, u_2^3 - u_2, u_1u_2 - u_2u_1\rangle$

q	n	$(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23})$	$f_{11}(y)$	$f_{12}(y)$	$f_{13}(y)$	$f_{21}(y)$	$f_{22}(y)$	$f_{23}(y)$	$\Phi(\mathcal{C})$	Remark
25	6	$(1, 1, -1, 1, 1, 1)$	1	41	$w^{20}w^91$	$w^{20}1$	1	$w^{20}1$	$[36, 31, 4]$	

TABLE 5.26: Hermitian LCD codes from Skew constacyclic codes over $\mathbb{F}_q[u_1, u_2, u_3]/\langle u_1^2 - u_1, u_2^2 - u_2, u_3^2 - 1, u_iu_j - u_iu_i\rangle$

q	n	$(\alpha_{111}, \alpha_{112}, \alpha_{121}, \alpha_{122}, \alpha_{211}, \alpha_{212}, \alpha_{221}, \alpha_{222}$	$f_{111}(y)$	$f_{112}(y)$	$f_{121}(y)$	$f_{122}(y)$	$f_{211}(y)$	$f_{212}(y)$	$f_{221}(y)$	$f_{222}(y)$	$\Phi(\mathcal{C})$	Remark
25	6	$(1, 1, 1, 1, -1, 1, 1, 1)$	$w^{20}1$	1	41	1	$w^{20}w^91$	$w^{20}1$	1	$w^{20}1$	$[48, 42, 4]$	

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