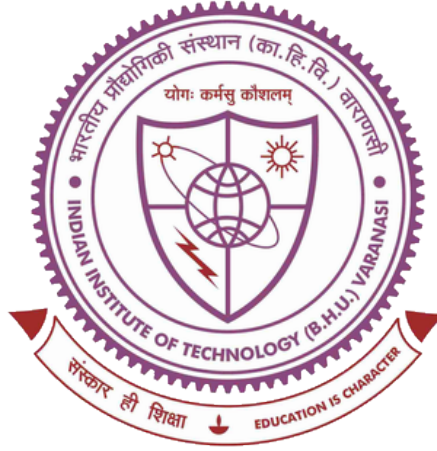


Study of Reaction-Advection-Diffusion Equations arising in Porous Media



Thesis submitted in partial fulfillment for the Award of degree
Doctor of Philosophy

By

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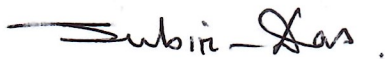
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I, "**Chetna Biswas**", certify that the work embodied in this thesis is my own bona fide work and carried out by me under the supervision of "**Prof. Subir Das**" from "**July 2019**" to "**April 2024**" at the "**Department of Mathematical Sciences**", Indian Institute of Technology (Banaras Hindu University), Varanasi. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not include them in this thesis and have not cited as my own work.

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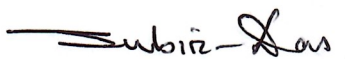


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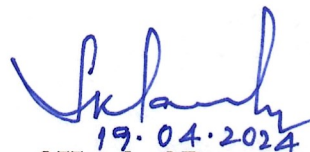
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Abbreviations

PDE	P artial D ifferential E quation
IDE	I ntegro D ifferential E quation
FPDE	F ractional P artial D ifferential E quation
LW	L egendre W avelets
RADE	R eaction- A dvection- D iffusion E quation
FRADE	F ractional-order R eaction- A dvection- D iffusion E quation
ADE	A dvection- D iffusion E quation
FRDE	F ractional-order R eaction- D iffusion E quation
SLCM	S hifted L egendre C ollocation M ethod
BFE	B urgers- F isher E quation