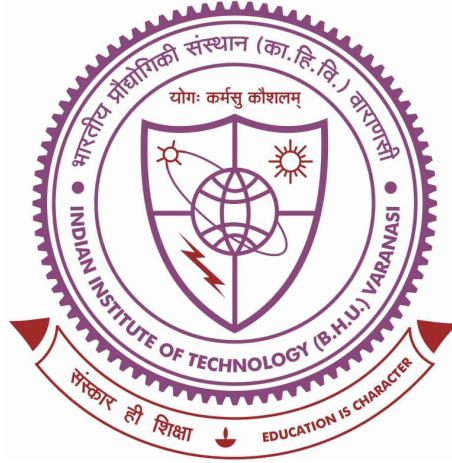


**Analysis of Discretization Methods for Time-Fractional
Diffusion and Diffusion-Wave Equations with Weak Initial
Singularity**



*The thesis submitted in partial fulfillment
for the Award of Degree*

Doctor of Philosophy

by

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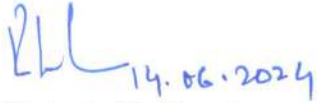
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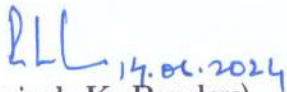
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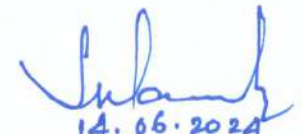
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Sarita Kumari

Dedicated to my parents

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Abbreviations

FC	Fractional Calculus
1D	One-Dimensional
2D	Two-Dimensional
PDE(s)	Partial Differential Equation(s)
TFCD	Time Fractional Caputo Derivative
TFCDs	Time Fractional Caputo Derivatives
R-L	Riemann-Liouville
GCFD	Generalized Caputo Fractional Derivative
TFDW(s)	Time Fractional Diffusion-Wave(s)
TFMDW(s)	Time Fractional Mixed Diffusion and Diffusion-Wave(s)
FDM(s)	Finite Differenc Method(s)
FD(s)	Fractional Derivative(s)
FDE(s)	Fractional Differential Equation(s)
ADI	Alternating Direction Implicit
OC	Order of Convergence
EOC	Expected Order of Convergence
LTE	Local Truncation Error

Symbols

\mathbb{R}	Set of real numbers
\mathbb{R}^+	Set of positive real numbers
\mathbb{N}	Set of natural numbers
\mathbb{Z}^+	Set of positive integers
$\gamma \geq 1$	Mesh grading parameter
Γ	Euler gamma function
$Re(\alpha)$	Real-part of α
$AC^n[a, b]$	n th-time absolutely continuous on $[a, b]$
$C[a, b]$	Continuous on $[a, b]$
$C^n[a, b]$	n th derivative continuous on $[a, b]$
$L^p(p \in [1, \infty])$	L^p space
$[x]$	greatest integer function
$\lceil x \rceil$	ceiling function

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