# Analysis of Discretization Methods for Time-Fractional Diffusion and Diffusion-Wave Equations with Weak Initial Singularity

***The thesis submitted in partial fulfillment***

***for the Award of Degree***

Doctor of Philosophy

by

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It is certified that the work contained in this thesis titled "**Analysis of Discretization Methods for Time-Fractional Diffusion and Diffusion-Wave Equations with Weak Initial Singularity**" by **Sarita Kumari** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

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I, **Sarita Kumari**, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of **Dr. Rajesh Kumar Pandey** from **July, 2019** to **June, 2024** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University) Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, etc., reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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**Sarita Kumari**

***Dedicated to my parents***

# Contents

# Abbreviations

| | |
|---|---|
| **FC** | **F**ractional **C**alculus |
| 1**D** | **O**ne-**D**imensional |
| 2**D** | **T**wo-**D**imensional |
| **PDE**(s) | **P**artial **D**ifferential **E**quation(s) |
| **TFCD** | **T**ime **F**ractional **C**aputo **D**erivative |
| **TFCDs** | **T**ime **F**ractional **C**aputo **D**erivatives |
| **R-L** | **R**iemann-**L**iouville |
| **GCFD** | **G**eneralized **C**aputo **F**ractional **D**erivative |
| **TFDW**(s) | **T**ime **F**ractional **D**iffusion-**W**ave(s) |
| **TFMDW**(s) | **T**ime **F**ractional **M**ixed **D**iffusion and **D**iffusion-**W**ave(s) |
| **FDM**(s) | **F**inite **D**ifferenc **M**ethod(s) |
| **FD**(s) | **F**ractional **D**erivative(s) |
| **FDE**(s) | **F**ractional **D**ifferential **E**quation(s) |
| **ADI** | **A**lternating **D**irection **I**mplicit |
| **OC** | **O**rder of **C**onvergence |
| **EOC** | **E**xpected **O**rder of **C**onvergence |
| **LTE** | **L**ocal **T**runcation **E**rror |

# Symbols

| | |
|---|---|
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{R}^+$ | Set of positive real numbers |
| $\mathbb{N}$ | Set of natural numbers |
| $\mathbb{Z}^+$ | Set of positive integers |
| $\gamma \geq 1$ | Mesh grading parameter |
| $\Gamma$ | Euler gamma function |
| $Re(\alpha)$ | Real-part of $\alpha$ |
| $AC^n[a,b]$ | $n$th-time absolutely continuous on $[a,b]$ |
| $C[a,b]$ | Continuous on $[a,b]$ |
| $C^n[a,b]$ | $n$th derivative continuous on $[a,b]$ |
| $L^p(p \in [1,\infty])$ | $L^p$ space |
| $[x]$ | greatest integer function |
| $\lceil x \rceil$ | ceiling function |

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