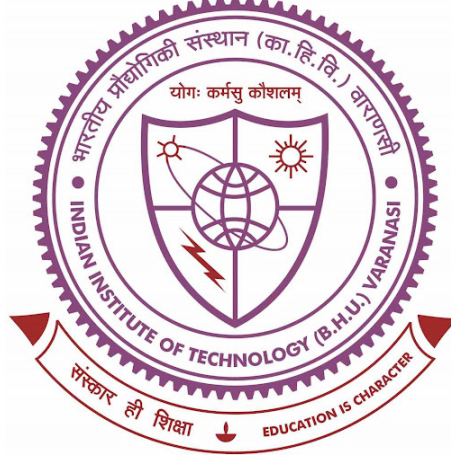

A Study of Graphs of Finite Groups



The thesis submitted in partial fulfillment

for the Award of Degree

DOCTOR OF PHILOSOPHY

by

AJAY KUMAR

DEPARTMENT OF MATHEMATICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY

(BANARAS HINDU UNIVERSITY)

VARANASI -221005

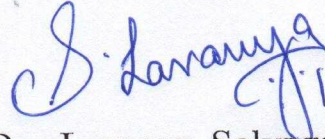
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16/03/2023

Dr. Lavanya Selvaganesh
Supervisor
Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)

Varanasi-221005
डा० लावण्या सेल्वगणेश
Dr. Lavanya Selvaganesh
सहायक आचार्य
Assistant Professor
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी/Varanasi-221005

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Ajay Kumar
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17121008

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S. Lavanya
16/03/2023

Dr. Lavanya Selvaganesh

Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

डा० लावण्या सेल्वगणेश
Dr. Lavanya Selvaganesh
सहायक आचार्य
Assistant Professor
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी/Varanasi-221005

S. K. Pandey
16.03.2023

Prof. S. K. Pandey

Professor and Head
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

विभागाध्यक्ष/HEAD
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी/Varanasi-221005

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ACKNOWLEDGEMENTS

Earning a Ph.D. in Mathematics from such a prestigious Institute is one of a lifetime experiences and I believe experiences are the only things that always count. Reaching to this stage of my life would never be possible without the loads of blessings of Almighty **Lord Shyam and Baba Vishwanath Ji** who always has been a ray of hope in all the difficult times. This journey has been a long memorable unforgettable times with ups and downs. It is a great pleasure and proud privilege to express my deep sense of gratitude to all whose inestimable support has made this dissertation possible.

First and foremost, I would like to express my gratitude, sincere appreciation and heartfelt thanks to my respected supervisor, **Dr. Lavanya Selvaganesh, Assistant Professor, Department of Mathematical Sciences, IIT(BHU), Varanasi**. I thank her for constant feedback, demand for clarity and posing interesting questions, advice and friendly support during my research. Her punctuality and strong work ethic has helped me develop both as a person and as a researcher. I would like thanks to **Professor Peter J. Cameron, Professor of Mathematics in the School of Mathematics and Statistics, University of St Andrews, Scotland** and **Professor T.Tamizh Chelvam, Emeritus Professor of Mathematics, Manonmaniam Sundaranar University, Tamil Nadu** for their inspirational talk, lectures which motivated me to do this work, for suggestions and sharing their experiences with us to improve our results. This dissertation in its present form is a consequence of their excellent guidance, regular encouragement and detailed and constructive comments without which it would not have been possible for me to complete the work.

Besides my supervisor, I would like to express my deep sense of gratitude to my RPEC members **Prof. Santwana Mukhopadhyay**, Department of Mathematical Sciences and **Dr. Somak Bhattacharyya**, Assistant Professor, Department of Electronics Engineering, for their guidance, discussion, ideas, feedback, and motivation during this period.

I want to extend my thanks to **Prof. S.K. Pandey**, Head of Department, **Prof. T. Som**, former Head of Department, **Prof. L.P. Singh**, former Head of Department and Conveners, DPGC of Department for their support throughout my research work. I also express my deepest gratitude to all faculty members of the department for their time-to-time discussion and encouragement.

I acknowledge my deep respect to **Mr. Mukesh Aggarwal** for teaching mathematics with great passion and to produce interest in field of mathematics. Also, I would like to thank **Mr. Selvaganesh**, for discussions, suggestions, and encouragement that have immensely helped in my research.

I express my cordial thanks to my wife **Mrs. Swati Gupta** for their enormous love, mental support and continuous encouragement in every tough situation of my life. Further, a very special thanks to my friends **Dr. Shobhit Kumar Srivastava** and **Dr. Rahul Chaturvedi** for their moral support and always standing behind me like my back bone in the tough time of this journey.

I am also thankful to my seniors and friends **Dr. Shiva Sharma**, **Mr. Yogesh Singh**, **Mr. Bhupendra Saini** for helping me a lot like a brother. I also give my thanks to **Dr. Abhay Rajpoot**, **Mr. Ajay Kumar Patel**, **Ms. Jesmina Pervin**, who, despite time constraints, has selflessly put in so much time and efforts to help claim my results as well as to spearhead my analysis.

I am thankful to my institution, Indian Institute of Technology(BHU), Varanasi, for providing all the necessary resources throughout my research. I am also thankful to all non teaching staff members of the department for their cooperation and help throughout my research tenure. I further gratefully acknowledge University Grant

Commision (UGC), India, for providing me the full financial support to carry out my research work.

No words are enough to describe the unconditional love and encouragement I have got from my parents, in-laws and other family members. In fact, I am truly indebted to my entire family for their love, support and for always believing in me. Knowing my interest in Mathematics, they backed up my decision to pursue Ph.D. on it. And, for the last five years, I have always received the much needed emotional support from them.

This acknowledgment would be incomplete if the name of great visionary **Ma-hamana Pt. Madan Mohan Malaviya** is not mentioned, who made this divine center of knowledge. Deepest regards to him.

Above all, praises and thanks to the God, the Almighty for His showers of blessings throughout my research work, who has made everything possible.



Ajay Kumar

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Symbols

G	Group
$G_1 \times G_2$	Direct product of groups G_1, G_2
H	Subgroup of a group G
$cl(u)$	Equivalence class of u of conjugate relation
$[u]$	Equivalence class of u of an equivalence relation
e	The identity element of G
G^*	$G \setminus \{e\}$
$o(G)$	Order of the group G
$o(a)$	Order of a in G
$\langle S \rangle$	Subgroup generated by a set S
$ X $	Cardinality of the set X
$\langle a \rangle$	Subgroup generated by an element a
$\pi(G)$	Set of all orders of elements in G
$\exp(G)$	LCM of all elements of $\pi(G)$
$w_d(G)$	Set of all elements of order d in G
\sim	Equivalence relation on G
\cong	Isomorphism
$K(G)$	Number of conjugacy classes of G
S_n	Symmetric group order $n!$
A_n	Alternating group order $\frac{n!}{2}$
\mathbb{Z}_n	Finite cyclic group order n
\mathbb{Z}_n^r	Direct product of r copies of \mathbb{Z}_n
D_{2n}	Dihedral group order $2n$
T_{4n}	Dicyclic group order $4n$
\overline{D}_{2n}	$D_{2n}^* \setminus \{x \in D_{2n} : o(x) = n\}$

\overline{T}_{4n}	$T_{4n}^* \setminus \{x \in T_{4n} : o(x) = 2n\}$
Q_8	Quaternion group order 8
M_{22}	Mathieu group of order 443520
M_{11}	Mathieu group of order 7920
\mathbb{R}	Set of all real numbers
\mathbb{Z}	Set of all integers
\mathbb{N}	Set of all natural numbers
$a b$	a divides b
$\phi(\cdot)$	Euler's phi function
Γ	Graph
Γ^q	Quotient graph of Γ
$V(\Gamma)$	Vertex set of Γ
$E(\Gamma)$	Edge set of Γ
$\delta(\Gamma)$	Minimum degree of Γ
$deg_{\Gamma}(u)$	Degree of a vertex u in Γ
$dom(\Gamma)$	Set of all dominant vertices in Γ
K_n	Complete graph on n vertices
K_{n_1, n_2, \dots, n_m}	Complete m-partite graph
$\omega(\Gamma)$	Clique number of Γ
S	Separating set
T	Minimal Separating set
$\kappa(\Gamma)$	Vertex connectivity
$\kappa'(\Gamma)$	Edge connectivity
$\Gamma_1 \times \Gamma_2$	Tensor product of Γ_1 and Γ_2
$\overline{C(G)}$	Complement of conjugate graph
$\vec{\mathcal{P}}(G)$	Directed power graph
$\mathcal{P}(G)$	Power graph
$S(G)$	Superpower graph

PREFACE

Algebraic graph theory is the study of the interplay between algebra (both abstract structure as well as linear algebra) and graph theory. On one hand, various concepts of abstract algebra have facilitated the construction of graphs and are explored as tools in computer science, while conversely, graph theory has also helped to characterize certain algebraic properties of abstract algebraic structures. Over the last decade, there has been an increased contribution toward the extensive investigation of graphs obtained from groups.

In this thesis, we highlight the rich interplay between the two topics viz groups and graphs obtained from finite groups, particularly the Complement of a conjugate graph and Superpower graphs. We begin by introducing all the required notions and definitions from group theory and graph theory, followed by a brief literature survey of results that motivated this thesis.

It is well established that multi-partite graphs have their own significance in graph theory. Joseph Varghese [1] has proved the necessary and sufficient condition for a complete multi-partite graph to be Hamiltonian. As a first work, we extend Joseph Varghese's results by studying the Hamiltonian and Eulerian properties of a graph generated with the group-theoretic concept of conjugacy relation on a finite group, referred to as the complement of a conjugate graph $\overline{C(G)}$. We also explore the other variations of the Hamiltonian property, namely, 1-Hamiltonian, pancyclicity, etc. Further, we present several structural characterizations of the considered graph class.

Next, we focus on the superpower graphs, $S(G)$, defined on finite Abelian groups. We begin by first characterizing the structure of such graphs and then we study the

relationship between the structure of tensor product of superpower graphs and the superpower graph of the direct product of underlying finite Abelian groups. Also, we find the tight bounds for the vertex connectivity of $S(G)$ and identify those groups for which the obtained bounds are attained by $S(G)$. In addition, we also establish the conditions for this class of graphs to be Hamiltonian along with some structural properties such as maximal dominating set, 1-Hamiltonian, Hamiltonian-connected, pancyclic and panconnected to name a few.

Among the class of non-Abelian groups, we first target the class of the dihedral and dicyclic groups to explore the properties of their superpower graphs, namely $S(D_{2n})$ and $S(T_{4n})$. It is well-known that these groups hold a fundamental place in group theory and they also appear as a subgroup of many groups. Our motive is to study the structural properties of the superpower graph of these groups and see how they differ from their corresponding power graphs. In this process, we compute sharp bounds for the vertex connectivity of $S(D_{2n})$ and $S(T_{4n})$ and identify the value of n for which the superpower graphs of respective groups attains the bound. In another attempt, we also obtain bounds for the edge connectivity of $S(D_{2n})$ and compute the edge connectivity of $S(D_{2n})$ for some special values of n through the minimum degree $S(D_{2n})$. We also investigate and characterize Hamiltonian-like properties for the groups and their superpower graphs under consideration.

Next, we discuss the structural characterizations of superpower graph $S(G)$ defined on any finite group G . In particular, explore the properties such as the perfectness, Eulerian graph and separating sets. We also discuss the order graph and comparability graph to their connection with the superpower graph. Next, we focus on generalizing the connectivity and Hamiltonian-like results for superpower graphs. To avoid repetition we will target on finite non-Abelian groups, particularly, those groups having an element of exponent order. We first begin by finding the dominating sets for this class and then we structurally characterizing this collection of

superpower graphs. We establish tight bounds for vertex connectivity and identify those groups and respective superpower graphs that attain the obtained bounds. Further, we also explore various Hamiltonian-like properties for this collection of superpower graphs.
