

Chapter 3

Discrete-time static output feedback controller design

Similar to the CT systems, SOF problem for DT systems is also a BMI problem. In this chapter, three new results are presented. First, the H_∞ and H_2 controller design for DT system is considered. New sufficient criteria are derived for designing H_∞ and H_2 controllers. The corresponding design criteria are presented in the form of LMIs. The development involves certain relaxations of the LMI variables through appropriate use of bounding inequalities leading to less conservative LMI design criteria. A transformation framework is also given for designing dynamic output feedback controllers of chosen order using the SOF design method.

Next, new sufficient LMI conditions are derived by using decomposition of an auxiliary matrix. The decomposition facilitates linearization of nonlinear term of reduced size to obtain LMI criteria. This leads to less conservative results than the design conditions developed previously. The efficacy of the developed conditions are illustrated through numerical examples.

To improve the transient behavior of the systems, pole-placement is one of the well-known technique generally used in industrial applications. Pole-placement for CT systems has been studied in the past and methodologies available for controller design. On the other hand, the constant damping ratio locus for DT systems is itself a non-convex region. A new convex approximation of the constant damping ratio loci by an elliptical segment is proposed here. Note that, being convex, the ellipse segment can be defined as an LMI region. Then, LMI criteria for controller (state feedback and SOF) synthesis are

derived for linear DT polytopic systems. A numerical example illustrates the efficacy of the proposed design approach.

3.1 SOF controller design without using a matrix decomposition approach

In this section, new sufficient LMI criteria are derived for designing H_∞ and H_2 SOF controllers. The nonlinear term arising from the multiplication of controller gain matrix and an auxiliary variable is approximated through a new LMI bounding condition so that the coupling terms can be effectively separated out. The derived LMI criteria are less conservative than the existing methods in designing the H_∞ and H_2 controllers. Numerical examples are used to illustrate the effectiveness of the proposed criteria.

3.1.1 Problem statement and preliminaries

Consider a DT LTI system represented as:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_w w(k), \\ z(k) &= C_z x(k) + D_{zu} u(k) + D_{zw} w(k), \\ y(k) &= Cx(k) + D_{yw} w(k) \end{aligned} \quad (3.1)$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input and $w(k) \in \mathbb{R}^r$ is the disturbance input to the system. $z(k) \in \mathbb{R}^q$ and $y(k) \in \mathbb{R}^p$ are the controlled and actual measured outputs, respectively. $A, B, B_w, C_z, D_{zu}, D_{zw}, C, D_{yw}$ are matrices of appropriate dimensions. For system (3.1), consider a SOF controller as:

$$u(k) = Ky(k) \quad (3.2)$$

where K is the controller gain matrix of appropriate dimension. The closed-loop system is given by

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = (\bar{A} + \bar{B}K\bar{C}) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \quad (3.3)$$

where

$$\bar{A} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ D_{zu} \end{bmatrix}, \bar{C} = \begin{bmatrix} C & D_{yw} \end{bmatrix}$$

The closed-loop transfer function matrix from $w(k)$ to $z(k)$ is

$$T_{wz}(\zeta) = C_{cl}(\zeta I - A_{cl})^{-1}B_{cl} + D_{cl} \quad (3.4)$$

where ζ^{-1} is the DT unit delay operator and $A_{cl} = A + BKC$, $B_{cl} = B_w + BKD_{yw}$, $C_{cl} = C_z + D_{zu}KC$ and $D_{cl} = D_{zw} + D_{zu}KD_{yw}$. Given the above system description, the objective of this work is to design the SOF controller K while the design performances are presented in latter sections. We invoke here the following two results that will be used subsequent developments.

3.1.2 H_∞ controller design

In this section, a general sufficient LMI criterion for H_∞ SOF controller design is presented that guarantee the H_∞ performance of the closed-loop DT system (3.3). For H_∞ performance of (3.3), the below Lemma is well known.

Lemma 3.1 (*[3]*) *The following statements are equivalent for $\gamma > 0$.*

1. $\|T_{wz}(\zeta)\|_\infty < \gamma$ and A_{cl} is Hurwitz.
2. There exists a matrix $P_\infty > 0$ satisfying

$$\begin{bmatrix} \bar{P}_\gamma & * \\ \bar{A} + \bar{B}K\bar{C} & -\bar{P}_\infty^{-1} \end{bmatrix} < 0, \quad (3.5)$$

where

$$\bar{P}_\gamma = \begin{bmatrix} -P_\infty & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \quad \bar{P}_\infty = \begin{bmatrix} P_\infty & 0 \\ 0 & I \end{bmatrix}$$

The H_∞ control performance is defined in the above statement. Smaller the γ indicates better the disturbance rejection in the output with respect to the disturbance $w(k)$. Condition (3.5) is a nonlinear matrix inequality criterion ensuring H_∞ control performance. The nonlinearity is due to the involvement of P_∞ and P_∞^{-1} . The objective is to obtain a sufficient LMI condition corresponding to (3.5) which can be solved efficiently in convex optimization framework [3]. In this regard, the following result is obtained.

Theorem 3.2 *The SOF controller (3.2) ensures $\|T_{wz}(\zeta)\| < \gamma$ for the closed-loop system (3.3), if, for a scalar β , there exists a solution $\{P_\infty > 0, Z_\infty, R_\infty, S_\infty, Q_\infty, M_\infty, J_\infty > 0\}$*

of the following LMI :

$$\begin{bmatrix} \bar{P}_\gamma & * & * & * & * \\ \Psi_1 & \Psi_2 & * & * & * \\ 0 & \beta Q_\infty & \Psi_3 & * & * \\ \bar{B}^T \bar{B} Z_\infty \bar{C} & 0 & 0 & \Psi_4 & * \\ 0 & 0 & 0 & S_\infty \bar{B} - \bar{B} R_\infty & -J_\infty \end{bmatrix} < 0, \quad (3.6)$$

where $\Psi_1 = S_\infty \bar{A} + \bar{B} Z_\infty \bar{C}$, $\Psi_2 = -Sym\{Q_\infty\} + J_\infty$, $\Psi_3 = -Sym\{\beta S_\infty\} + P_\infty$, $\Psi_4 = -Sym\{\bar{B}^T \bar{B} R_\infty\}$. The corresponding SOF controller can be obtained as $K = R_\infty^{-1} Z_\infty$.

Proof : Consider the H_∞ performance criterion (3.5). Pre- and post-multiplying (3.5) by $diag\{I, S_\infty\}$ and $diag\{I, S_\infty^T\}$, respectively, and thereby using Lemma 1.14 ($-S_\infty P_\infty^{-1} S_\infty^T \leq -Sym\{S_\infty M_\infty^T\} + M_\infty P_\infty M_\infty^T$) for arbitrary M_∞ , a sufficient condition for (3.5) can be written as:

$$\begin{bmatrix} \bar{P}_\gamma & * \\ S_\infty \bar{A} + S_\infty \bar{B} K \bar{C} & -Sym\{S_\infty M_\infty^T\} + M_\infty P_\infty M_\infty^T \end{bmatrix} < 0 \quad (3.7)$$

In order to take care of the term $M_\infty P_\infty M_\infty^T$, (3.7) is rewritten as:

$$\begin{bmatrix} \bar{P}_\gamma & * \\ S_\infty \bar{A} + S_\infty \bar{B} K \bar{C} & -Sym\{S_\infty M_\infty^T\} \end{bmatrix} + \begin{bmatrix} 0 \\ M_\infty \end{bmatrix} P_\infty \begin{bmatrix} 0 \\ M_\infty \end{bmatrix}^T < 0 \quad (3.8)$$

Now, using Lemma 1.15 on (3.8) (with \mathcal{T} is the first term in the LHS of the above, $M = \begin{bmatrix} 0 & M_\infty^T \end{bmatrix}$, $R = P_\infty$ and $L = L_\infty$), a sufficient condition of (3.8) can be given as:

$$\begin{bmatrix} \begin{bmatrix} \bar{P}_\gamma & * \\ S_\infty \bar{A} + S_\infty \bar{B} K \bar{C} & -Sym\{S_\infty M_\infty^T\} \end{bmatrix} & * \\ L_\infty \begin{bmatrix} 0 & M_\infty^T \end{bmatrix} & \Psi_3 \end{bmatrix} < 0 \quad (3.9)$$

Now, substituting $L_\infty = \beta S_\infty$ and defining $Q_\infty \triangleq S_\infty M_\infty^T$ and Ψ_3 as in (3.6), one gets,

$$\begin{bmatrix} \bar{P}_\gamma & * & * \\ S_\infty \bar{A} + S_\infty \bar{B} K \bar{C} & -Sym\{Q_\infty\} & * \\ 0 & \beta Q_\infty & \Psi_3 \end{bmatrix} < 0, \quad (3.10)$$

In view of the fact that $K = R_\infty^{-1}Z_\infty$, and addition-subtraction of $\bar{B}Z_\infty\bar{C}$ in the (1, 2) term and its transpose in the (2,1) term of (3.10) yields

$$\begin{bmatrix} \bar{P}_\gamma & * & * \\ \Psi_1 & -Sym\{Q_\infty\} & * \\ 0 & \beta Q_\infty & \Psi_3 \end{bmatrix} + Sym \left\{ \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \hat{W} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}^T \right\} < 0, \quad (3.11)$$

where Ψ_1 is defined in (3.6), $\hat{W} = (S_\infty\bar{B} - \bar{B}R_\infty)R_\infty^{-1}Z_\infty\bar{C}$. Now, one can write a sufficient condition for (3.11) using Lemma 1.14 as:

$$\begin{bmatrix} \bar{P}_\gamma & * & * \\ \Psi_1 & \Psi_2 & * \\ 0 & \beta Q_\infty & \Psi_3 \end{bmatrix} + \begin{bmatrix} \hat{W}^T \\ 0 \\ 0 \end{bmatrix} J_\infty^{-1} \begin{bmatrix} \hat{W}^T \\ 0 \\ 0 \end{bmatrix}^T < 0, \quad J_\infty > 0 \quad (3.12)$$

where $\Psi_2 = -Sym\{Q_\infty\} + J_\infty$. Again, applying Lemma 1.15 on (3.12), one can write

$$\begin{bmatrix} \begin{bmatrix} \bar{P}_\gamma & * & * \\ \Psi_1 & \Psi_2 & * \\ 0 & \beta Q_\infty & \Psi_3 \end{bmatrix} & * \\ \bar{L}_\infty R_\infty^{-1} Z_\infty \bar{C} \begin{bmatrix} I & 0 & 0 \end{bmatrix} & -Sym\{\bar{L}_\infty\} + \bar{J}_\infty \end{bmatrix} < 0 \quad (3.13)$$

where $\bar{J}_\infty = (S_\infty\bar{B} - \bar{B}R_\infty)^T J_\infty^{-1} (S_\infty\bar{B} - \bar{B}R_\infty)$. Finally, applying schur complement [3] and substituting $\bar{L}_\infty = \bar{B}^T \bar{B} R_\infty$ in (3.13), one obtains (3.6). \square

Remark 3.3 *To demonstrate the less conservativeness obtained in Theorem 3.2 compared to the main result of [89], one can replace the variable M_∞ with αI and $\bar{L}_\infty = \beta L$ in (3.6) and obtain the LMI condition as:*

$$\begin{bmatrix} \Xi_{11} & * & * & * \\ G\hat{A} + \hat{B}V\hat{C} & \Xi_{22} + J & * & * \\ \hat{B}^T \hat{B}V\hat{C} & 0 & -Sym\{\beta \hat{B}^T \hat{B}U\} & * \\ 0 & 0 & G\hat{B} - \hat{B}U & -\frac{J}{\beta^2} \end{bmatrix} < 0.$$

The above LMI is the same as the one obtained in Theorem 1 of [89]. Clearly, the present result is less restrictive than that of [89].

For synthesizing mere stabilizing controllers, the below result can be obtained from Theorem 3.2. Its proof is omitted.

Corollary 3.4 *The DT system eqrefD4eq1 is stabilizable by the SOF controller (3.2) if for a scalar β , there exists a solution $\{P > 0, Z, R, S, Q, M, J > 0\}$ of the following LMI:*

$$\begin{bmatrix} -P & * & * & * & * \\ \Psi_{11} & \Psi_{22} & * & * & * \\ 0 & \beta Q & \Psi_{33} & * & * \\ B^T B Z C & 0 & 0 & \Psi_{44} & * \\ 0 & 0 & 0 & SB - BR & -J \end{bmatrix} < 0, \quad (3.14)$$

where $\Psi_{11} = SA + BZC$, $\Psi_{22} = -\text{Sym}\{Q\} + J$, $\Psi_{33} = -\text{Sym}\{\beta S\} + P$, $\Psi_{44} = -B^T BR - R^T B^T B$. The corresponding controller gain can be obtained as $K = R^{-1}Z$.

3.1.3 H_2 controller design

LMI conditions for designing SOF controller for ensuring the H_2 performance of the closed-loop system (3.3). For H_2 performance of (3.3), assume $D_{yw} = 0$. For such a system, the necessary and sufficient criterion for the H_2 performance is given in the below result.

Lemma 3.5 [3, page 141] *The following statements are equivalent for $\mu > 0$.*

1. $\|T_{wz}(\zeta)\|_2^2 < \mu$ and A_{cl} is Hurwitz.
2. There exist matrices $P_0 > 0$ and $\mathcal{W} > 0$ satisfying

$$\text{trace}(\mathcal{W}) < \mu, \quad \begin{bmatrix} \mathcal{W} & * \\ B_w + BKD_{yw} & P_0^{-1} \end{bmatrix} > 0 \quad (3.15)$$

$$\begin{bmatrix} P_0 & * \\ \begin{bmatrix} A + BKC \\ C_z + D_z KC \end{bmatrix} & \bar{P}_0^{-1} \end{bmatrix} > 0 \quad (3.16)$$

$$\text{where } \bar{P}_0 = \begin{bmatrix} P_0 & 0 \\ 0 & I \end{bmatrix}.$$

The objective here is to minimize the H_2 control performance μ . In order to find a K satisfying the μ performance, the LMI conditions (3.15) and (3.16) are to be solved. The result derived below following the similar procedure as in Theorem (3.2), provides a sufficient condition for statement 2) above.

Theorem 3.6 *The SOF controller (3.2) ensures $\|T_{wz}(\zeta)\|_2^2 < \mu$ for the closed-loop system (3.3), if, for the scalars ρ_1, ρ_2 and ρ_3 , there exists a solution $\{P_0 > 0, \bar{Z}, \bar{R}, S_1, S_2, Q_1, Q_2, M_1, M_2, J_1 > 0, J_2 > 0, J_3 > 0, \mathcal{W} > 0\}$ of the following LMI :*

$$\text{trace}(\mathcal{W}) < \mu$$

$$\begin{bmatrix} -\mathcal{W} & * & * & * & * \\ \Xi_1 & \Xi_2 & * & * & * \\ 0 & \rho_1 Q_1 & \Xi_3 & * & * \\ \Xi_5 & 0 & 0 & \Xi_4 & * \\ 0 & 0 & 0 & \Xi_{10} & -J_1 \end{bmatrix} < 0, \quad (3.17)$$

$$\begin{bmatrix} -P_0 & * & * & * & * & * & * & * \\ \Xi_6 & \Xi_7 & * & * & * & * & * & * \\ C_z & 0 & -I + J_3 & * & * & * & * & * \\ 0 & \rho_2 Q & 0 & \Xi_8 & * & * & * & * \\ \Xi_9 & 0 & 0 & 0 & \Xi_4 & * & * & * \\ \rho_3 \Xi_9 & 0 & 0 & 0 & 0 & \rho_3 \Xi_4 & * & * \\ 0 & 0 & 0 & 0 & 0 & \Xi_{11} & -J_2 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & D_{zu} & -J_3 \end{bmatrix} < 0 \quad (3.18)$$

where $\Xi_1 = S_1 B_w + B \bar{Z} D_{yw}$, $\Xi_2 = -\text{Sym}\{Q_1\} + J_1$, $\Xi_3 = -\text{Sym}\{\rho_1 S_1\} + P_0$, $\Xi_4 = -B^T B \bar{R} - \bar{R}^T B^T B$, $\Xi_5 = B^T B \bar{Z} D_{yw}$, $\Xi_6 = S_2 A + B \bar{Z} C$, $\Xi_7 = -\text{Sym}\{Q_2\} + J_2$, $\Xi_8 = -\text{Sym}\{\rho_2 S_2\} + P_0$, $\Xi_9 = B^T B \bar{Z} C$, $\Xi_{10} = S_1 B - B \bar{R}$, $\Xi_{11} = S_2 B - B \bar{R}$. The corresponding controller gain can be obtained as $K = \bar{R}^{-1} \bar{Z}$.

Proof : First, an LMI condition for (3.15) is derived. Pre- and post-multiplying (3.15) by $\text{diag}\{I, S_1\}$ and $\text{diag}\{I, S_1^T\}$, respectively, and thereby using Lemma 1.14 ($-S_1 P_0^{-1} S_1^T \leq -\text{Sym}\{S_1 M_1^T\} + M_1 P_0 M_1^T$) for arbitrary M_1 , a sufficient condition for (3.15) can be written as:

$$\begin{bmatrix} -\mathcal{W} & * \\ S_1 B_w + S_1 B K D_{yw} & -\text{Sym}\{S_1 M_1^T\} \end{bmatrix} + \begin{bmatrix} 0 \\ M_1 \end{bmatrix} P_0 \begin{bmatrix} 0 \\ M_1 \end{bmatrix}^T < 0 \quad (3.19)$$

Applying Lemma 1.15 on (3.19) and substituting $L = \rho_1 S$ and defining $Q_1 \triangleq S_1 M_1^T$, one gets

$$\begin{bmatrix} -\mathcal{W} & * & * \\ S_1 B_w + B\bar{Z}D_{yw} & -Sym\{Q_1\} & * \\ 0 & \rho_1 Q_1 & \Xi_3 \end{bmatrix} + Sym \left\{ \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} W_1 \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}^T \right\} < 0 \quad (3.20)$$

where Ξ_3 is defined in (3.18). $W_1 = (S_1 B - B\bar{R})\bar{R}^{-1}\bar{Z}D_{yw}$. Now, employing bounding Lemma 1.14 on (3.20), one can write

$$\begin{bmatrix} -\mathcal{W} & * & * \\ S_1 B_w + B\bar{Z}D_{yw} & \Xi_2 & * \\ 0 & \rho_1 Q_1 & \Xi_3 \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} W_1^T J_1^{-1} W_1 \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}^T < 0 \quad (3.21)$$

where Ξ_2 is defined in (3.18). Again, applying Lemma 1.15 on (3.21). Finally, substituting $L = B^T B\bar{R}$ and applying Schur complement, one obtains LMI condition (3.17).

Next, LMI condition for (3.16) is derived. Pre- and post-multiplying (3.16) by $diag\{I, S_2\}$ and $diag\{I, S_2^T\}$, respectively, and thereby using Lemma 1.14 ($-S_2 P_0^{-1} S_2^T \leq -Sym\{S_2 M_2^T\} + M_2 P_0 M_2^T$) for arbitrary M_2 , a sufficient condition for (3.16) is

$$\begin{bmatrix} -P_0 & * & * \\ S_2 A + S_2 BKC & -Sym\{S_2 M_2^T\} & * \\ C_z + D_{zu}KC & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ M_2 \\ 0 \end{bmatrix} P_0 \begin{bmatrix} 0 & M_2^T & 0 \end{bmatrix} < 0 \quad (3.22)$$

Applying Lemma 1.15 on (3.22) and substituting $L = \rho_2 S_2$ and defining $Q_2 \triangleq S_2 M_2^T$ one gets,

$$\begin{bmatrix} -P_0 & * & * & * \\ S_2 A + S_2 BKC & -Sym\{Q_2\} & * & * \\ C_z + D_{zu}KC & 0 & -I & * \\ 0 & \rho_2 Q_2 & 0 & \Xi_8 \end{bmatrix} < 0 \quad (3.23)$$

where Ξ_8 is defined in (3.18). In view that $K = \bar{R}^{-1}\bar{Z}$ and addition-subtraction of $B\bar{Z}C$ and $C^T\bar{Z}^T B^T$ in the (1,2) and (2,1) terms, respectively, yields

$$\begin{bmatrix} -P_0 & * & * & * \\ \Xi_6 & -Sym\{Q_2\} & * & * \\ C_z & 0 & -I & * \\ 0 & \rho_2 Q_2 & 0 & \Xi_8 \end{bmatrix} + Sym \left\{ \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} W_1 \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \right\} + Sym \left\{ \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} W_2 \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \right\} < 0 \quad (3.24)$$

where $W_1 = (S_2B - B\bar{R})\bar{R}^{-1}\bar{Z}C$, $W_2 = D_{zu}\bar{R}^{-1}\bar{Z}C$. Now, using Lemma 1.14, a sufficient condition for (3.24) is:

$$\begin{bmatrix} -P_0 & * & * & * \\ \Xi_6 & \Xi_7 & * & * \\ C_z & 0 & -I + J_3 & * \\ 0 & \rho_2 Q_2 & 0 & \Xi_8 \end{bmatrix} + \begin{bmatrix} W_3^T & W_3^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{J}_2 & 0 \\ 0 & \bar{J}_3 \end{bmatrix} \begin{bmatrix} W_3 & 0 & 0 & 0 \\ W_3 & 0 & 0 & 0 \end{bmatrix} < 0 \quad (3.25)$$

where $\bar{J}_2 = (S_2B - B\bar{R})^T J_2^{-1} (S_2B - B\bar{R})$, $\bar{J}_3 = D_{zu}^T J_3^{-1} D_{zu}$, $W_3 = \bar{R}^{-1}\bar{Z}C$. Now, applying Lemma 1.15 with $L = \text{diag}\{L_1, L_2\}$.

$$\begin{bmatrix} \begin{bmatrix} -P_0 & * & * \\ \Xi_6 & \Xi_7 & * \\ C_z & 0 & -I + J_3 \end{bmatrix} & * \\ \begin{bmatrix} 0 & \rho_2 Q_2 & 0 \\ L_1 W_3 & 0 & 0 \\ L_2 W_3 & 0 & 0 \end{bmatrix} & \text{diag}\{\Xi_8, -\text{Sym}\{L_1\} + \bar{J}_2, -\text{Sym}\{L_2\} + \bar{J}_3\} \end{bmatrix} < 0 \quad (3.26)$$

Finally, substituting $L_1 = B^T B\bar{R}$ and $L_2 = \rho_3 B^T B\bar{R}$ in (3.25) and applying Schur complement, one obtains (3.18). \square

Remark 3.7 *Theorems 3.2 and 3.6 provide new LMI criteria for SOF controller design for linear DT systems. The proposed LMI conditions for H_∞ and H_2 controller designs can be solved, for example, using LMI control toolbox [158]. For solving these LMIs, one has to obtain the parameters β in Theorem 3.2, and ρ_1 , ρ_2 and ρ_3 in Theorem 3.6. Obtaining these scalar parameters may require use of suitable algorithms for optimization of γ and μ . For solving examples in this paper, the following approach is used—(i) First, the feasibility problem of the LMIs is solved to obtain a set of initial scalar parameters, (ii) Then, the numerical optimization algorithm 'fminsearch' [147] is used to obtain a locally convergent solution. Through gridding of the initial values of the scalar parameters, a large range of these parameters can be covered for obtaining a possible globally optimal solution. The same is used in Example 3.8 later in section 3.1.5.*

3.1.4 Dynamic controller design

Consider the following DOF controller for system (3.1).

$$\begin{aligned}x_c(k+1) &= A_c x_c(k) + B_c y(k) \\ u(k) &= C_c x_c(k) + D_c y(k)\end{aligned}\tag{3.27}$$

where $x_c(k) \in \mathbb{R}^{n_c}$ is the controller state with n_c being the chosen controller order, A_c, B_c, C_c and D_c are appropriate dimensional controller matrices to be designed. Let us define a new state variable as the augmented plant state and a fictitious error state as:

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ Tx(k) - x_c(k) \end{bmatrix},\tag{3.28}$$

where $T \in \mathbb{R}^{n_c \times n}$ is a full row-rank matrix. For full-order DOF controller, ($n_c = n$), a straightforward choice is $T = I$. Then, the closed-loop system dynamics corresponding to the state definition (3.28) can be written as:

$$\begin{aligned}\tilde{x}(k+1) &= \tilde{A}\tilde{x}(k) + \tilde{B}_w w(k) + \tilde{B}\tilde{u}(k) \\ \tilde{z}(k) &= \tilde{C}_z \tilde{x}(k) + \tilde{D}_{zu} \tilde{u}(k) + \tilde{D}_{zw} w(k) \\ \tilde{y}(k) &= \tilde{C}\tilde{x}(k) + \tilde{D}_{yw} w(k)\end{aligned}\tag{3.29}$$

with $\tilde{u}(k) = \tilde{K}\tilde{y}(k)$ and

$$\begin{aligned}\tilde{A} &= \begin{bmatrix} A & 0 \\ TA & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 \\ TB & -I \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \\ T & -I \end{bmatrix}, \quad \tilde{B}_w = \begin{bmatrix} B_w \\ TB_w \end{bmatrix}, \quad \tilde{C}_z = \begin{bmatrix} C_z & 0 \end{bmatrix}, \\ \tilde{D}_{zu} &= \begin{bmatrix} D_{zu} & 0 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}, \quad \tilde{D}_{yw} = \begin{bmatrix} D_{yw} \\ 0 \end{bmatrix}, \quad \tilde{D}_{zw} = D_{zw}\end{aligned}$$

The system (3.29) is in the form of (3.1) with the \tilde{K} is the SOF controller as in (3.2). Hence, the DOF controller parameters with the parameters embedded in \tilde{K} can be designed using the same SOF design criteria developed previously.

3.1.5 Numerical examples

In this section, several numerical examples are shown to demonstrate the effectiveness of the proposed design methods.

Example 3.8 First, the SOF stabilization problem is considered. Existing criteria in [26]¹, [122] (Condition (15)), [68] (Theorem 1), [82] (Theorem 1), [89]² are considered for a comparison with the proposed one (Corollary 1). These methods are referred as M_1 , M_2 , M_3 , M_4 , M_5 , M_6 , respectively. The design methods M_1 , M_2 , M_3 and M_4 do not require choosing any scalar parameter for solving the respective LMI criteria, and hence these are LMI problems solved in a single step. On the other hand, methods M_5 and M_6 involve obtaining scalar parameters for solving the LMIs. It indeed requires obtaining locally optimized scalar values by solving the LMI problem several times. Hence, these methods (M_5 and M_6) take more computational time as compared to the single step methods.

Randomly generated 1000 controllable and observable systems (A, B, C) for different (n, m, p) are tested and success rates are tabulated in Table 3.1. The condition of M_5 requires two scalar parameters (α and β) for designing the controller. These parameters are obtained using the numerical optimization algorithm, `fminsearch` of the optimization toolbox of MATLAB. Since `fminsearch` is a local search algorithm, the initial values of the parameters are gridded as $[-15, -10, -5, 0, 5, 10, 15]$. Similarly, the scalar parameter β in the proposed method is initialized through the same gridding.

From Table 3.1, it can be observed that the present method (M_6) yields similar result to that of M_5 though it is simpler since lesser number of scalar parameters to be chosen. Moreover, it is clearly superior to the criteria of M_1 , M_2 , M_3 and M_4 .

¹The SOF stabilization condition has been provided in Lemma 3 [68].

²The SOF stabilization condition has not been explicitly given but it can be obtained from Theorem 1 [89] as:

$$\begin{bmatrix} -P & * & * & * \\ GA + BVC & -Sym\{\alpha G\} + \alpha^2 P & * & * \\ B^T BVC & 0 & -Sym\{\beta B^T BU\} & * \\ 0 & 0 & GB - BU & -\frac{J}{\beta^2} \end{bmatrix} < 0, P > 0, J > 0$$

The corresponding controller gain can be obtained as: $K = U^{-1}V$

Table 3.1: Comparison of SOF stabilization problem for 1000 randomly generated cases

n=3								
(m,p)	(1,1)	(1,2)	(2,1)	(2,2)	(2,3)	(3,2)	(1,3)	(3,1)
M_1	94.4%	99.1%	97.2%	99.9%	100%	99.9%	100%	98.6%
M_2	73.3%	98.5%	92.9%	99.9%	100%	99.9%	100%	96.5%
M_3	94%	99%	96.8%	99.9%	100%	99.9%	100%	97.8%
M_4	69.4%	71.8%	72.3%	71.2%	100%	72.7%	100%	74%
M_5	100%	99.5%	99.9%	99.9%	100%	100%	100%	100%
M_6	100%	99.8%	99.9%	99.9%	100%	100%	100%	100%
n=5								
(m,p)	(1,2)	(2,2)	(2,3)	(3,2)	(1,3)	(3,1)	(2,4)	(4,2)
M_1	81.1%	96.4%	99.5%	98%	93%	88%	99.9%	98.9%
M_2	77.6%	93.8%	99.9%	99%	97.1%	76.7%	100%	99.7%
M_3	80.7%	95.7%	99.3%	97%	93%	85.4%	99.9%	98.1%
M_4	29.4%	42.7%	51.2%	44.2%	39.8%	40.2%	63.1%	46.2%
M_5	99%	99%	100%	99%	99%	98%	100%	100%
M_6	98.9%	99.3%	100%	99.4%	99.2%	98%	100%	100%
n=7								
(m,p)	(1,2)	(2,3)	(3,2)	(1,3)	(3,1)	(2,4)	(4,2)	(4,1)
M_1	41.7%	91.7%	89.7%	60.5%	62%	98%	91.4%	88.2%
M_2	33.8%	90%	75.9%	68%	23.5%	98.9%	91.1%	88.9%
M_3	41.1%	91%	87.2%	60.5%	56.6%	98%	89.2%	84.2%
M_4	6.7%	18.8%	18.9%	7.1%	15.1%	23%	19.5%	41.7%
M_5	94%	99.8%	99.1%	98%	98.8%	99%	100%	98.9%
M_6	93.9%	99.6%	99.3%	98%	98.7%	99.2%	100%	99%

Example 3.9 Consider system (3.1) with [130] :

$$A = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix}, B = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix}, D_{yw} = 0,$$

$$B_w = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix}, D_{zu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C_z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The objective is to design an H_2 controller for the above system using Theorem 3.6. The obtained result along with the existing ones are shown in Table 3.2. It can be observed that the present result yields the same performance measure as compared to the method of [26]. Further, for this system, one can design centralized and decentralized strictly

Table 3.2: SOF H_2 Control for Example 3.9

Methods	μ
Algorithm [163]	0.35
Theorem 7 [26]	0.18
Theorem 3.6 (proposed)	0.18

proper full-order DOF controller with a new measurement matrix as:

$$D_{yw} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A comparative study of the proposed result with the existing ones is tabulated in Table 3.3. It can be observed that the proposed method yields considerably less conservative results.

Table 3.3: DOF H_2 Control For Example 3.9

Methods	Minimum value of μ for different control structure	
	Centralized	Decentralized
Theorem 2 [130]	8.40	45.65
Theorem 7 [26]	3.86	18.73
Theorem 3.6 (Proposed)	0.6649	3.2845
$T = I, n_c = n,$ (ρ_1, ρ_2, ρ_3)	(0.5304, 0.6286, 3.4028)	(899.0583, 163.1345, 973.5714)

Example 3.10 Next, consider system (3.1) with [89] :

$$A = \begin{bmatrix} \delta & 0.8 & -0.4 \\ -0.5 & 0.4 & 0.5 \\ 1.2 & 1.1 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 1.3 \end{bmatrix}, B_w = \begin{bmatrix} 0.1 \\ 0.4 \\ 0.1 \end{bmatrix}, C_z = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 1.2 & 1 \\ 0 & -3 & 1 \end{bmatrix}, D_{yw} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, D_z = 0, D_{zw} = 0.3$$

where, δ takes different values. Theorem 3.2 is used for designing H_∞ controller for the above system. The obtained γ values are given in Table 3.4 along with the existing results. From Table 3.4, it can be seen that the proposed method is less conservative than the existing ones.

For this example, full-order and first-order DOF controller designs are also considered, and shown in the Table 3.4. The present result for the full-order controller is clearly superior than that of [27] and [89].

Table 3.4: Output Feedback for H_∞ Control for Example 3.10

SOF				
Methods	$\delta = 0.5$	$\delta = 1.2$	$\delta = 1.7$	$\delta = 1.9$
Lemma 3 [43]	6.0869	9.5372	73.0178	infeasible
Theorem 1 [89]	4.5816	6.5679	7.6225	15.0828
Theorem 3.2 (Proposed)	4.5451	5.3190	6.4845	12.2117
(β)	(0.1126)	(0.0095)	(0.5908)	(0.1391)
Full-Order DOF				
Methods	$\delta = 0.5$	$\delta = 1.2$	$\delta = 1.7$	$\delta = 1.9$
Lemma 4 [27]	5.1890	6.1487	11.5262	infeasible
Theorem 2 [89]	4.5925	5.3996	6.4968	13.2388
Theorem 3.2 (Proposed)	0.3301	0.3219	0.9644	7.3138
$T = I, n_c = n, (\beta)$	(0.0001)	(0.0105)	(0.0189)	(0.6094)
First-Order DOF				
Methods	$\delta = 0.5$	$\delta = 1.2$	$\delta = 1.7$	$\delta = 1.9$
Theorem 3.2 (Proposed)	0.3301	0.3219	0.9656	7.3284
$T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, n_c = 1, (\beta)$	(0.0001)	(0.0105)	(0.0190)	(0.6053)

3.2 SOF controller design using a matrix decomposition approach

This section presents new sufficient LMI conditions for designing static H_2 and H_∞ controllers using decomposition of an auxiliary matrix for DT systems. In [57], a diagonal decomposition of Lyapunov matrix is considered for CT systems without involving the off-diagonal terms for designing SOF controllers. By considering the off-diagonal terms in Lyapunov matrix in [145], unlike the developments in [57], new results were given for CT systems by decomposing the Lyapunov matrix. However, the direct application of the approach in [145] for DT systems is seen to yield conservative results. To alleviate this, we use decomposition of an auxiliary matrix variable instead of the Lyapunov one and establish sufficient conditions for designing H_2 and H_∞ SOF controllers for DT systems.

The nonlinear term $GBKC$ (G is an auxiliary matrix, B and C are input and output matrices, respectively, and K is the SOF gain matrix) is taken care of through decomposition of G . A variant of well known projection lemma [144] is then used to approximate the decomposed nonlinear terms. Such rework is seen to yield less conservative criteria for H_2 and H_∞ controller designs than the criteria discussed in the previous section. Further, the SOF control design is used for designing the full-order and reduced-order DOF controllers through an error transformation defined in [164]. This present result with two scalar parameters, is also shown to yield less conservative results for discrete-time systems than in [164], which contains only a single scalar parameter. Numerical examples illustrate the effectiveness of the new developments.

Consider the DT systems (3.1) with SOF (3.2) for obtaining the new sufficient conditions for SOF Controller design for ensuring H_2 and H_∞ performances. Conditions (1.46), (1.47) and (1.48) are BMIs and therefore NP-hard [1] due to the involvement of the terms $GBKC$ and $\bar{G}BKC$. In comparison with the design techniques discussed in the previous section [145], the above nonlinear terms are handled through decomposition of G and \bar{G} in this work.

3.2.1 Decomposition of G and \bar{G}

Consider matrices $Q \in \mathcal{N}(B)$, where $\mathcal{N}(B)$ represents the null space of B and $R = B^\dagger + L_B Q$, where $B^\dagger = (B^T B)^{-1} B^T$ and $L_B \in \mathbb{R}^{m_1 \times (n-m_1)}$ is a given matrix. Then, any

matrix G can be decomposed as:

$$G = \begin{bmatrix} Q \\ R \end{bmatrix}^T \begin{bmatrix} X_Q & X_S \\ X_T & X_R \end{bmatrix} \begin{bmatrix} Q \\ R \end{bmatrix}, \quad (3.30)$$

where $X_S \in \mathbb{R}^{(n-m_1) \times m_1}$, $X_Q \in \mathbb{R}^{(n-m_1) \times (n-m_1)}$, $X_T \in \mathbb{R}^{m_1 \times (n-m_1)}$ and $X_R \in \mathbb{R}^{m_1 \times m_1}$.

In view of that $QB = 0$, $RB = I$ and $G = Q^T X_Q Q + R^T X_T Q + Q^T X_S R + R^T X_R R$, one obtains $GB = R^T X_R + Q^T X_S$. Then with the change of variable $X_R K := Y$, one can write

$$GBK = R^T Y + Q^T X_S X_R^{-1} Y \quad (3.31)$$

Similarly, \bar{G} can also be decomposed using Q , R , \bar{X}_Q , \bar{X}_S , \bar{X}_T and \bar{X}_R .

Remark 3.11 *Similar decomposition can be made exploiting the null and the range space of C considering the duality problem obtained by taking congruence transformation w.r.t. $X = P^{-1}$ on both sides of the conditions (1.46), (1.47) in Lemma 1.12 for designing the H_2 controller and $\bar{X} = \bar{P}^{-1}$ on both sides of the condition (1.48) in Lemma 1.12 for designing the H_∞ controller. There, $BKCG^T$ and $BKC\bar{G}^T$ are the nonlinear terms which are to be taken care of through similar decomposition as above.*

Remark 3.12 *Similar decomposition can be made exploiting the null space of C and $R = C^\dagger + QL_C$ with L_C is an $(n-p) \times p$ matrix and $C^\dagger = C^T(CC^T)^{-1}$, considering the duality problem obtained by taking congruence transformation w.r.t. $X = P^{-1}$ on both sides of the conditions (1.46), (1.47) in Lemma 1.12 for designing the H_2 controller and $\bar{X} = \bar{P}^{-1}$ on both sides of the condition (1.48) in Lemma 1.12 for designing the H_∞ controller. There, $BKCG^T$ and $BKC\bar{G}^T$ are the nonlinear terms which are to be taken care of through similar decomposition as above.*

In view of the above, a decomposition of the auxiliary variable G is used in this work to obtain the main results. In [57], a diagonal decomposition of Lyapunov matrix is considered for SOF controller design for continuous-time systems. However, the off-diagonal elements of the Lyapunov matrix have been omitted therein. This imposed restriction in the developed condition. In the recent work of [145], the off-diagonal terms considered for obtaining LMI criteria for SOF design for continuous-time systems. In the present work, sufficient conditions are developed for designing H_2 and H_∞ based SOF controllers for DT systems through decomposition of the auxiliary matrix G instead of the Lyapunov matrix. This yields less conservative results as shown in the examples.

Remark 3.13 The matrix $Q \in \mathbb{R}^{(n-m_1) \times n}$ is one whose columns form the basis of the $\text{null}(B)$, i.e., $QB = 0$. Further, R is a given matrix such that $RB = I$ (R is the left inverse of B). Notice that R can be parametrized as $R = B^\dagger + L_B Q$ [57], where L_B is an $m_1 \times (n - m_1)$ matrix and $B^\dagger = (B^T B)^{-1} B^T$ is the Moore-Penrose pseudo inverse of B . This can be seen since $(B^\dagger + L_B Q)B = I$ and, conversely, if $RB = I$ then $B(R - B^\dagger) = 0$. Thus, the choice of R is not unique.

Next, we derive sufficient LMI conditions corresponding to (1.46), (1.47) and (1.48) for designing the H_2 and H_∞ SOF controllers based on the decomposition considered in the preceding section.

3.2.2 H_2 controller design

Theorem 3.14 The DT system (3.1) is stable with the SOF controller (3.2) with controller gain K_2 and a performance $\|T_{wz}(\zeta)\|_2^2 < \mu$ is ensured if, for the scalars ρ_1 and ρ_2 , there exist matrices $P = P^T > 0$, $\mathcal{W} = \mathcal{W}^T > 0$, X_Q , X_R , X_S , X_T and Y such that following LMIs are satisfied.

$$\text{trace}(\mathcal{W}) < \mu, \quad \begin{bmatrix} -\mathcal{W} & * \\ G B_w & -\text{Sym}\{G\} + P \end{bmatrix} < 0, \quad (3.32)$$

$$\begin{bmatrix} -P & * & * & * \\ \Psi_1 & -\text{Sym}\{G\} + P & * & * \\ \Psi_2 & 0 & -I & * \\ \Psi_3 & \rho_2 X_S^T Q & \rho_2 D_{zu}^T & -\Psi_4 \end{bmatrix} < 0 \quad (3.33)$$

where $\Psi_1 = GA + R^T Y C - \rho_1 Q^T X_S B^T$, $\Psi_2 = C_z - \rho_1 D_{zu} B^T$, $\Psi_3 = \rho_1 X_R B^T + Y C$, $\Psi_4 = \text{Sym}\{\rho_2 X_R\}$. The controller gain can be computed as $K_2 = X_R^{-1} Y$.

Proof : In view of $D_{cl} = 0$, one obtains (3.32). Replacing (3.31) into (1.47), adding-subtracting $\rho_1 Q^T X_S B^T$ and $\rho_1 D_{zu} B^T$ in the $\{2,1\}$ and $\{3,1\}$ terms of (1.47), respectively, with the corresponding symmetric terms, one can write

$$\begin{bmatrix} -P & * & * \\ \Psi_1 & -\text{Sym}\{G\} + P & * \\ \Psi_2 & 0 & -I \end{bmatrix} + \text{Sym}\{\Psi_5 X_R^{-1} \Psi_6^T\} < 0 \quad (3.34)$$

where $\Psi_5 = \begin{bmatrix} 0 & X_S^T Q & D_{zu}^T \end{bmatrix}^T$, $\Psi_6 = \begin{bmatrix} \Psi_3 & 0 & 0 \end{bmatrix}^T$. Finally, applying Lemma 1.16 on (3.34) with $L = X_R$, one obtains (3.33). Therefore, LMI condition (3.32) and (3.33) are sufficient for conditions (1.46) and (1.47), respectively. \square

3.2.3 H_∞ controller design

Theorem 3.15 *The DT system (3.1) is stabilizable by the SOF controller (3.2) with controller gain K_∞ and $\|T_{wz}(\zeta)\|_\infty < \gamma$ is ensured if, for the scalars α_1 and α_2 , there exist matrices $\bar{P} = \bar{P}^T > 0$, \bar{X}_Q , \bar{X}_S , \bar{X}_T , \bar{X}_R and \bar{Y} satisfying the below LMI.*

$$\begin{bmatrix} -\bar{P} & * & * & * & * \\ 0 & -\gamma I & * & * & * \\ \Gamma_1 & \Gamma_2 & -\text{Sym}\{\bar{G}\} + \bar{P} & * & * \\ \Gamma_3 & D_{zw} & 0 & -\gamma I & * \\ \Gamma_4 & \bar{Y}D_{yw} & \alpha_1 \bar{X}_S^T Q & \alpha_1 D_{zu}^T & -\Gamma_5 \end{bmatrix} < 0, \quad (3.35)$$

where $\Gamma_1 = \bar{G}A + R^T \bar{Y}C - \alpha_2 Q^T \bar{X}_S B^T$, $\Gamma_2 = \bar{G}B_w + R^T \bar{Y}D_{yw}$, $\Gamma_3 = C_z - \alpha_2 D_{zu} B^T$, $\Gamma_4 = \bar{Y}C + \alpha_2 \bar{X}_R B^T$, $\Gamma_5 = \text{Sym}\{\alpha_1 \bar{X}_R\}$. Moreover, the corresponding controller gain can be computed as $K_\infty = \bar{X}_R^{-1} \bar{Y}$.

Proof : Here, it is to show that (3.35) is sufficient for (1.48). Replacing (3.31) into (1.48), and then introducing $\alpha_2 Q^T \bar{X}_S B^T$ and $\alpha_2 D_{zu} B^T$ in the {3,1} and {4,1} terms of (1.48), respectively, with the corresponding symmetric terms, one can write a sufficient condition of (1.48) as:

$$\begin{bmatrix} -\bar{P} & * & * & * \\ 0 & -\gamma I & * & * \\ \Gamma_1 & \Gamma_2 & -\text{Sym}\{\bar{G}\} + \bar{P} & * \\ \Gamma_3 & D_{zw} & 0 & -\gamma I \end{bmatrix} + \text{Sym}\{\Gamma_6 \bar{X}_R^{-1} \Gamma_7^T\} < 0 \quad (3.36)$$

where $\Gamma_6 = \begin{bmatrix} 0 & 0 & \bar{X}_S^T Q & D_{zu}^T \end{bmatrix}^T$, $\Gamma_7 = \begin{bmatrix} \Gamma_4 & \bar{Y}D_{yw} & 0 & 0 \end{bmatrix}^T$. Finally, applying Lemma 1.16 on (3.36) with $L = \bar{X}_R$, one obtains (3.35). \square

Remark 3.16 *In [164] and [129], the nonlinear term $GBKC \in \mathbb{R}^{n \times n}$ is approximated through adding-subtracting the terms $BZ_\infty C$ and BVC (Z_∞ and V are defined therein), respectively, and then linearizing the variables using bounding inequalities. On the contrary, in this work, a reduced version of the nonlinear term, i.e., $Q^T X_S X_R^{-1} Y \in \mathbb{R}^{n \times p_2}$*

in (3.31), is approximated through adding-subtracting $\rho_1 Q^T X_S B^T$. Clearly, the linearization used in Theorem 3.14 above is benefited for being approximation of nonlinear term of reduced size. Due to this, the proposed conditions are less conservative, which is corroborated through the numerical examples given in section 3.2.4. Similar benefit is also exploited in Theorem 3.15.

3.2.4 Numerical examples

In this section, numerical examples are provided to show the efficacy of the proposed conditions. Note that, for chosen $\rho_1, \rho_2, \alpha_1, \alpha_2$, the criteria in Theorems 3.14 and 3.15 become LMIs. These scalar parameters are obtained by using the numerical search algorithm *fminsearch* of the of MATLAB [165]. L_B is chosen as zero matrix to solve the below examples.

Example 3.17 Consider system (3.1) with [26] :

$$A = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix}, B = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix}, D_{zu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix}, C_z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The objective is to use Theorem 3.14 and design an H_2 controller for the above system.

Table 3.5: SOF H_2 Control for Example 3.17

Methods	μ
Algorithm [163]	0.35
Algorithm 2 [110]	0.27
Theorem 7 [26]	0.18
Theorem 3.6 (Proposed)	0.18
Theorem 3.14 (Proposed)	0.17

The obtained and the existing results are shown in Table 3.5. The corresponding values of ρ_1 and ρ_2 are 0.1804 and 0.0774, respectively. It can be seen that the present result yields better performance measure as compared to the methods of [163], [26], [110] and Theorem 3.6. Note that, nonlinear optimization tool, i.e., HIFOO-D cannot deal with problems concerning design based on H_2 controller.

Table 3.6: H_∞ Output Feedback Control for Example 3.18

SOF				
Methods	$\bar{\delta} = 0.5$	$\bar{\delta} = 1.2$	$\bar{\delta} = 1.7$	$\bar{\delta} = 1.9$
Lemma 3 [43]	6.0869	9.5372	73.0178	infeasible
Theorem 1 [89]	4.5816	6.5679	7.6225	15.0828
Theorem 3.2 (Proposed)	4.5451	5.3190	6.4845	12.2117
Theorem 1 [129]	3.6072	4.9744	4.1234	7.3284
Theorem 3.15 (Proposed)	3.0644	4.4416	5.5845	10.3336
Full-Order DOF ($T = I, n_c = n$)				
Methods	$\bar{\delta} = 0.5$	$\bar{\delta} = 1.2$	$\bar{\delta} = 1.7$	$\bar{\delta} = 1.9$
Lemma 4 [27]	5.1890	6.1487	11.5262	infeasible
Theorem 2 [89]	4.5925	5.3996	6.4968	13.2388
Theorem 3.2 (Proposed)	0.3301	0.3219	0.9644	7.3138
Theorem 3.15 (Proposed)	0.3300	0.3200	0.7659	5.0329
First-Order DOF ($n_c = 1$)				
Theorem 3.2 ($T = [1 \ 0 \ 0]$)	0.3301	0.3219	0.9656	7.3284
Theorem 3.15 (Proposed) ($T = [1 \ 0 \ 0]$)	0.3300	0.3205	0.7762	5.7300

Example 3.18 Let us consider system (3.1) with [89]

$$A = \begin{bmatrix} \bar{\delta} & 0.8 & -0.4 \\ -0.5 & 0.4 & 0.5 \\ 1.2 & 1.1 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 1.3 \end{bmatrix}, B_w = \begin{bmatrix} 0.1 \\ 0.4 \\ 0.1 \end{bmatrix}, C = \begin{bmatrix} -1 & 1.2 & 1 \\ 0 & -3 & 1 \end{bmatrix}$$

$$D_{yw} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, C_z = [-1 \ 0 \ 2], D_{zu} = 0, D_{zw} = 0.3,$$

where $\bar{\delta}$ takes different values. A comparison of the present results with existing ones [43, 89, 129] and Theorem 3.2 for the SOF controller is shown in Table 3.6. The values of (α_1, α_2) are $(0.6683, 0.2765)$ for $\bar{\delta} = 0.5$, $(1.6448, 0.1290)$ for $\bar{\delta} = 1.2$, $(3.4308, 0.0582)$ for $\bar{\delta} = 1.7$ and $(1.6100, 0.0279)$ for $\bar{\delta} = 1.9$, respectively. From Table 3.6, it is observed that the proposed method is less conservative than [43] [89] and Theorem 3.2 for all values of $\bar{\delta}$. In comparison of the results of [129], it yields smaller γ for $\bar{\delta} = 0.5, 1.2$ but larger γ for $\bar{\delta} = 1.7, 1.9$. For First-order DOF in Theorem 3.2, with $T = [1 \ 0 \ 0]$, the values of (α_1, α_2) are $(5990, 3.3)$ for $\bar{\delta} = 0.5$, $(4300, -5.5)$ for $\bar{\delta} = 1.2$, $(33.45, 0.0416)$ for $\bar{\delta} = 1.7$ and $(1.40, 0.17)$ for $\bar{\delta} = 1.9$. Full-order DOF controllers are also designed for a comparison as shown in Table 3.6. The corresponding values of (α_1, α_2) are $(1874.9, 1.5)$ for $\bar{\delta} = 0.5$, $(19861, 18)$ for $\bar{\delta} = 1.2$, $(32.230, 0.1431)$ for $\bar{\delta} = 1.7$ and $(1.1720, 0.1758)$ for $\bar{\delta} = 1.9$.

From the obtained results, it can be shown that full-order as well as reduced-order DOF controllers can be easily designed through SOF design methods and the results conform to the characteristic that higher-order controllers have larger degrees of freedom. It can be also seen from the results that the present design with two scalar parameters yields less conservative result than the method in Theorem 3.2, which employs only a single scalar parameter for obtaining the results.

Example 3.19 By virtue of this example, in which we consider the selective benchmark control problems from the COMPl_eib library [166], we show the effectiveness of the present technique over the ones existing in literature. H_∞ SOF discrete-time controller design using Theorem 3.15 is considered for the comparison. The system data used here, is taken from the COMPl_eib library [166]. Some complimentary data for these examples is listed in [145]. The continuous-time plants are discretized with a sampling time of 0.5 seconds for building the example cases. The methods of HIFOO-D [111], Theorem 3.2 and [129] are considered for comparison. HIFOO-D is a freeware program and is considered as the effective tool for solving fixed-order stabilization and optimal H_∞ output feedback discrete-time control problems. The number of LMI variables (NLV) in Theorem 3.2, [129] and Theorem 3.15 are $4n^2 + m_1^2 + m_1p_2 + n$, $1.5n^2 + p_1^2 + m_1p_2 + 2np_1 + 0.5n + m_1^2$, $1.5n^2 + m_1p_2 + 0.5n$, respectively. Note that, HIFOO-D is a non-Lyapunov based optimization tools so, it doesnt involve any NLVs. In addition to the above, the LMIs require to tune scalar optimization variables as one (β) in Theorem 3.2, two (ϵ, θ) in [129] and two (α_1, α_2) in the present method. These parameters are obtained using fminsearch algorithm of MATLAB.

Table 3.7: γ obtained for benchmark examples from *COMPl_eib* plants

Plants			HIFOO-D [111]		Theorem 3.2		Theorem 1 [129]		Theorem 3.15	
Name	n, m_1, p_2	m_2, p_1	γ	NLV	γ (β)	NLV	γ (ϵ, θ)	NLV	γ (α_1, α_2)	NLV
EB1	10,1,1	2,2	9.750	-	11.8807 (0.001)	412	9.141 (0.9882,17.1104)	201	8.9047 (0.1068,0.3972)	156
EB2	10,1,1	2,2	4.0808	-	4.342 (0.0011)	412	4.102 (0.6802,11.4813)	201	3.9405 (2.5975,1.6853)	156
DIS1	8,4,4	1,8	17.359	-	19.005 (0.001)	296	15.9902 (1.6513,3.4304)	324	12.347 (2.921,0.064)	116
DIS2	3,2,2	3,3	3.501	-	3.8981 (0.2036)	47	3.7874 (0.0445,4.0688)	50	3.2553 (5.909,-0.038)	19
DIS3	6,4,4	6,6	4.503	-	25.9006 (1.5353)	182	4.461 (2.2291,1.744)	197	4.4062 (0.7972,0.7343)	73
DIS4	6,4,6	6,6	5.902	-	Infeasible	190	Infeasible	205	5.5856 (35.8326,0.3318)	81
AC3	5,2,4	5,5	17.056	-	87.132 (0.0001)	117	15.4875 (22.3628,0.3746)	127	16.6369 (0.9124,6.6077)	48
AC6	7,2,4	7,7	58.961	-	148.8927 (0.0011)	215	151.8127 (-0.0203,66.055)	236	54.3081 (2138.9,4)	85
AC7	9,1,2	4,1	0.065	-	0.0542 (0.001)	336	0.0247 (0.001,1734.2)	148	0.0240 (13.4486,0.0351)	128

The initial values for these variables in the *fminsearch* program are taken as $\beta = 0.1$ for the method of Theorem 3.2, $\epsilon = \theta = 0.1$ for the method of [129] and $\alpha_1 = \alpha_2 = 0.1$ for the proposed method. The optimized values of the scalar variables are also given in Table 3.7 along with the γ values. The dimensions associated with the system matrices and the NLV for each case are also included.

It can be observed from Table 3.7 that the present result using Theorem 3.15 is less conservative than HIFOO-D [111], Theorem 3.2 and [129] for almost all the plants except for the plant AC3 where the γ value is slightly larger than the method of [129]. Also, the NLVs in the proposed method are lesser compared to Theorem 3.2 and [129] for all the plants.

It is shown here that appropriate use of bounding inequalities lead to less conservative LMI design criteria. Every system has an inherent performance index associated with it. Actually, we are computing the performance of the system based on the approximated (sufficient LMI condition) controller design criteria. The computed performance indices are always higher than the actual value associated with the system. So, we define the

term conservativeness based on how close is the computed value of performance towards the actual value of the system. The Smaller the value of the computed performance, lesser is the conservativeness. In this work, the same has been considered to define the term conservativeness. We have considered different SOF design criteria available in the literature for the particular system and made a comparative study of proposed design criteria with the existing ones. The obtained values of the system performance are smaller than the other design methods. This is precisely what we mean by the *reduction of conservativeness* based on the LMI design criteria.

3.3 Pole-placement in damping region

Pole placement with specified damping and decay rate is one of the primary criteria for ensuring good transient response. Systematic solution for DT systems is an open problem due to the non-convexity of the constant damping locus. This section addresses the problem of placing the closed-loop poles of linear DT uncertain polytopic systems in constant damping region. The non-convex damping region is approximated by an elliptical segment that represents an LMI region. LMI criteria are then derived to design state and output feedback (static as well as dynamic) controllers. The effectiveness of the proposed approach is demonstrated with numerical examples.

3.3.1 Convex approximation of damping region

This section presents a convex approximation of the constant damping region with minimum decay rate for DT systems. Before proceeding to such an approximation, let us inspect the contracted area based full-ellipse approximation next. Such an approximation carried out earlier in [32].

3.3.1.1 Full-ellipse approximation

In this section, a full-ellipse approximation ($\hat{\mathcal{D}}_1$) of the \mathcal{D}_D -region is considered. The approximation is shown in Fig. 3.1. An inner ellipse with its minor axis aligned along the imaginary axis can be constructed. The main hindrance of such an approximation is the over-consideration of the decay rate that may lead to large control gains.

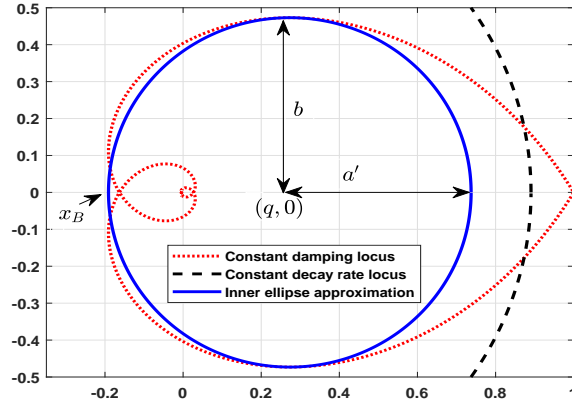


Figure 3.1: Full-ellipse representation for $\hat{\mathcal{D}}_1$ region (solid line)

The ellipse parameters can be obtained as the following. The real and imaginary parts of (1.53) are

$$Re(z) = e^{-\cotan(\cos^{-1}(\zeta))|\omega|T} \cos(\omega T), \quad (3.37)$$

$$Im(z) = e^{-\cotan(\cos^{-1}(\zeta))|\omega|T} \sin(\omega T). \quad (3.38)$$

The minor axis of the inner ellipse is the maximum of $Im(z)$ in (3.38). The point is located at $\omega = T^{-1}\cos^{-1}(\zeta)$. Substituting ω into (3.37) and (3.38), the center (q) and minor axis (b) of the inner ellipse can be computed as:

$$q = \zeta e^{-\cotan(\cos^{-1}(\zeta))|\cos^{-1}(\zeta)|}, \quad (3.39)$$

$$b = e^{-\cotan(\cos^{-1}(\zeta))|\cos^{-1}(\zeta)|} \sin(\cos^{-1}(\zeta)). \quad (3.40)$$

Finally, the point x_B is located where $Re(z)$ in (3.37) equals its minimum value. This yields $\omega = T^{-1}(\cos^{-1}(\zeta) + \frac{\pi}{2})$ and

$$x_B = -e^{-\cotan(\cos^{-1}(\zeta))|\cos^{-1}(\zeta) + \frac{\pi}{2}|} \sin(\cos^{-1}(\zeta)). \quad (3.41)$$

The major axis a' can be obtained as $a' = |x_B| + q$. A numerical evaluation of the error in the approximation is tabulated in Table 3.8 for different ζ values. Note that the errors are considerable due to the over-approximation of the minimum decay rate locus. Since the ellipse should be independent of the decay rate α , the maximum error between corresponding points on the full ellipse and damping curve occurs at $\alpha = 0$.

Table 3.8: Approximation error for $\hat{\mathcal{D}}_1$

ζ	Area of full ellipse $\hat{\mathcal{D}}_1 (A_{e_1})$	Area of $\mathcal{D}_D(\alpha, \theta) (A_d)$	Error (in area) (%)
0.5	0.6315	0.8599	26.56
0.7	0.3776	0.5192	27.28
0.9	0.1922	0.2470	22.18

$$\text{Error (in area)}(\%) = \frac{|A_{e_1} - A_d|}{A_d} \times 100\%$$

 Table 3.9: Approximation error for $\hat{\mathcal{D}}_2$

ζ	Area of ellipse segment $\hat{\mathcal{D}}_2 (A_{e_2})$	Area of $\mathcal{D}_D(\alpha, \theta) (A_d)$	Error (in area) (%)
$\alpha = 0$			
0.5	0.9474	0.8599	10.17
0.7	0.5905	0.5192	13.73
0.9	0.2914	0.2470	17.98
$\alpha = -0.1$			
0.5	0.9390	0.8599	9.20
0.7	0.5849	0.5192	12.66
0.9	0.2885	0.2470	16.80
$\alpha = -0.2$			
0.5	0.9307	0.8599	8.23
0.7	0.5794	0.5192	11.59
0.9	0.2856	0.2470	15.63

$$\text{Error (in area)} (\%) = \frac{(A_{e_2} - A_d)}{A_d} \times 100\%$$

3.3.1.2 Ellipse segment approximation

An ellipse segment approximation ($\hat{\mathcal{D}}_2$) of the region \mathcal{D}_D shown in Fig. 3.2 is proposed here. The center (q) and minor axis (b) of the ellipse are the same as in (3.39) and (3.40). The major axis a of the ellipse can be obtained as: $a = e^{\alpha T} - q$. A numerical evaluation of the area error is shown in Table 3.9. The error increases with increase in the damping ratio (ζ) but decreases with decay rate. Also, a comparison with Table 3.8 shows that

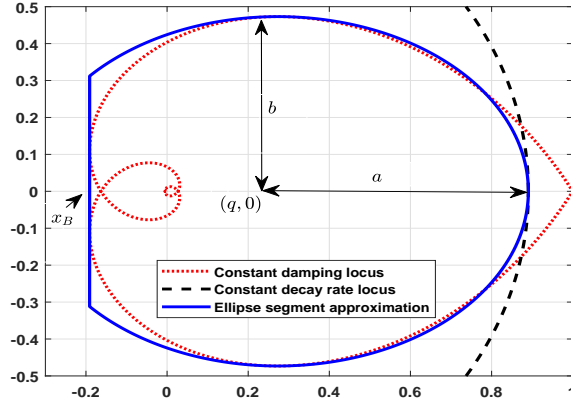


Figure 3.2: Ellipse segment $\hat{\mathcal{D}}_2$ region (solid line)

the errors are much lesser in case of $\hat{\mathcal{D}}_2$.

In the next section, we define the pole placement problem for discrete-time systems with poles to be placed in $\hat{\mathcal{D}}_2$.

3.3.2 Problem statement and preliminaries

Consider a DT polytopic system as:

$$\begin{aligned} x(k+1) &= A(\rho)x(k) + B(\rho)u(k), \\ y(k) &= Cx(k), \end{aligned} \quad (3.42)$$

where $x(k) \in \mathbb{R}^n$ represents the state vector, $u(k) \in \mathbb{R}^m$ is the control input vector, $y(k) \in \mathbb{R}^p$ is the measured output. $A(\rho)$ and $B(\rho)$ are parameter dependent affine matrices of appropriate dimensions that are continuous in the parameter argument ρ and belong to the following uncertainty polytope.

$$\Omega = \left([A(\rho), B(\rho), C] = \sum_{i=1}^{\bar{N}} \rho_i [A_i, B_i, C], \sum_{i=1}^{\bar{N}} \rho_i = 1 \right) \quad (3.43)$$

where \bar{N} is the number of polytope vertices and $\rho_i \geq 0$.

Remark 3.20 *The matched assumption made in the measured output matrix of the system (4.17), i.e., C is constant, may restrict the application of the presented result to general class of polytopic systems. However, many systems that are modeled in polytopic form possess this feature such as autonomous vehicles (land robots [84, 148, 149], airborne vehicles [85, 150], underwater vehicles [86]), electric motors [151–154] and DC-DC*

converters [34, 87, 155]. Hence, the class of polytopic systems considered here has wide applications.

For system (4.17), consider a feedback controller, $u(k) = K_f(x(k))$, where $K_f(\cdot)$ represents linear set of controllers, such as state feedback, and output feedback of static and dynamic types. The closed-loop system can be represented as:

$$x(k+1) = A_{cl}(\rho)x(k), \quad (3.44)$$

where $A_{cl}(\rho)$ is the closed-loop system matrix. The main objective of this section is to derive LMI criteria for designing the controller $u(k)$ such that the closed-loop poles are placed in $\hat{\mathcal{D}}_2$ region.

3.3.2.1 The LMI Region $\hat{\mathcal{D}}_2$

In this section, the LMI region for $\hat{\mathcal{D}}_2$ is introduced. The system (3.44) is said to be $\hat{\mathcal{D}}_2$ -stable if all the closed-loop eigenvalues lie in the $\hat{\mathcal{D}}_2$ region. In this regard, the following definition of LMI region [28] is invoked.

Definition 3.21 *A subset $\hat{\mathcal{D}}_2$ of the complex plane is called an LMI region if there exist a symmetric matrix $\mathcal{Q} \in \mathbb{R}^{n \times n}$ and a matrix $\mathcal{M} \in \mathbb{R}^{n \times n}$ that satisfy $\hat{\mathcal{D}}_2 = \{z \in \mathcal{C} : f_{\hat{\mathcal{D}}_2}(z) < 0\}$ with $f_{\hat{\mathcal{D}}_2}(z) = \mathcal{Q} + z\mathcal{M} + \bar{z}\mathcal{M}^T$ where \mathcal{C} denotes the set of complex numbers, \bar{z} is the complex conjugate of z and $f_{\hat{\mathcal{D}}_2}(\cdot)$ is the characteristic function (CF) of $\hat{\mathcal{D}}_2$.*

The characteristic function of $\hat{\mathcal{D}}_2$ is intersection of two subsets — (i) a full ellipse and (ii) the right-half plane of a vertical line through x_B , which is the point where $Re(z)$ in (3.37) equals its minimum value. This yields $\omega = T^{-1}(\cos^{-1}(\zeta) + \frac{\pi}{2})$ and

$$x_B = -e^{-\cotan(\cos^{-1}(\zeta))|\cos^{-1}(\zeta) + \frac{\pi}{2}|} \sin(\cos^{-1}(\zeta)).$$

The characteristic function of the ellipse is

$$f_{\hat{\mathcal{D}}_2'}(z) = \begin{bmatrix} -2a^2 & * \\ r_1\bar{z} + r_2z - 2qb & -2b^2 \end{bmatrix},$$

where $r_1 = a + b$ and $r_2 = b - a$.

For right half-plane of the vertical line, the characteristic function is

$$f_{\hat{\mathcal{D}}_2''}(z) = z + \bar{z} - x_B.$$

Then the characteristic function for $\hat{\mathcal{D}}_2$ can be written as:

$$f_{\hat{\mathcal{D}}_2} = f_{\hat{\mathcal{D}}'_2} \cap f_{\hat{\mathcal{D}}''_2}. \quad (3.45)$$

The LMI condition corresponding to the characteristic function $f_{\hat{\mathcal{D}}'_2}$ can be written following [28] with $P = P^T > 0$ as:

$$\begin{bmatrix} -2a^2P & * \\ r_1PA_{cl}^T(\rho) + r_2A_{cl}(\rho)P - 2qbP & -2b^2P \end{bmatrix} < 0, \quad (3.46)$$

and the condition for $f_{\hat{\mathcal{D}}''_2}(z)$ is

$$A_{cl}(\rho)P + PA_{cl}^T(\rho) - 2x_B P > 0. \quad (3.47)$$

LMI criteria for controller synthesis satisfying (3.46) and (3.47) are derived in the next section.

3.3.3 Controller design

First, state feedback and then output feedback controller design problems are considered in this section.

3.3.3.1 State feedback controller

Consider a state feedback controller as:

$$u(k) = K_s x(k), \quad (3.48)$$

where K_s is the state feedback gain. It is assumed that all the states are measurable, i.e., $C = I$. The following result presents an LMI criterion for state feedback controller synthesis.

Theorem 3.22 *The closed-loop poles of the DT system (4.17) subject to the state feedback controller (3.48) are placed in the $\hat{\mathcal{D}}_2$ region if there exist matrices $P = P^T > 0$ and Z , $\forall i = 1, \dots, \bar{N}$ satisfying the following LMIs.*

$$\begin{bmatrix} -2a^2P & * \\ r_1PA_i^T + r_1Z^TB_i^T + r_2A_iP + r_2B_iZ - 2qbP & -2b^2P \end{bmatrix} < 0, \quad (3.49)$$

$$A_iP + PA_i^T + B_iZ + Z^TB_i^T - 2x_B P > 0. \quad (3.50)$$

The controller gain can be computed as $K_s = ZP^{-1}$.

Proof: By substituting $A_d(\rho) = A(\rho) + B(\rho)K_s$ in conditions (3.46), (3.47) and replacing K_sP by variable Z , one obtains (3.49) and (3.50). \square

3.3.3.2 SOF controller

It is well known that the Static Output Feedback controller design problem is NP-hard [1]. In this section, we present an LMI for designing SOF controller given by

$$u(k) = K_o y(k), \quad (3.51)$$

where $K_o \in \mathbb{R}^{m \times p}$ is the SOF gain. The LMI criterion derived follows the work in [145], where more details on challenges on SOF controller design methods are discussed. Similar decomposition of the Lyapunov matrix as in [145] is considered. The benefit of using the decomposition is to reduce the size of the nonlinear matrix terms so that these can be better approximated. To this end, the following algebraic manipulations are introduced for the subsequent developments.

Consider the matrices $Q \in \mathcal{N}(C)$ and $R = C^\dagger + QL_C$ [57], where L_C is an $(n-p) \times p$ matrix, $C^\dagger = C^T(CC^T)^{-1}$ is the Moore-Penrose pseudo inverse of C and $\mathcal{N}(\cdot)$ represent the null-space. Then, any $P > 0$ can be decomposed as

$$P = \begin{bmatrix} Q^T \\ R^T \end{bmatrix}^T \begin{bmatrix} P_Q & P_S \\ P_S^T & P_R \end{bmatrix} \begin{bmatrix} Q^T \\ R^T \end{bmatrix}, \quad \begin{bmatrix} P_Q & P_S \\ P_S^T & P_R \end{bmatrix} > 0,$$

where $P_Q \in \mathbb{R}^{(n-p) \times (n-p)}$, $P_S \in \mathbb{R}^{(n-p) \times p}$ and $P_R \in \mathbb{R}^{p \times p}$. In view of $CQ = 0$ and R is so chosen that $CR = I$. Then, one can write $CP = P_R R^T + P_S^T Q^T$. For employing Theorem 3.23 as discussed later, decomposition of the Lyapunov matrix variable P is made through Q as the null matrix of C and $R = C^T(CC^T)^{-1}$. Further, there exist matrices $Y \in \mathbb{R}^{m \times p}$ and invertible matrix $P_R \in \mathbb{R}^{p \times p}$ such that

$$K_o CP = Y P_R^{-1} P_S^T Q^T + Y R^T \quad (3.52)$$

with $K_o = Y P_R^{-1}$ holds for all K_o and P_R . Next, the Lemmas 1.14 and 1.15 discussed in the chapter 1 would be helpful in deriving the SOF synthesis criterion. Lemma 1.14 is a well-known bounding inequality. Lemma 1.15 is used to linearize similar inequality conditions in the derivation of controller synthesis criterion. The following theorem presents an LMI criterion for SOF synthesis.

Theorem 3.23 *The controller (3.51) with SOF gain K_o places the closed-loop poles of system (4.17) in the LMI region $\hat{\mathcal{D}}_2$ if, for the scalar parameters α_1 and α_2 , there exist positive definite matrices $P = P^T$, Z_{R_1} , Z_{R_2} , Z_{R_3} , J and matrices $P_R = P_R^T$, P_Q , P_S , Y and M satisfying the LMI conditions (3.53) given at the top of the next page. The corresponding controller gain can be computed as $K_o = Y P_R^{-1}$.*

$$\begin{bmatrix} -2a^2P & * & * & * & * & * & * \\ \Xi_1 & -2b^2P + J & * & * & * & * & * \\ \Xi_2 & 0 & -J & * & * & * & * \\ 0 & P_S^T Q^T & 0 & -r_1^{-2}Z_{R_1} & * & * & * \\ P_S^T Q^T & 0 & 0 & 0 & -r_2^{-2}Z_{R_2} & * & * \\ \alpha_2 \Xi_3^T & 0 & 0 & 0 & 0 & -\Xi_4 & * \\ 0 & 0 & \alpha_2 \Xi_3^T & 0 & 0 & 0 & -\Xi_5 \end{bmatrix} < 0, \quad \begin{bmatrix} -\Xi_6 & * & * \\ P_S^T Q^T & -Z_{R_3} & * \\ \alpha_2 \Xi_3^T & 0 & -\Xi_7 \end{bmatrix} < 0, \quad \begin{bmatrix} P_Q & P_S \\ P_S^T & P_R \end{bmatrix} > 0, \forall i = 1, \dots, \bar{N}, \quad (3.53)$$

where $\Xi_1 = r_1 \psi_1 - 2qbP + \bar{M}$, $\bar{M} = MP$, $\psi_1 = PA_i^T + RY^T B_i^T - \alpha_1 QP_S R^T$, $\Xi_2 = r_2 \psi_1^T - \bar{M}$, $\Xi_3 = B_i Y + \alpha_1 R P_R$, $\Xi_4 = \text{Sym}\{\alpha_2 P_R\} - Z_{R_1}$, $\Xi_5 = \text{Sym}\{\alpha_2 P_R\} - Z_{R_2}$, $\Xi_6 = \text{Sym}\{\psi_1\} - 2x_B P$, $\Xi_7 = \text{Sym}\{\alpha_2 P_R\} - Z_{R_3}$.

Proof : Adding-subtracting MP in the $\{2,1\}$ term of (3.46) along with the corresponding symmetric term, one can write

$$\begin{bmatrix} -2a^2P & * \\ r_1 P A_{cl}^T(\rho) - 2qbP + MP & -2b^2P \end{bmatrix} + \text{Sym} \left\{ \begin{bmatrix} 0 \\ I \end{bmatrix} (r_2 A_{cl}(\rho)P - MP) \begin{bmatrix} I & 0 \end{bmatrix} \right\} < 0 \quad (3.54)$$

Now, setting $U = \begin{bmatrix} 0 & I \end{bmatrix}^T$, $V = \begin{bmatrix} r_2 A_{cl}(\rho)P - MP & 0 \end{bmatrix}$, $W = J$ and then applying bounding Lemma 1.14 on (3.54) and applying Schur complement formula [3], one obtains

$$\begin{bmatrix} -2a^2P & * & * \\ r_1 P A_{cl}^T(\rho) - 2qbP + MP & -2b^2P + J & * \\ r_2 A_{cl}(\rho)P - MP & 0 & -J \end{bmatrix} < 0. \quad (3.55)$$

Then, replacing $A_{cl}(\rho)$ by $A(\rho) + B(\rho)K_o C$ and substituting the value of $K_o C P$ from

(3.52) in (3.55), one gets using (3.43)

$$\sum_{i=1}^{\bar{N}} \rho_i \left\{ \begin{bmatrix} 0 & r_1 \Xi_3 P_R^{-1} P_S^T Q^T & r_2 Q P_S P_R^{-T} \Xi_3^T \\ r_1 Q P_S P_R^{-T} \Xi_3^T & 0 & 0 \\ r_2 \Xi_3 P_R^{-1} P_S^T Q^T & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2a^2 P & * & * \\ \Xi_1 & -2b^2 P + J & * \\ \Xi_2 & 0 & -J \end{bmatrix} \right\} < 0, \quad (3.56)$$

where $\Xi_3 = B_i Y + \alpha_1 R P_R$. Next, the following bounding inequalities are derived to obtain bound of the nonlinear term in (3.56). Consider the following expression

$$\begin{aligned} & \begin{bmatrix} 0 & Q P_S \\ Q P_S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 P_R^{-T} & 0 \\ 0 & r_2 P_R^{-T} \end{bmatrix} \begin{bmatrix} \Xi_3 & 0 \\ 0 & 0 \\ 0 & \Xi_3 \end{bmatrix}^T + \\ & \begin{bmatrix} \Xi_3 & 0 \\ 0 & 0 \\ 0 & \Xi_3 \end{bmatrix} \begin{bmatrix} r_1 P_R^{-1} & 0 \\ 0 & r_2 P_R^{-1} \end{bmatrix} \begin{bmatrix} 0 & Q P_S \\ Q P_S & 0 \\ 0 & 0 \end{bmatrix}^T. \end{aligned} \quad (3.57)$$

Then, setting $U = \begin{bmatrix} 0 & Q P_S \\ Q P_S & 0 \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} \Xi_3^T & 0 & 0 \\ 0 & 0 & \Xi_3^T \end{bmatrix}$, $W =$

$\text{Diag}\{P_R^T Z_{R_1}^{-1} P_R^T, P_R^T Z_{R_2}^{-1} P_R^T\}$ with $Z_{R_1}, Z_{R_2} > 0$, and thereafter employing the bounding Lemma 1.14 on (3.57), one obtains the following bounding.

$$\begin{aligned} & \begin{bmatrix} 0 & r_1 \Xi_3 P_R^{-1} P_S^T Q^T & r_2 Q P_S P_R^{-T} \Xi_3^T \\ r_1 Q P_S P_R^{-T} \Xi_3^T & 0 & 0 \\ r_2 \Xi_3 P_R^{-1} P_S^T Q^T & 0 & 0 \end{bmatrix} \\ & \leq \begin{bmatrix} 0 & Q P_S \\ Q P_S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{Z_{R_1}}{r_1^2} & 0 \\ 0 & \frac{Z_{R_2}}{r_2^2} \end{bmatrix}^{-1} \begin{bmatrix} 0 & Q P_S \\ Q P_S & 0 \\ 0 & 0 \end{bmatrix}^T \\ & + \begin{bmatrix} \Xi_3 & 0 \\ 0 & 0 \\ 0 & \Xi_3 \end{bmatrix} \begin{bmatrix} P_R^{-T} Z_{R_1} P_R^{-T} & 0 \\ 0 & P_R^{-T} Z_{R_2} P_R^{-T} \end{bmatrix} \begin{bmatrix} \Xi_3 & 0 & 0 \\ 0 & 0 & \Xi_3 \end{bmatrix}^T. \end{aligned} \quad (3.58)$$

Now, using the bounding (3.58) in (3.56) and employing , one obtains

$$\begin{aligned}
& \begin{bmatrix} -2a^2P & * & * \\ \Xi_1 & -2b^2P + J & * \\ \Xi_2 & 0 & -J \end{bmatrix} + \begin{bmatrix} 0 & QP_S \\ QP_S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{Z_{R_1}}{r_1^2} & 0 \\ 0 & \frac{Z_{R_2}}{r_2^2} \end{bmatrix}^{-1} \begin{bmatrix} 0 & QP_S \\ QP_S & 0 \\ 0 & 0 \end{bmatrix}^T \\
& + \begin{bmatrix} \Xi_3 & 0 \\ 0 & 0 \\ 0 & \Xi_3 \end{bmatrix} \begin{bmatrix} P_R^{-T} Z_{R_1} P_R^{-T} & 0 \\ 0 & P_R^{-T} Z_{R_2} P_R^{-T} \end{bmatrix} \begin{bmatrix} \Xi_3 & 0 & 0 \\ 0 & 0 & \Xi_3 \end{bmatrix}^T < 0, \forall i = 1, \dots, \bar{N}. \quad (3.59)
\end{aligned}$$

Finally, applying Schur complement on (3.59) and employing Lemma 1.15 and substituting $L = \text{Diag}\{\alpha_2 P_R^T, \alpha_2 P_R^T\}$, one obtains first LMI condition of (3.53) which is a sufficient condition for (3.46). Similarly, to show that the second LMI in (3.53) is sufficient for (3.47), rewriting it again as:

$$A_{cl}(\rho)P + PA_{cl}^T(\rho) - 2x_B P > 0. \quad (3.60)$$

Replacing $A_{cl}(\rho)$ by $A(\rho) + B(\rho)K_oC$ and substituting the value of K_oCP from (3.52) in (3.60), one gets $\Xi_6 + \text{Sym}\{QP_S P_R^{-T} \Xi_3^T\} > 0, \forall i = 1, \dots, \bar{N}$. Now, using Lemma 1.14 with $U = -QP_S P_R^{-T}$, $V = \Xi_3^T$ and $W = P_R^T Z_{R_3}^{-1} P_R^T$ and employing Schur complement, one obtains

$$\begin{bmatrix} -\Xi_6 & * \\ P_S^T Q^T & -Z_{R_3} \end{bmatrix} + \begin{bmatrix} \Xi_3 \\ 0 \end{bmatrix} P_R^{-T} Z_{R_3} P_R^{-T} \begin{bmatrix} \Xi_3 \\ 0 \end{bmatrix}^T < 0, \forall i = 1, \dots, \bar{N}. \quad (3.61)$$

Finally, applying Lemma 1.15 on (3.61) and substituting $L = \alpha_2 P_R^T$, one obtains the second LMI in (3.53). \square

3.3.3.3 Dynamic output feedback (DOF) controller

DOF controllers are often designed using SOF design methods due to flexibility of choosing structured matrix variables in SOF case [57]. Here, an observer based design method is introduced for designing DOF controllers of chosen order.

Consider the DOF controller for system (4.17) as

$$\begin{aligned}
x_c(k+1) &= A_c x_c(k) + B_c y(k), \\
u(k) &= C_c x_c(k) + D_c y(k), \quad (3.62)
\end{aligned}$$

where $x_c(k) \in \mathbb{R}^{n_c}$ is the controller state with n_c being the chosen controller order, A_c, B_c, C_c and D_c are controller matrices to be designed. Let us define a new state variable as the augmented plant state and a fictitious error state as

$$\tilde{x}(k) = \begin{bmatrix} x^T(k) & (Gx(k) - x_c(k))^T \end{bmatrix}^T, \quad (3.63)$$

where $G \in \mathbb{R}^{n_c \times n}$ is a full row-rank matrix. For full-order dynamic controller ($n_c = n$), a straightforward choice is $G = I$. Then, the closed-loop system is

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\rho)\tilde{x}(k) + \tilde{B}(\rho)\tilde{u}(k), \\ \tilde{y}(k) &= \tilde{C}\tilde{x}(k), \end{aligned} \quad (3.64)$$

with $\tilde{u}(k) = \tilde{K}\tilde{y}(k)$ and $\tilde{A}(\rho) = \begin{bmatrix} A(\rho) & 0 \\ GA(\rho) & 0 \end{bmatrix}$, $\tilde{B}(\rho) = \begin{bmatrix} B(\rho) & 0 \\ GB(\rho) & -I \end{bmatrix}$, $\tilde{C} = \begin{bmatrix} C & 0 \\ G & -I \end{bmatrix}$, $\tilde{K} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$. The above closed-loop system is in the form of (3.44) with \tilde{K} is the SOF controller as in (3.51). Hence, the DOF controller with its parameters embedded in \tilde{K} can be designed using the SOF criterion in Theorem 3.23.

3.3.3.4 Discrete-time PI controller

Consider the following PI controller for system (4.17) as

$$u(k) = K_p y_1(k) + K_I \sum_{i=0}^k y_2(i),$$

where K_p and K_I are the proportional and integral gains, respectively. For the state feedback controller $y_1(k) = x(k)$ and $y_2(k) = C_a x(k)$, whereas for output feedback $y_1(k) = y(k)$ and $y_2(k) = C_a x(k)$ with $C_a = [0 \ I]$ involves the states to be tracked (states arranged accordingly). Define a new variable $z(k) = \sum_{i=0}^k y_2(i)$. Then $z(k+1) = z(k) + y_2(k+1) = z(k) + C_a x(k+1)$. From (4.17), we get $z(k+1) = (C_a A(\rho) + C_a B(\rho) K_p C) x(k) + (I + C_a B(\rho) K_I) z(k)$. Now, augmenting the states, we get

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \left(\begin{bmatrix} A(\rho) & 0 \\ C_a & I \end{bmatrix} + \begin{bmatrix} B(\rho) \\ C_a B(\rho) \end{bmatrix} \begin{bmatrix} K_p & K_I \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x(k) \\ z(k) \end{bmatrix}.$$

Finally, the closed-loop system can be expressed as

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = (\bar{A}(\rho) + \bar{B}(\rho)\bar{K}\bar{C}) \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \quad (3.65)$$

where $\bar{A}(\rho) = \begin{bmatrix} A(\rho) & 0 \\ C_a & I \end{bmatrix}$, $\bar{B}(\rho) = \begin{bmatrix} B(\rho) \\ C_a B(\rho) \end{bmatrix}$, $\bar{K} = \begin{bmatrix} K_p & K_I \end{bmatrix}$ and $\bar{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$. The system (2) is in the form of (3.44) with the \bar{K} is the controller gain. Hence, the PI controller parameters with the parameters embedded in \bar{K} can be designed using the state feedback criteria in Theorem 3.22 and SOF criteria in Theorem 3.23, respectively.

3.3.4 Numerical examples

In this section, the effectiveness of the proposed design criteria is illustrated through numerical examples. In this regard, note that, (3.53) is an LMI for chosen scalars α_1 and α_2 that are obtained through *fminsearch* program of MATLAB [165] to search for a feasible solution. Note that the present approach looks for a feasible solution and the obtained pole location are not optimal. However, one can further augment more performance criteria in LMI framework such as H_∞ performance in SOF framework for DT systems [164] to enhance performance.

Example 3.24 Consider the linear discrete-time uncertain system drawn from [89] with $T = 0.1$ sec.

$$A = \begin{bmatrix} \delta & 0.8 & -0.4 \\ -0.5 & 0.4 & 0.5 \\ 1.2 & 1.1 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 1.3 \end{bmatrix}, C = \begin{bmatrix} -1 & 1.2 & 1 \\ 0 & -3 & 1 \end{bmatrix}$$

where δ is the uncertain parameter and $0.47 \leq \delta \leq 0.59$. Firstly, choosing $\sigma = -0.2$, a state feedback controller using Theorem 3.22. The obtained values of closed-loop poles with their corresponding controller gains for different values of ζ are tabulated in Table 3.10.

Similarly, SOF controller is then designed using Theorem 3.23 to place the poles of the closed-loop system in the desired LMI regions using the same design parameters for different values of damping ratio ζ . The scalar parameters α_1 and α_2 are obtained using *fminsearch* algorithm of MATLAB. The initial values for these variables in the *fminsearch* program are chosen as $\alpha_1 = \alpha_2 = 0.00001$. The obtained values of closed-loop poles and the controller gains are tabulated in the Table 3.11. Next, using Theorem 3.23, DOF controllers are also designed. The same design parameters are considered as used in previous design case. The closed-loop poles with the designed dynamic controller (\tilde{K}) are

Table 3.10: Designed State Feedback Controllers for Example 3.24

Controller Gain	Closed-loop poles
$\zeta = 0.5$	
$K_s = \begin{bmatrix} -0.0806 & -0.5081 & -0.2948 \\ -0.6631 & -0.8266 & -0.0922 \end{bmatrix}$	1) 0.3997, 0.0874, 0.2102 2) $0.3036 \pm 0.1563i$, 0.2102
$\zeta = 0.7$	
$K_s = \begin{bmatrix} -0.0603 & -0.4865 & -0.2575 \\ -0.6241 & -0.8242 & -0.0216 \end{bmatrix}$	1) 0.5341, 0.0842, 0.2507 2) $0.3692 \pm 0.0374i$, 0.2506
$\zeta = 0.9$	
$K_s = \begin{bmatrix} -0.0534 & -0.4783 & -0.2300 \\ -0.6083 & -0.8224 & 0.0326 \end{bmatrix}$	1) 0.6526, 0.0517, 0.2654 2) 0.5992, 0.2248, 0.2658

Table 3.11: Designed SOF Controllers for Example 3.24

Controller Gain	Closed-loop poles
$\zeta = 0.5$	
$K_o = \begin{bmatrix} 0.0105 & -0.2082 \\ 0.4995 & -0.3797 \end{bmatrix}$	1) 0.7027, $0.0796 \pm 0.2315i$ 2) 0.6494, $0.1663 \pm 0.1876i$
$\zeta = 0.7$	
$K_o = \begin{bmatrix} 0.0099 & -0.1938 \\ 0.5015 & -0.3517 \end{bmatrix}$	1) 0.7530, $0.0701 \pm 0.1741i$ 2) 0.7074, $0.1529 \pm 0.01098i$
$\zeta = 0.9$	
$K_o = \begin{bmatrix} 0.0309 & -0.2385 \\ 0.5514 & -0.4507 \end{bmatrix}$	1) 0.6528, $0.0440 \pm 0.0749i$ 2) 0.5767, 0.2849, -0.0007

tabulated in Table 3.12. It can be seen that the closed-loop poles of the system are placed in the damping region $\hat{\mathcal{D}}_2$, which is effectively within the \mathcal{D}_D region for all the controller design cases.

Table 3.12: Designed DOF Controllers for Example 3.24

Controller Gain (\tilde{K})						Closed-loop poles	
$\zeta = 0.5$							
$A_c =$	0.0971	0.0393	-0.1493	$B_c =$	0.3327	-0.0787	1) 0.6924, 0.1516 \pm 0.0767i, 0.1563, 0.0161, 0.0212 2) 0.5970, 0.0851 \pm 0.1195i, 0.1312, 0.0212, 0.0147
	0.0601	0.0594	-0.1216		0.2496	-0.0724	
$C_c =$	-0.0715	-0.0331	0.1645	$D_c =$	-0.3254	0.0793	
	-0.1344	-0.0715	0.2000		0.1641	0.0799	
	0.0286	-0.0203	-0.0619		-0.3909	-0.0380	
$\zeta = 0.7$							
$A_c =$	0.1811	0.0612	-0.1630	$B_c =$	0.3136	-0.0019	1) 0.6870, 0.4775, 0.1817, 0.0433, 0.0513, 0.0581 2) 0.6650, 0.2417, 0.1829, 0.0395, 0.0513, 0.0574
	0.1031	0.1138	-0.1356		0.2311	-0.0184	
$C_c =$	-0.1242	-0.0558	0.2088	$D_c =$	-0.3072	0.0070	
	-0.1875	-0.1063	0.2312		0.1709	0.0126	
	0.0521	-0.0175	-0.0699		-0.3639	-0.0093	
$\zeta = 0.9$							
$A_c =$	0.1926	0.1379	-0.1931	$B_c =$	0.3372	0.0087	1) 0.7475, 0.4863, 0.2483, 0.0324, 0.0391, 0.0501 2) 0.7292, 0.2935, 0.2144, 0.0191, 0.0383, 0.0501
	0.1133	0.1825	-0.1471		0.2149	-0.0048	
$C_c =$	-0.1400	-0.1318	0.2159	$D_c =$	-0.3150	-0.0031	
	-0.1884	-0.2077	0.2433		0.1375	0.0190	
	0.0316	0.0210	-0.0648		-0.3484	-0.0185	

Example 3.25 Consider system (4.17), $T = 1$ sec, $\bar{N} = 2$ and the following matrices:

$$A_1 = \begin{bmatrix} 0.5220 & 0 & -0.6029 \\ 0 & 1.1052 & 0 \\ 0 & 0 & 0.0498 \end{bmatrix}, A_2 = \begin{bmatrix} 0.2592 & 0 & -0.3808 \\ 0 & 1.1052 & 0 \\ 0 & 0 & 0.0498 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.9457 & -0.3673 \\ 0 & 1.8275 \\ 1.0253 & 1.0253 \end{bmatrix}, B_2 = \begin{bmatrix} 0.9302 & -0.2114 \\ 0 & 1.8275 \\ 1.0253 & 1.0253 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ -1.1 & 0.005 & 0.1 \end{bmatrix}$$

First, design of state feedback controller is carried out. Selecting the design parameter $\sigma = -0.2$, state feedback controller is designed using Theorem 3.22 for different damping ratio (ζ). The obtained values of the closed-loop poles with the corresponding controller gains (K_s) at the two vertices of the polytope are shown in Table 3.13. It can be seen that the closed-loop poles of the system are placed in the damping region $\hat{\mathcal{D}}_2$, even within \mathcal{D}_D region.

Next, selecting $\sigma = -0.2$, SOF controllers are designed using Theorem 3.23 for different damping ratio (ζ). The obtained values of the closed-loop poles with the corresponding controller gains (K_o) at the two vertices of the polytope are shown in Table 3.14. The initial values of the scalars α_1 and α_2 are chosen as $\alpha_1 = \alpha_2 = 0.0001$ in the `fminsearch`

Table 3.13: Designed State Feedback Controllers for Example 3.25

Controller Gain	Closed-loop poles
$\zeta = 0.5$	
$K_s = \begin{bmatrix} -0.1063 & 0.2904 & 0.3660 \\ 0.0354 & -0.5856 & -0.0959 \end{bmatrix}$	1) 0.6997, 0.3554, 0.2000 2) 0.6426, 0.1542, 0.1955
$\zeta = 0.7$	
$K_s = \begin{bmatrix} -0.0869 & 0.2368 & 0.4054 \\ 0.0435 & -0.4988 & -0.1125 \end{bmatrix}$	1) 0.7483, 0.3850, 0.2509 2) 0.7098, 0.1623, 0.2494
$\zeta = 0.9$	
$K_s = \begin{bmatrix} -0.0861 & 0.2165 & 0.4222 \\ 0.0517 & -0.4610 & -0.1221 \end{bmatrix}$	1) 0.7751, 0.3851, 0.2697 2) 0.7412, 0.1563, 0.2697

Table 3.14: Designed SOF Controllers for Example 3.25

Controller Gain	Closed-loop poles
$\zeta = 0.5$	
$K_o = \begin{bmatrix} -0.1812 & 0.1576 \\ -0.3421 & -0.0652 \end{bmatrix}$	1) 0.7027, 0.0838 \pm 0.1439i 2) 0.5798, 0.0208 \pm 0.1287i
$\zeta = 0.7$	
$K_o = \begin{bmatrix} -0.4410 & 0.0690 \\ -0.2495 & -0.0217 \end{bmatrix}$	1) 0.7147, 0.3832, 0.0473 2) 0.6615, 0.1805, 0.0453
$\zeta = 0.9$	
$K_o = \begin{bmatrix} -0.6437 & 0.0787 \\ -0.2032 & -0.0202 \end{bmatrix}$	1) 0.7663, 0.4260, 0.0291 2) 0.7323, 0.2135, 0.0176

program of MATLAB. DOF controllers are also designed using Theorem 3.23. The DOF design condition is framed in the form of SOF design using the error transformation as given in Section 3.3.3.C. The closed-loop poles with the designed dynamic controller (\tilde{K}) are tabulated in Table 3.15. It can be seen that the closed-loop poles of the system are placed in the damping region $\hat{\mathcal{D}}_2$, which is effectively within the \mathcal{D}_D region for all the controller design cases.

Table 3.15: Designed DOF Controllers for Example 3.25

Controller Gain (\tilde{K})						Closed-loop poles	
$\zeta = 0.5$							
$A_c =$	0.0971	0.0393	-0.1493	$, B_c =$	0.3327	-0.0787	1) 0.6924, 0.1516 \pm 0.0767i, 0.1563, 0.0161, 0.0212 2) 0.5970, 0.0851 \pm 0.1195i, 0.1312, 0.0212, 0.0147
	0.0601	0.0594	-0.1216		0.2496	-0.0724	
$C_c =$	-0.0715	-0.0331	0.1645	$, D_c =$	-0.3254	0.0793	
	-0.1344	-0.0715	0.2000		0.1641	0.0799	
	0.0286	-0.0203	-0.0619		-0.3909	-0.0380	
$\zeta = 0.7$							
$A_c =$	0.1811	0.0612	-0.1630	$, B_c =$	0.3136	-0.0019	1) 0.6870, 0.4775, 0.1817, 0.0433, 0.0513, 0.0581 2) 0.6650, 0.2417, 0.1829, 0.0395, 0.0513, 0.0574
	0.1031	0.1138	-0.1356		0.2311	-0.0184	
$C_c =$	-0.1242	-0.0558	0.2088	$, D_c =$	-0.3072	0.0070	
	-0.1875	-0.1063	0.2312		0.1709	0.0126	
	0.0521	-0.0175	-0.0699		-0.3639	-0.0093	
$\zeta = 0.9$							
$A_c =$	0.1926	0.1379	-0.1931	$, B_c =$	0.3372	0.0087	1) 0.7475, 0.4863, 0.2483, 0.0324, 0.0391, 0.0501 2) 0.7292, 0.2935, 0.2144, 0.0191, 0.0383, 0.0501
	0.1133	0.1825	-0.1471		0.2149	-0.0048	
$C_c =$	-0.1400	-0.1318	0.2159	$, D_c =$	-0.3150	-0.0031	
	-0.1884	-0.2077	0.2433		0.1375	0.0190	
	0.0316	0.0210	-0.0648		-0.3484	-0.0185	

3.4 Summary

In this chapter, new sufficient LMI conditions are derived for designing static H_2 and H_∞ controllers for DT systems. Two design criteria have been proposed. First design criteria are developed through appropriate use of bounding inequalities. It involves certain relaxations of the LMI variables leading to less conservative LMI design criteria. The second design criteria are derived using decomposition of an auxiliary matrix. The decomposition facilitates linearization of nonlinear term of reduced size to obtain linear matrix inequality criteria. This leads to less conservative results than the design conditions developed previously. A transformation framework is also proposed for designing dynamic output feedback (DOF) controller using the SOF design method. The efficacy of the developed conditions are illustrated through the numerical examples. A new ellipse-segment approximation is proposed to approximate the non-convex region of the constant damping ratio locus in DT systems. LMI criteria for state and output feedback (static as well as dynamic) controllers are derived to locate the closed-loop poles of DT uncertain polytopic systems in desired damping region. Through examples, it is shown that the closed-loop poles satisfy the specified damping and decay rate.

In the next chapter, we study the application of the proposed design on some real-

time prototypes such as coupled tank system, 2-DOF helicopter and a boost converter. Simulations and experimental results are provided to show its application and effectiveness.