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# Appendix A

## Generation of coherence in an exactly solvable nonlinear nanomechanical system

### S-I Matrix elements in $G_-$ , $G_0$ and $G_+$ regions

The matrix elements corresponding to Eq. (2.22) in the region  $G_-$ :

$$\begin{aligned} A_{11} = & E_n(l) + \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\ & \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \cos \alpha, \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} A_{12} = & \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\ & \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned}
 A_{13} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \cos \alpha, \quad (\text{A.3})
 \end{aligned}$$

$$\begin{aligned}
 A_{14} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.4})
 \end{aligned}$$

$$\begin{aligned}
 A_{21} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.5})
 \end{aligned}$$

$$\begin{aligned}
 A_{22} &= E_n(l) - \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha), \quad (\text{A.6})
 \end{aligned}$$

$$\begin{aligned}
 A_{23} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.7})
 \end{aligned}$$

$$\begin{aligned}
 A_{24} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha), \quad (\text{A.8})
 \end{aligned}$$

$$\begin{aligned}
A_{31} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \cos \alpha, \quad (\text{A.9})
\end{aligned}$$

$$\begin{aligned}
A_{32} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.10})
\end{aligned}$$

$$\begin{aligned}
A_{33} &= E_n(l) + \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \cos \alpha, \quad (\text{A.11})
\end{aligned}$$

$$\begin{aligned}
A_{34} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.12})
\end{aligned}$$

$$\begin{aligned}
A_{41} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.13})
\end{aligned}$$

$$\begin{aligned}
A_{42} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha), \quad (\text{A.14})
\end{aligned}$$

$$\begin{aligned}
 A_{43} = & \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 & \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.15})
 \end{aligned}$$

$$\begin{aligned}
 A_{44} = & E_n(l) - \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 & \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha), \quad (\text{A.16})
 \end{aligned}$$

The matrix elements corresponding to Eq. (2.22) in the region  $G_0$ :

$$\begin{aligned}
 A_{11} = & a_n(l) + \omega_0/2 + Q \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 \right. \right. \\
 & \left. \left. + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \cos \alpha, \quad (\text{A.17})
 \end{aligned}$$

$$A_{12} = Q \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.18})$$

$$A_{21} = Q \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \quad (\text{A.19})$$

$$\begin{aligned}
 A_{22} = & a_n(l) - \omega_0/2 + Q \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha), \\
 & \quad \quad \quad (\text{A.20})
 \end{aligned}$$

The matrix elements corresponding to Eq. (2.22) in the region  $G_+$ :

$$\begin{aligned}
 A_{11} = & E_n(l) + \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 & \left. + \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} \cos \alpha, \quad (\text{A.21})
 \end{aligned}$$

$$\begin{aligned}
A_{12} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. + \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \frac{1}{2} \sin \alpha, \right. \quad (A.22)
\end{aligned}$$

$$\begin{aligned}
A_{13} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \frac{1}{2} \cos \alpha, \right. \quad (A.23)
\end{aligned}$$

$$\begin{aligned}
A_{14} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \frac{1}{2} \sin \alpha, \right. \quad (A.24)
\end{aligned}$$

$$\begin{aligned}
A_{21} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. + \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \frac{1}{2} \sin \alpha, \right. \quad (A.25)
\end{aligned}$$

$$\begin{aligned}
A_{22} &= E_n(l) - \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. + \left\{ \frac{\pi}{2} \left( -B_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \frac{1}{2} (-\cos \alpha), \right. \quad (A.26)
\end{aligned}$$

$$\begin{aligned}
A_{23} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \frac{1}{2} \sin \alpha, \right. \quad (A.27)
\end{aligned}$$

$$\begin{aligned}
 A_{24} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha),
 \end{aligned} \tag{A.28}$$

$$\begin{aligned}
 A_{31} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} \cos \alpha,
 \end{aligned} \tag{A.29}$$

$$\begin{aligned}
 A_{32} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} \sin \alpha,
 \end{aligned} \tag{A.30}$$

$$\begin{aligned}
 A_{33} &= E_n(l) + \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. + \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} \cos \alpha,
 \end{aligned} \tag{A.31}$$

$$\begin{aligned}
 A_{34} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. + \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+3}^{(2n+1)}(l) \right\} \right] \frac{1}{2} \sin \alpha,
 \end{aligned} \tag{A.32}$$

$$\begin{aligned}
 A_{41} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
 &\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} \sin \alpha,
 \end{aligned} \tag{A.33}$$

$$\begin{aligned}
A_{42} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. - \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha), \\
A_{43} &= \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. + \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} \sin \alpha, \tag{A.34}
\end{aligned}$$

$$\begin{aligned}
A_{44} &= E_n(l) - \omega_0/2 + \frac{Q}{2} \left[ \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\} \right. \\
&\quad \left. + \left\{ \pi \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(l) B_{2r+4}^{(2n+2)}(l) \right\} \right] \frac{1}{2} (-\cos \alpha), \tag{A.35}
\end{aligned}$$

## S-II Eigenvectors in $G_-$ region

Eigenvectors of Hamiltonian corresponding to  $G_-$  region in Eq. (2.27) is given as:-

$$\begin{aligned}
&\frac{(b_1 - c_1 \pm \sqrt{(b_1 - c_1)^2 + (d_1 - e_1)^2})}{\sqrt{2(d_1 - e_1)^2 \pm 2(b_1 - c_1 \pm \sqrt{(b_1 - c_1)^2 + (d_1 - e_1)^2})^2}}, \\
&\frac{(d_1 - e_1)}{\sqrt{2(d_1 - e_1)^2 \pm 2(b_1 - c_1 \pm \sqrt{(b_1 - c_1)^2 + (d_1 - e_1)^2})^2}}, \\
&\frac{(-b_1 + c_1 \mp \sqrt{(b_1 - c_1)^2 + (d_1 - e_1)^2})}{\sqrt{2(d_1 - e_1)^2 \pm 2(b_1 - c_1 \pm \sqrt{(b_1 - c_1)^2 + (d_1 - e_1)^2})^2}}, \\
&\frac{(d_1 - e_1)}{\sqrt{2(d_1 - e_1)^2 \pm 2(b_1 - c_1 \pm \sqrt{(b_1 - c_1)^2 + (d_1 - e_1)^2})^2}} \tag{A.36}
\end{aligned}$$

$$\begin{aligned}
 & \frac{(b_1 + c_1 \pm \sqrt{(b_1 + c_1)^2 + (d_1 + e_1)^2})}{\sqrt{2(d_1 + e_1)^2 + 2(b_1 + c_1 \pm \sqrt{(b_1 + c_1)^2 + (d_1 + e_1)^2})^2}}, \\
 & \frac{(d_1 + e_1)}{\sqrt{2(d_1 + e_1)^2 + 2(b_1 + c_1 \pm \sqrt{(b_1 + c_1)^2 + (d_1 + e_1)^2})^2}}, \\
 & , \\
 & \frac{(b_1 + c_1 \pm \sqrt{(b_1 + c_1)^2 + (d_1 + e_1)^2})}{\sqrt{2(d_1 + e_1)^2 + 2(b_1 + c_1 \pm \sqrt{(b_1 + c_1)^2 + (d_1 + e_1)^2})^2}}, \\
 & \frac{(d_1 + e_1)}{\sqrt{2(d_1 + e_1)^2 + 2(b_1 + c_1 \pm \sqrt{(b_1 + c_1)^2 + (d_1 + e_1)^2})^2}}
 \end{aligned} \tag{A.37}$$

### S-III Coefficients of density matrix in $G_-$ region

$$\begin{aligned}
 \zeta_1(t) = e^{-ia_1 t} & \left\{ \frac{1}{2} \cos(t\lambda_1) + \frac{1}{2} \cos(t\lambda_2) - \frac{1}{2} ib_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} + \frac{\sin(t\lambda_2)}{\lambda_2} \right) \right. \\
 & \left. + \frac{1}{2} ic_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} - \frac{\sin(t\lambda_2)}{\lambda_2} \right) \right\},
 \end{aligned} \tag{A.38}$$

$$\begin{aligned}
 \zeta_2(t) = e^{-ia_1 t} & \left\{ -\frac{1}{2} ie_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} + \frac{\sin(t\lambda_2)}{\lambda_2} \right) + \frac{1}{2} id_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} - \frac{\sin(t\lambda_2)}{\lambda_2} \right) \right\},
 \end{aligned} \tag{A.39}$$

$$\begin{aligned}
 \zeta_3(t) = e^{-ia_1 t} & \left\{ -\frac{1}{2} ic_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} + \frac{\sin(t\lambda_2)}{\lambda_2} \right) + \frac{1}{2} \cos(t\lambda_2) - \frac{1}{2} \cos(t\lambda_1) \right. \\
 & \left. + \frac{1}{2} ib_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} - \frac{\sin(t\lambda_2)}{\lambda_2} \right) \right\},
 \end{aligned} \tag{A.40}$$

$$\begin{aligned}
 \zeta_4(t) = e^{-ia_1 t} & \left\{ -\frac{1}{2} id_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} + \frac{\sin(t\lambda_2)}{\lambda_2} \right) + \frac{1}{2} ie_1 \left( \frac{\sin(t\lambda_1)}{\lambda_1} - \frac{\sin(t\lambda_2)}{\lambda_2} \right) \right\}.
 \end{aligned} \tag{A.41}$$

The following notations are used:

$$\lambda_{1,2} = \sqrt{(b_1 \mp c_1)^2 + (d_1 \mp e_1^2)^2}. \quad (\text{A.42})$$

### S-IV Calculation of $d|c_1(t)|/dt$

$$\begin{aligned}
F = \frac{d|c_1(t)|}{dt} &= \frac{-\tau}{2}|c_1(t)| + \frac{|c_1(0)|^2 e^{-\tau t}}{|c_1(t)|} \left( \left( \cosh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) \right. \right. \\
&+ \frac{\kappa_1}{|\kappa_x|^2} \sinh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) - \frac{\kappa_2}{|\kappa_x|^2} \cosh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) \left. \right) \\
&\left( \frac{\kappa'}{2} \left( \sinh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) + \frac{\kappa_1}{|\kappa_x|^2} \cosh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) \right. \right. \\
&- \frac{\kappa_2}{|\kappa_x|^2} \sinh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) \left. \right) + \frac{\kappa''}{2} \left( -\cosh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) \right. \\
&- \frac{\kappa_1}{|\kappa_x|^2} \sinh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) - \frac{\kappa_2}{|\kappa_x|^2} \cosh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) \left. \right) \left. \right) \\
&+ \left( \sinh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) + \frac{\kappa_1}{|\kappa_x|^2} \cosh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) \right. \\
&+ \frac{\kappa_2}{|\kappa_x|^2} \sinh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) \left. \right) \left( \frac{\kappa'}{2} \left( \cosh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) \right. \right. \\
&+ \frac{\kappa_1}{|\kappa_x|^2} \sinh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) + \frac{\kappa_2}{|\kappa_x|^2} \cosh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) \left. \right) \\
&+ \frac{\kappa''}{2} \left( -\sinh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) - \frac{\kappa_1}{|\kappa_x|^2} \cosh\left(\frac{\kappa' t}{2}\right) \cos\left(\frac{\kappa'' t}{2}\right) \right. \\
&\left. \left. + \frac{\kappa_2}{|\kappa_x|^2} \sinh\left(\frac{\kappa' t}{2}\right) \sin\left(\frac{\kappa'' t}{2}\right) \right) \right) \left. \right)
\end{aligned} \quad (\text{A.43})$$

Where  $\kappa' = \sqrt{\frac{\sqrt{(\tau^2 - 2Ng - \delta)^2 + (2\tau\delta)^2} + (\tau^2 - 2Ng - \delta)}{2}}$   $\left( \kappa'' = \sqrt{\frac{\sqrt{(\tau^2 - 2Ng - \delta)^2 + (2\tau\delta)^2} - (\tau^2 - 2Ng - \delta)}{2}} \right)$   
are real (imaginary) part of  $\kappa_x$ .  $\kappa_1 = (\tau\kappa' + \delta\kappa'')$ ,  $\kappa_2 = (\delta\kappa' - \tau\kappa'')$ .

## S-V Reduced density matrix $\rho_s(t)$

The reduced density matrix of the system in  $G_0$  region coupled to  $N$  bosonic reservoirs can be obtained from the total state  $|\Psi_0(0)\rangle = |\psi_0(0)\rangle \otimes \prod_{n=1}^N |\bar{0}\rangle$  which evolves in time as:

$$|\Psi(t)\rangle = [c_0(0)|0\rangle + c_1(t)|1\rangle] \otimes \prod_{n=1}^N |\bar{0}\rangle_{n,r} + |0\rangle \otimes \sum_{n=1}^N \sum_k c_{n,k}(t) |1_k\rangle_{n,r}. \quad (\text{A.44})$$

$$|\bar{0}\rangle_{n,r} = \prod_{k=1} |0_k\rangle_{n,r}, \prod_{k=1} \langle 0_k | 0_k \rangle_{n,r} = 1, \prod_{n=1}^N \langle \bar{0} | \bar{0} \rangle_{n,r} = 1, \quad (\text{A.45})$$

Normalization condition is:

$$\begin{aligned} \langle \Psi(t) | \Psi(t) \rangle &= ([c_0^*(0)\langle 0 | + c_1^*(t)\langle 1 |] \otimes \prod_{n'=1}^N \langle \bar{0} |_{n',r} + \langle 0 | \otimes \sum_{n'=1}^N \sum_{k'} c_{n',k}^*(t) \langle 1_{k'} |_{n',r}) [c_0(0)|0\rangle \\ &+ c_1(t)|1\rangle] \otimes \prod_{n=1}^N |\bar{0}\rangle_{n,r} + |0\rangle \otimes \sum_{n=1}^N \sum_k c_{n,k}(t) |1_k\rangle_{n,r} \end{aligned} \quad (\text{A.46})$$

which comes out to be

$$|c_0(0)|^2 + |c_1(t)|^2 + \sum_{n=1}^N \sum_k |c_{n,k}(t)|^2 = 1. \quad (\text{A.47})$$

The state of the system at any time  $t$  can be written in the form

$$\begin{aligned} |\Psi(t)\rangle &= \left[ c_0(0) \otimes \prod_{n=1}^N \prod_{k=1} |0_k\rangle_{n,r} + \mathcal{S} \otimes \sum_{n=1}^N \sum_k c_{n,k}(t) |1_k\rangle_{n,r} \right] \otimes |0\rangle \\ &+ c_1(t) |1\rangle \otimes \prod_{n=1}^N \prod_{k=1} |0_k\rangle_{n,r} \end{aligned} \quad (\text{A.48})$$

and corresponding density matrix can be written as

$$\begin{aligned}
\rho(t) &= |\Psi(t)\rangle\langle\Psi(t)| \\
&= \left[ c_0(0) \otimes \prod_{n=1}^N \prod_{k=1}^N |0_k\rangle_{n,r} + \mathcal{I} \otimes \sum_{n=1}^N \sum_k c_{n,k}(t) |1_k\rangle_{n,r} \right] \left[ c_0^*(0) \otimes \prod_{n'=1}^N \prod_{k'=1}^N \langle 0_{k'}|_{n',r} \right. \\
&+ \mathcal{I} \otimes \sum_{n'=1}^N \sum_{k'} c_{n',k'}^*(t) \langle 1_{k'}|_{n',r} \left. \right] |0\rangle\langle 0| + \left[ c_0(0) \otimes \prod_{n=1}^N \prod_{k=1}^N |0_k\rangle_{n,r} \right. \\
&+ \mathcal{I} \otimes \sum_{n=1}^N \sum_k c_{n,k}(t) |1_k\rangle_{n,r} \left. \right] c_1^*(t) \otimes \prod_{n'=1}^N \prod_{k'=1}^N \langle 0_{k'}|_{n',r} |0\rangle\langle 1| + c_1(t) \otimes \prod_{n=1}^N \prod_{k=1}^N |0_k\rangle_{n,r} \\
&\left[ c_0^*(0) \otimes \prod_{n=1}^N \prod_{k'=1}^N \langle 0_{k'}|_{n',r} + \mathcal{I} \otimes \sum_{n'=1}^N \sum_{k'} c_{n',k'}^*(t) \langle 1_{k'}|_{n',r} \right] |1\rangle\langle 0| \\
&+ |c_1(t)|^2 \otimes \prod_{n=1}^N \prod_{k=1}^N |0_k\rangle_{n,r} \prod_{n'=1}^N \prod_{k'=1}^N \langle 0_{k'}|_{n',r} |1\rangle\langle 1|, \tag{A.49}
\end{aligned}$$

Now, tracing out the reservoir part, we get reduced density matrix in terms of Probability amplitude  $c_0(0), c_1(t)$  as:

$$\begin{aligned}
\rho_s(t) &= \text{tr}_r(|\Psi(t)\rangle\langle\Psi(t)|) = (1 - |c_1(t)|^2) |0\rangle\langle 0| + c_0(0) c_1^*(t) |0\rangle\langle 1| \\
&+ c_1(t) c_0^*(0) |1\rangle\langle 0| + |c_1(t)|^2 |1\rangle\langle 1|. \tag{A.50}
\end{aligned}$$

## S-VI Eigenvalues and eigenvectors related to section 2.7

Eigenvalues corresponding to Hamiltonian Eq.2.85

$$\begin{aligned}
E_{\hat{H},1,n}(l) &= \frac{1}{2} \left( E_{1,n}(l) + E_{2,n}(l) + \sqrt{4Y^2 + (E_{1,n}(l) - E_{2,n}(l) + 2J)^2} \right), \\
E_{\hat{H},2,n}(l) &= \frac{1}{2} \left( E_{1,n}(l) + E_{2,n}(l) - \sqrt{4Y^2 + (E_{1,n}(l) - E_{2,n}(l) + 2J)^2} \right). \tag{A.51}
\end{aligned}$$

The coefficient of eigenvectors represented in Eq.2.87 are:

$$\zeta_1 = \frac{L}{\sqrt{L^2 + 1}} = \sin \Theta, \text{ and } \zeta_2 = \frac{1}{\sqrt{L^2 + 1}} = \cos \Theta, \tag{A.52}$$

where,

$$L = \frac{E_{1,n}(l) - E_{2,n}(l) + 2J}{2Y} + \frac{\sqrt{4Y^2 + (E_{1,n}(l) - E_{2,n}(l) + 2J)^2}}{2Y}, \quad (\text{A.53})$$

and

$$J = -\Delta\omega_0(\alpha_1^2 - \beta_1^2), \quad Y = -2\Delta\omega_0\alpha_1\beta_1 \quad (\text{A.54})$$

$$\begin{aligned} E_{1,n}(l) &= a(l) + \sqrt{\frac{1}{4}\omega_0^2 + \frac{1}{4}Q^2\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle^2 + \frac{1}{2}\omega_0Q\cos\alpha\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle}, \\ E_{2,n}(l) &= a(l) - \sqrt{\frac{1}{4}\omega_0^2 + \frac{1}{4}Q^2\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle^2 + \frac{1}{2}\omega_0Q\cos\alpha\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle}, \end{aligned} \quad (\text{A.55})$$

The coefficient of eigenvectors of  $\rho(t)$  represented in Eq.2.89 are:

$$v_1 = -\frac{\tan(\Theta)\exp(-i\omega_{12}t)}{\sqrt{\tan(\Theta)^2 + 1}}, \quad (\text{A.56})$$

$$v_2 = \frac{1}{\sqrt{\tan(\Theta)^2 + 1}}, \quad (\text{A.57})$$

and the eigenvalues of  $\rho(t)$  are

$$E_1(\rho(t)) = p_1, \quad E_2(\rho(t)) = p_2. \quad (\text{A.58})$$

$$\begin{aligned} \lambda &= \frac{2}{Q\sin\alpha\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle} \left\{ \frac{1}{2}\omega_0 + \frac{1}{2}Q\cos\alpha\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle \right. \\ &\quad \left. + \sqrt{\frac{1}{4}\omega_0^2 + \frac{1}{4}Q^2\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle^2 + \frac{1}{2}\omega_0Q\cos\alpha\langle ce_n(l)|\cos 2\varphi|ce_n(l)\rangle} \right\}. \end{aligned} \quad (\text{A.59})$$

The value of the integral:

$$\begin{aligned} \langle ce_{2n+1}(l)|V|ce_{2n+1}(l)\rangle &= \Delta l \left\{ \frac{\pi}{2} \left( A_1^{(2n+1)}(l) \right)^2 + \pi \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(l) A_{2r+3}^{(2n+1)}(l) \right\}, \\ \langle ce_{2n}(l)|V|ce_{2n}(l)\rangle &= \Delta l \left\{ \pi A_0^{(2n)}(l) A_2^{(2n)}(l) + \pi \sum_{r=0}^{\infty} A_{2r}^{(2n)}(l) A_{2r+2}^{(2n)}(l) \right\}. \end{aligned} \quad (\text{A.60})$$

Parameters for small barrier limit:

$$E_{1,n}(l) = n^2 + \sqrt{\frac{1}{4}\omega_0^2 + \frac{1}{4}Q^2 + \frac{1}{2}Q\omega_0 \cos \alpha}, \quad (\text{A.61})$$

$$E_{2,n}(l) = n^2 - \sqrt{\frac{1}{4}\omega_0^2 + \frac{1}{4}Q^2 + \frac{1}{2}Q\omega_0 \cos \alpha}, \quad (\text{A.62})$$

$$\lambda = \frac{2}{Q \sin \alpha} \left\{ \frac{1}{2}\omega_0 + \frac{1}{2}Q \cos \alpha + \sqrt{\frac{1}{4}\omega_0^2 + \frac{1}{4}Q^2 + \frac{1}{2}Q\omega_0 \cos \alpha} \right\}. \quad (\text{A.63})$$



# Appendix B

## Hybrid quantum-classical chaotic NEMS

### S-I Normalization condition for the wave function represented in Eq.(3.16) when $G_n\{\varphi\}$ is diagonal.

The eigenstates and eigenvalues of the  $n$ th Floquet operator have the form:

$$\begin{aligned} |\varphi_n^+\rangle &= \eta_n|0\rangle + \xi_n|1\rangle, \\ |\varphi_n^-\rangle &= \xi_n|0\rangle - \eta_n|1\rangle, \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} \eta_n &= \frac{1}{\sqrt{1+k_n^2}}, \\ \xi_n &= -\frac{k_n}{\sqrt{1+k_n^2}}, \\ k_n &= \frac{\omega_0 + \sqrt{\chi_n^2 + \omega_0^2}}{\chi_n}. \end{aligned} \tag{B.2}$$

$k_n = k_n^*$ , so  $\eta_n = \eta_n^*$  and  $\xi_n = \xi_n^*$ . The normalization condition of the wave function after  $N$  kicks, i. e. , at  $t = NT$  has the form:

$$\begin{aligned}
\langle \psi(t = TN) | \psi(t = TN) \rangle &= |A_{11}|^2 \{ |\eta_1|^2 (|\eta_N|^2 + |\xi_N|^2) \} + A_{11}^* A_{12} \{ |\eta_1|^2 (\xi_N \eta_N^* \\
&- \xi_N^* \eta_N) \} + A_{11}^* A_{21} \{ \eta_1^* \xi_1 \exp(2i\varphi_1) (|\eta_N|^2 + |\xi_N|^2) \} \\
&+ A_{11}^* A_{22} \{ \eta_1^* \xi_1 (\xi_N \eta_N^* - \xi_N^* \eta_N) \} + A_{11} A_{12}^* \{ |\eta_1|^2 (-\xi_N \eta_N^* + \xi_N^* \eta_N) \} \\
&+ |A_{12}|^2 \{ |\eta_1|^2 (|\eta_N|^2 + |\xi_N|^2) \} + A_{12}^* A_{21} \{ \eta_1^* \xi_1 \exp(2i\varphi_1) (-\xi_N \eta_N^* + \xi_N^* \eta_N) \} \\
&+ A_{12}^* A_{22} \{ \eta_1^* \xi_1 \exp(2i\varphi_1) (|\xi_N|^2 + |\eta_N|^2) \} + A_{21}^* A_{11} \{ \eta_1 \xi_1^* \exp(-2i\varphi_1) (|\xi_N|^2 \\
&+ |\eta_N|^2) \} + A_{21}^* A_{12} \{ \eta_1 \xi_1^* \exp(-2i\varphi_1) (\xi_N \eta_N^* - \eta_N \xi_N^*) \} \\
&+ |A_{21}|^2 \{ |\xi_1|^2 (|\eta_N|^2 + |\xi_N|^2) \} + A_{21}^* A_{22} \{ |\xi_1|^2 (\eta_N^* \xi_N - \xi_N^* \eta_N) \} \\
&+ A_{11} A_{22}^* \{ \eta_1 \xi_1^* \exp(-2i\varphi_1) (\eta_N \xi_N^* - \eta_N^* \xi_N) \} + A_{12} A_{22}^* \{ \eta_1 \xi_1^* \exp(-2i\varphi_1) (|\xi_N|^2 + |\eta_N|^2) \} \\
&+ A_{21} A_{22}^* \{ |\xi_1|^2 (\eta_N \xi_N^* - \eta_N^* \xi_N) + |A_{22}|^2 \{ |\xi_1|^2 (|\eta_N|^2 + |\xi_N|^2) \}.
\end{aligned}$$

In the particular case,  $A_{11} = A_{22}^*$  and  $A_{12} = A_{21}^*$

after simplification one can get the form:

$$\begin{aligned}
\langle \psi(t = TN) | \psi(t = TN) \rangle &= |A_{11}|^2 + |A_{12}|^2 + A_{11}^* A_{21} \{ \eta_1^* \xi_1 \exp(2i\varphi_1) \} \\
&+ A_{12}^* A_{22} \{ \eta_1^* \xi_1 \exp(2i\varphi_1) \} + A_{21}^* A_{11} \{ \eta_1 \xi_1^* \exp(-2i\varphi_1) \} + A_{12} A_{22}^* \{ \eta_1 \xi_1^* \exp(2i\varphi_1) \}.
\end{aligned} \tag{B.3}$$

If all kicks are identical, the off-diagonal elements of the matrix  $\mathcal{A}$  are zero. Therefore:

$$|A_{11}|^2 = 1, |A_{12}|^2 = 0, |A_{21}|^2 = 0, |A_{22}|^2 = 1$$

$$\langle \psi(t = TN) | \psi(t = TN) \rangle = 1. \tag{B.4}$$

## S-II Normalization condition for the wave function represented in Eq.(3.16) when $G_n\{\varphi\}$ is non-diagonal.

We prove the normalization of the evolved wave function in the general case of three  $N = 3$  different kicks. The elements of the  $G$  matrix in Eq.(3.16) have the form:

$$G_2\{\varphi\} = \begin{bmatrix} \exp(-i\varphi_2)(\eta_2\eta_1 + \xi_2\xi_1) & \exp(-i\varphi_2)(\eta_2\xi_1 - \xi_2\eta_1) \\ \exp(i\varphi_2)(\xi_2\eta_1 - \eta_2\xi_1) & \exp(i\varphi_2)(\eta_2\eta_1 + \xi_2\xi_1) \end{bmatrix}, \quad (\text{B.5})$$

$$G_3\{\varphi\} = \begin{bmatrix} \exp(-i\varphi_3)(\eta_3\eta_2 + \xi_3\xi_2) & \exp(-i\varphi_3)(\eta_3\xi_2 - \xi_3\eta_2) \\ \exp(i\varphi_3)(\xi_3\eta_2 - \eta_3\xi_2) & \exp(i\varphi_3)(\eta_3\eta_2 + \xi_3\xi_2) \end{bmatrix}. \quad (\text{B.6})$$

Therefore for  $\mathcal{A}$  matrix we deduce

$$\mathcal{A} = \prod_{n=2}^3 G_n\{\varphi\}, \quad (\text{B.7})$$

$$\begin{aligned} A_{11} &= \exp(-i\varphi_2 - i\varphi_3)(\eta_2\eta_1 + \xi_2\xi_1)(\eta_3\eta_2 + \xi_3\xi_2) \\ &+ \exp(-i\varphi_2 + i\varphi_3)(\eta_2\xi_1 - \xi_2\eta_1)(\xi_3\eta_2 - \eta_3\xi_2), \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} A_{12} &= \exp(-i\varphi_2 - i\varphi_3)(\eta_2\eta_1 + \xi_2\xi_1)(\eta_3\xi_2 - \xi_3\eta_2) \\ &+ \exp(-i\varphi_2 + i\varphi_3)(\eta_2\xi_1 - \xi_2\eta_1)(\eta_3\eta_2 + \xi_3\xi_2), \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} A_{21} &= \exp(i\varphi_2 - i\varphi_3)(\eta_3\eta_2 + \xi_2\xi_3)(\eta_1\xi_2 - \xi_1\eta_2) \\ &+ \exp(i\varphi_2 + i\varphi_3)(\eta_2\eta_1 + \xi_2\xi_1)(\xi_3\eta_2 - \eta_3\xi_2), \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned}
A_{22} &= \exp(i\varphi_2 - i\varphi_3)(\xi_2\eta_1 + \eta_2\xi_1)(\eta_3\xi_2 - \xi_3\eta_2) \\
&+ \exp(i\varphi_2 + i\varphi_3)(\eta_2\eta_1 + \xi_2\xi_1)(\eta_3\eta_2 + \xi_3\xi_2), \tag{B.11}
\end{aligned}$$

$$\begin{aligned}
\langle \psi(t = TN) | \psi(t = TN) \rangle &= |A_{11}|^2 |\eta_1|^2 + |A_{12}|^2 |\eta_1|^2 \\
&+ |A_{21}|^2 |\xi_1|^2 + A_{11}^* A_{21} \{ \eta_1^* \xi_1 \exp(2i\varphi_1) \} + A_{12}^* A_{22} \{ \eta_1^* \xi_1 \exp(2i\varphi_1) \} \\
&+ |A_{22}|^2 |\xi_1|^2 + A_{21}^* A_{11} \{ \eta_1 \xi_1^* \exp(-2i\varphi_1) \} \\
&+ A_{12} A_{22}^* \{ \eta_1 \xi_1^* \exp(2i\varphi_1) \}, \tag{B.12}
\end{aligned}$$

$$\begin{aligned}
|A_{11}|^2 &= ((\eta_2\xi_1 - \eta_1\xi_2)(-\eta_3\xi_2 + \eta_2\xi_3) \cos[\varphi_2 - \varphi_3] + (\eta_1\eta_2 + \xi_1\xi_2)(\eta_2\eta_3 \\
&+ \xi_2\xi_3) \cos[\varphi_2 + \varphi_3])^2 + ((\eta_2\xi_1 - \eta_1\xi_2)(-\eta_3\xi_2 + \eta_2\xi_3) \sin[\varphi_2 - \varphi_3] \\
&+ (\eta_1\eta_2 + \xi_1\xi_2)(\eta_2\eta_3 + \xi_2\xi_3) \sin[\varphi_2 + \varphi_3])^2, \tag{B.13}
\end{aligned}$$

$$\begin{aligned}
|A_{12}|^2 &= ((\eta_2\xi_1 - \eta_1\xi_2)(\eta_2\eta_3 + \xi_2\xi_3) \cos[\varphi_2 - \varphi_3] + (\eta_1\eta_2 + \xi_1\xi_2)(\eta_3\xi_2 \\
&- \eta_2\xi_3) \cos[\varphi_2 + \varphi_3])^2 + ((\eta_2\xi_1 - \eta_1\xi_2)(\eta_2\eta_3 + \xi_2\xi_3) \sin[\varphi_2 - \varphi_3] \\
&- (\eta_1\eta_2 + \xi_1\xi_2)(-\eta_3\xi_2 + \eta_2\xi_3) \sin[\varphi_2 + \varphi_3])^2, \tag{B.14}
\end{aligned}$$

$$\begin{aligned}
|A_{21}|^2 &= ((-\eta_2\xi_1 + \eta_1\xi_2)(\eta_2\eta_3 + \xi_2\xi_3) \cos[\varphi_2 - \varphi_3] + (\eta_1\eta_2 + \xi_1\xi_2)(-\eta_3\xi_2 \\
&+ \eta_2\xi_3) \cos[\varphi_2 + \varphi_3])^2 + ((-\eta_2\xi_1 + \eta_1\xi_2)(\eta_2\eta_3 + \xi_2\xi_3) \sin[\varphi_2 - \varphi_3] \\
&+ (\eta_1\eta_2 + \xi_1\xi_2)(-\eta_3\xi_2 + \eta_2\xi_3) \sin[\varphi_2 + \varphi_3])^2, \tag{B.15}
\end{aligned}$$

$$\begin{aligned}
 |A_{22}|^2 &= ((\eta_2\xi_1 - \eta_1\xi_2)(-\eta_3\xi_2 + \eta_2\xi_3) \cos[\varphi_2 - \varphi_3] + (\eta_1\eta_2 + \xi_1\xi_2)(\eta_2\eta_3 \\
 &+ \xi_2\xi_3) \cos[\varphi_2 + \varphi_3])^2 + ((\eta_2\xi_1 - \eta_1\xi_2)(-\eta_3\xi_2 + \eta_2\xi_3) \sin[\varphi_2 - \varphi_3] \\
 &+ (\eta_1\eta_2 + \xi_1\xi_2)(\eta_2\eta_3 + \xi_2\xi_3) \sin[\varphi_2 + \varphi_3])^2, \tag{B.16}
 \end{aligned}$$

$$\begin{aligned}
 A_{21}A_{11}^* &= -\exp(2i\varphi_2)((\eta_2^2 + \xi_2^2)(\eta_3\xi_1 - \eta_1\xi_3) \cos[\varphi_3] - i(\eta_2^2(\eta_3\xi_1 + \eta_1\xi_3) \\
 &- \xi_2^2(\eta_3\xi_1 + \eta_1\xi_3) + 2\eta_2\xi_2(-\eta_1\eta_3 + \xi_1\xi_3)) \sin[\varphi_3])((\eta_2^2 + \xi_2^2)(\eta_1\eta_3 \\
 &+ \xi_1\xi_3) \cos[\varphi_3] + i(\eta_1(\eta_2^2\eta_3 - \eta_3\xi_2^2 + 2\eta_2\xi_2\xi_3) + \xi_1(2\eta_2\eta_3\xi_2 - \eta_2^2\xi_3 \\
 &+ \xi_2^2\xi_3)) \sin[\varphi_3]), \tag{B.17}
 \end{aligned}$$

$$\begin{aligned}
 A_{21}^*A_{11} &= \exp(-2i\varphi_2)((-\eta_2\xi_1 + \eta_1\xi_2)(\eta_1\eta_2 + \xi_1\xi_2)(\xi_2(-\eta_3 + \xi_3) \\
 &+ \eta_2(\eta_3 + \xi_3))(\eta_2(\eta_3 - \xi_3) + \xi_2(\eta_3 + \xi_3)) + (\eta_3\xi_2 - \eta_2\xi_3)(\eta_2\eta_3 + \xi_2\xi_3) \\
 &(-(\xi_1(-\eta_2 + \xi_2) + \eta_1(\eta_2 + \xi_2))(\eta_1(\eta_2 - \xi_2) + \xi_1(\eta_2 + \xi_2)) \cos[2\varphi_3] \\
 &+ i(\eta_1^2 + \xi_1^2)(\eta_2^2 + \xi_2^2) \sin[2\varphi_3])), \tag{B.18}
 \end{aligned}$$

$$\begin{aligned}
 A_{12}^*A_{22} &= \exp(2i\varphi_2)((\eta_2^2 + \xi_2^2)(\eta_3\xi_1 - \eta_1\xi_3) \cos[\varphi_3] - i(\eta_2^2(\eta_3\xi_1 + \eta_1\xi_3) \\
 &- \xi_2^2(\eta_3\xi_1 + \eta_1\xi_3) + 2\eta_2\xi_2(-\eta_1\eta_3 + \xi_1\xi_3)) \sin[\varphi_3])((\eta_2^2 + \xi_2^2)(\eta_1\eta_3 \\
 &+ \xi_1\xi_3) \cos[\varphi_3] + i(\eta_1(\eta_2^2\eta_3 - \eta_3\xi_2^2 + 2\eta_2\xi_2\xi_3) + \xi_1(2\eta_2\eta_3\xi_2 - \eta_2^2\xi_3 \\
 &+ \xi_2^2\xi_3)) \sin[\varphi_3]), \tag{B.19}
 \end{aligned}$$

$$\begin{aligned}
A_{22}^* A_{12} = & \exp(-2i\varphi_2) ((\eta_2 \xi_1 - \eta_1 \xi_2)(\eta_1 \eta_2 + \xi_1 \xi_2)(\xi_2(-\eta_3 + \xi_3) \\
& + \eta_2(\eta_3 + \xi_3))(\eta_2(\eta_3 - \xi_3) + \xi_2(\eta_3 + \xi_3)) + (\eta_3 \xi_2 - \eta_2 \xi_3)(\eta_2 \eta_3 \\
& + \xi_2 \xi_3)((\xi_1(-\eta_2 + \xi_2) + \eta_1(\eta_2 + \xi_2))(\eta_1(\eta_2 - \xi_2) + \xi_1(\eta_2 + \xi_2)) \cos[2\varphi_3] \\
& - i(\eta_1^2 + \xi_1^2)(\eta_2^2 + \xi_2^2) \sin[2\varphi_3]).
\end{aligned}
\tag{B.20}$$

The normalization equation takes the form:

$$\langle \psi(t = TN) | \psi(t = TN) \rangle = (\eta_1^2 + \xi_1^2)^2 (\eta_2^2 + \xi_2^2)^2 (\eta_3^2 + \xi_3^2).
\tag{B.21}$$

The normalization condition holds

$$\langle \psi(t = TN) | \psi(t = TN) \rangle = 1.
\tag{B.22}$$

### S-III Expectation value of $\langle \sigma_\alpha \rangle$ , $\alpha = x, y, z$

The analytical expressions for expectation values of the spin components used in calculations:

$$\begin{aligned}
\langle \sigma_x \rangle = & |A_{11}|^2 |\eta_1|^2 (\eta_N^* \xi_N + \xi_N^* \eta_N) + A_{11}^* A_{12} |\eta_1|^2 (-|\eta_N|^2 + |\xi_N|^2) \\
& + A_{11}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) (\eta_N^* \xi_N + \eta_N \xi_N^*) + A_{11}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) (-|\eta_N|^2 \\
& + |\xi_N|^2) + A_{12}^* A_{11} |\eta_1|^2 (-|\eta_N|^2 + |\xi_N|^2) + |A_{12}|^2 |\eta_1|^2 (-\eta_N^* \xi_N - \xi_N^* \eta_N) \\
& + A_{12}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) (-|\eta_N|^2 + |\xi_N|^2) + A_{12}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) (-\eta_N^* \xi_N \\
& - \eta_N \xi_N^*) + A_{21}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) (\eta_N^* \xi_N + \xi_N^* \eta_N) + A_{21}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) (-|\eta_N|^2 \\
& + |\xi_N|^2) + |A_{21}|^2 |\xi_1|^2 (\eta_N \xi_N^* + \xi_N \eta_N^*) + A_{21}^* A_{22} |\xi_1|^2 (-|\eta_N|^2 + |\xi_N|^2) \\
& + A_{22}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) (-|\eta_N|^2 + |\xi_N|^2) + A_{22}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) (-\eta_N^* \xi_N \\
& - \eta_N \xi_N^*) + A_{21} A_{22}^* |\xi_1|^2 (-|\eta_N|^2 + |\xi_N|^2) + |A_{22}|^2 |\xi_1|^2 (-\eta_N \xi_N^* - \xi_N \eta_N^*),
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
\langle \sigma_y \rangle = & |A_{11}|^2 |\eta_1|^2 (-i\eta_N^* \xi_N + i\xi_N^* \eta_N) + A_{11}^* A_{12} |\eta_1|^2 (i|\eta_N|^2 + i|\xi_N|^2) \\
& + A_{11}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) (-i\eta_N^* \xi_N + i\eta_N \xi_N^*) + A_{11}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) (i|\eta_N|^2 \\
& + i|\xi_N|^2) + A_{12}^* A_{11} |\eta_1|^2 (-i|\eta_N|^2 - i|\xi_N|^2) + |A_{12}|^2 |\eta_1|^2 (-i\eta_N^* \xi_N + i\xi_N^* \eta_N) \\
& + A_{12}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) (-i|\eta_N|^2 - i|\xi_N|^2) + A_{12}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) (-i\eta_N^* \xi_N + i\eta_N \xi_N^*) \\
& + A_{21}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) (-i\eta_N^* \xi_N + i\xi_N^* \eta_N) + A_{21}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) (i|\eta_N|^2 + i|\xi_N|^2) \\
& + |A_{21}|^2 |\xi_1|^2 (i\eta_N \xi_N^* - i\xi_N \eta_N^*) + A_{21}^* A_{22} |\xi_1|^2 (i|\eta_N|^2 + i|\xi_N|^2) \\
& + A_{22}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) (-i|\eta_N|^2 - i|\xi_N|^2) + A_{22}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) (-i\eta_N^* \xi_N \\
& + i\eta_N \xi_N^*) + A_{21} A_{22}^* |\xi_1|^2 (-i|\eta_N|^2 - i|\xi_N|^2) + |A_{22}|^2 |\xi_1|^2 (i\eta_N \xi_N^* - i\xi_N \eta_N^*),
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
\langle \sigma_z \rangle = & |A_{11}|^2 |\eta_1|^2 (|\eta_N|^2 - |\xi_N|^2) + A_{11}^* A_{12} |\eta_1|^2 (\xi_N \eta_N^* + \xi_N^* \eta_N) \\
& + A_{11}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) (|\eta_N|^2 - |\xi_N|^2) + A_{11}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) (\xi_N \eta_N^* + \xi_N^* \eta_N) \\
& + A_{12}^* A_{11} |\eta_1|^2 (\xi_N \eta_N^* + \xi_N^* \eta_N) + |A_{12}|^2 |\eta_1|^2 (|\xi_N|^2 - |\eta_N|^2) \\
& + A_{12}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) (\xi_N^* \eta_N + \eta_N^* \xi_N) + A_{12}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) (|\xi_N|^2 - |\eta_N|^2) \\
& + A_{21}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) (|\eta_N|^2 - |\xi_N|^2) + A_{21}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) (\eta_N^* \xi_N + \xi_N \eta_N) \\
& + |A_{21}|^2 |\xi_1|^2 (|\eta_N|^2 - |\xi_N|^2) + A_{21}^* A_{22} |\xi_1|^2 (\eta_N^* \xi_N + \xi_N^* \eta_N) \\
& + A_{22}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) (\eta_N \xi_N^* + \eta_N^* \xi_N) + A_{22}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) (|\xi_N|^2 - |\eta_N|^2) \\
& + A_{21} A_{22}^* |\xi_1|^2 (\xi_N^* \eta_N + \eta_N^* \xi_N) + |A_{22}|^2 |\xi_1|^2 (|\xi_N|^2 - |\eta_N|^2).
\end{aligned} \tag{B.25}$$

## S-IV Elements of density matrix for Eq.(3.19)

The elements of the reduced density matrix, analytical expressions used for calculation of the coherence.

$$\begin{aligned}
\rho_{11} = & |A_{11}|^2 |\eta_1|^2 |\eta_N|^2 + A_{11}^* A_{12} |\eta_1|^2 \eta_N^* \xi_N + A_{11}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) |\eta_N|^2 \\
& + A_{11}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N^* \xi_N + A_{12}^* A_{11} |\eta_1|^2 \eta_N \xi_N^* + |A_{12}|^2 |\eta_1|^2 |\xi_N|^2 \\
& + A_{12}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N \xi_N^* + A_{12}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) |\xi_N|^2 \\
& + A_{21}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) |\eta_N|^2 + A_{21}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) \eta_N^* \xi_N \\
& + |A_{21}|^2 |\xi_1|^2 |\eta_N|^2 + A_{21}^* A_{22} |\xi_1|^2 \xi_N \eta_N^* + A_{11} A_{22}^* \exp(-2i\varphi_1) \eta_N \xi_N^* \\
& + A_{12} A_{22}^* \eta_1 \xi_1^* \exp(-2i\varphi_1) |\xi_N|^2 + A_{21} A_{22}^* |\xi_1|^2 \eta_N \xi_N^* + |A_{22}|^2 |\xi_1|^2 |\xi_N|^2,
\end{aligned} \tag{B.26}$$

$$\begin{aligned}
\rho_{12} = & |A_{11}|^2 |\eta_1|^2 \eta_N \xi_N^* + A_{11}^* A_{12} |\eta_1|^2 |\xi_N|^2 + A_{11}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N \xi_N^* \\
& + A_{11}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) |\xi_N|^2 - A_{12}^* A_{11} |\eta_1|^2 |\eta_N|^2 - |A_{12}|^2 |\eta_1|^2 \xi_N \eta_N^* \\
& - A_{12}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) |\eta_N|^2 - A_{12}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N^* \xi_N \\
& + A_{21}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) \xi_N^* \eta_N + A_{21}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) |\xi_N|^2 \\
& + |A_{21}|^2 |\xi_1|^2 \eta_N \xi_N^* + A_{21}^* A_{22} |\xi_1|^2 |\xi_N|^2 - A_{11} A_{22}^* \eta_1 \xi_1^* \exp(-2i\varphi_1) |\eta_N|^2 \\
& - A_{12} A_{22}^* \eta_1 \xi_1^* \exp(-2i\varphi_1) \eta_N^* \xi_N - A_{21} A_{22}^* |\xi_1|^2 |\eta_N|^2 - |A_{22}|^2 |\xi_1|^2 \xi_N \eta_N^*,
\end{aligned} \tag{B.27}$$

$$\begin{aligned}
\rho_{21} = & |A_{11}|^2 |\eta_1|^2 \eta_N^* \xi_N - A_{11}^* A_{12} |\eta_1|^2 |\eta_N|^2 + A_{11}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N^* \xi_N \\
& - A_{11}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) |\eta_N|^2 + A_{12}^* A_{11} |\eta_1|^2 |\xi_N|^2 - |A_{12}|^2 |\eta_1|^2 \xi_N^* \eta_N \\
& + A_{12}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) |\xi_N|^2 - A_{12}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N \xi_N^* \\
& + A_{21}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) \xi_N \eta_N^* - A_{21}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) |\eta_N|^2 \\
& + |A_{21}|^2 |\xi_1|^2 \eta_N^* \xi_N - A_{21}^* A_{22} |\xi_1|^2 |\eta_N|^2 + A_{11} A_{22}^* \eta_1 \xi_1^* \exp(-2i\varphi_1) |\xi_N|^2 \\
& - A_{12} A_{22}^* \eta_1 \xi_1^* \exp(-2i\varphi_1) \eta_N \xi_N^* + A_{21} A_{22}^* |\xi_1|^2 |\xi_N|^2 - |A_{22}|^2 |\xi_1|^2 \xi_N^* \eta_N,
\end{aligned} \tag{B.28}$$

$$\begin{aligned}
\rho_{22} = & |A_{11}|^2 |\eta_1|^2 |\xi_N|^2 - A_{11}^* A_{12} |\eta_1|^2 \eta_N \xi_N^* + A_{11}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) |\xi_N|^2 \\
& - A_{11}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N \xi_N^* - A_{12}^* A_{11} |\eta_1|^2 \eta_N^* \xi_N + |A_{12}|^2 |\eta_1|^2 |\eta_N|^2 \\
& - A_{12}^* A_{21} \eta_1^* \xi_1 \exp(2i\varphi_1) \eta_N^* \xi_N + A_{12}^* A_{22} \eta_1^* \xi_1 \exp(2i\varphi_1) |\eta_N|^2 \\
& + A_{21}^* A_{11} \eta_1 \xi_1^* \exp(-2i\varphi_1) |\xi_N|^2 - A_{21}^* A_{12} \eta_1 \xi_1^* \exp(-2i\varphi_1) \eta_N \xi_N^* \\
& + |A_{21}|^2 |\xi_1|^2 |\xi_N|^2 - A_{21}^* A_{22} |\xi_1|^2 \xi_N^* \eta_N^* - A_{11} A_{22}^* \exp(-2i\varphi_1) \eta_N^* \xi_N \\
& + A_{12} A_{22}^* \eta_1 \xi_1^* \exp(-2i\varphi_1) |\eta_N|^2 - A_{21} A_{22}^* |\xi_1|^2 \eta_N^* \xi_N + |A_{22}|^2 |\xi_1|^2 |\eta_N|^2.
\end{aligned} \tag{B.29}$$

## S-V Normalization constants for eigenstates of Floquet operator for Eq.(3.26) and matrix elements for Eq.(3.28)

The eigenstates of Floquet operator for spin-1 is defined as:

$$\begin{aligned}
|\varphi_n^1\rangle = & -\eta_n|0\rangle + \xi_n|1\rangle + \zeta_n|2\rangle, \text{ and } |\varphi_n^2\rangle = x_n|0\rangle + y_n|1\rangle + z_n|2\rangle \text{ and } |\varphi_n^3\rangle = u_n|0\rangle + v_n|1\rangle + \\
w_n|2\rangle, & \text{ where the normalization constants for above eigenstates is defined as: } \eta_n = -\frac{1}{\sqrt{2}} = \\
-\zeta_n, & \xi_n = 0, x_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2 + c_n^2}}, y_n = \frac{b_n}{\sqrt{a_n^2 + b_n^2 + c_n^2}}, z_n = \frac{c_n}{\sqrt{a_n^2 + b_n^2 + c_n^2}}, u_n = \frac{d_n}{\sqrt{d_n^2 + e_n^2 + f_n^2}}, v_n = \\
\frac{e_n}{\sqrt{d_n^2 + e_n^2 + f_n^2}}, & w_n = \frac{f_n}{\sqrt{d_n^2 + e_n^2 + f_n^2}}, a_n = \frac{2\chi_n^2 + 2\sqrt{2}\chi_n\Omega + \Omega^2}{(\sqrt{2}\chi_n + \Omega)^2}, b_n = \frac{\delta - \sqrt{4\chi_n^2 + \delta^2 + 4\sqrt{2}\chi_n\Omega + 2\Omega^2}}{\sqrt{2}\chi_n + \Omega}, c_n = 1, \\
d_n = \frac{2\chi_n^2 + 2\sqrt{2}\chi_n\Omega + \Omega^2}{(\sqrt{2}\chi_n + \Omega)^2}, & e_n = \frac{\delta + \sqrt{4\chi_n^2 + \delta^2 + 4\sqrt{2}\chi_n\Omega + 2\Omega^2}}{\sqrt{2}\chi_n + \Omega}, f_n = 1.
\end{aligned}$$

Matrix elements of Eq.3.28 are given as.

$$G_{11} = \exp(-i\varphi_n^1)(\eta_n \eta_{n-1} + \xi_n \xi_{n-1} + \eta_n \eta_{n-1}), \tag{B.30}$$

$$G_{12} = \exp(-i\varphi_n^1)(-\eta_n x_{n-1} + \xi_n y_{n-1} + \eta_n z_{n-1}), \tag{B.31}$$

$$G_{13} = \exp(-i\varphi_n^1)(-\eta_n u_{n-1} + \xi_n v_{n-1} + \eta_n w_{n-1}), \quad (\text{B.32})$$

$$G_{21} = \exp(-i\varphi_n^2)(-x_n \eta_{n-1} + y_n \xi_{n-1} + z_n \eta_{n-1}), \quad (\text{B.33})$$

$$G_{22} = \exp(i\varphi_n^2)(x_n x_{n-1} + y_n y_{n-1} + z_n z_{n-1}), \quad (\text{B.34})$$

$$G_{23} = \exp(i\varphi_n^2)(x_n u_{n-1} + y_n v_{n-1} + z_n w_{n-1}), \quad (\text{B.35})$$

$$G_{31} = \exp(-i\varphi_n^3)(-u_n \eta_{n-1} + v_n \xi_{n-1} + w_n \eta_{n-1}), \quad (\text{B.36})$$

$$G_{32} = \exp(i\varphi_n^2)(u_n x_{n-1} + v_n y_{n-1} + w_n z_{n-1}), \quad (\text{B.37})$$

$$G_{33} = \exp(i\varphi_n^2)(u_n u_{n-1} + v_n v_{n-1} + w_n w_{n-1}). \quad (\text{B.38})$$



# Appendix C

## Scrambling and quantum feedback in a nanomechanical system

### S-I Appendix: Calculation of thermally averaged OTOC

$$C_\rho$$

From Eq.(4.16) after re-scaling the total Hamiltonian will be written as:

$$H_{tot} = (\omega_{0R} + \Omega_0)(\sigma_1^z + \sigma_2^z) + \Omega_n(\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) \quad (\text{C.1})$$

or in the matrix form, in the standard basis,

$$\begin{pmatrix} 2(\Omega_0 + \omega_{0R}) & 0 & 0 & 0 \\ 0 & 0 & \Omega_n & 0 \\ 0 & \Omega_n & 0 & 0 \\ 0 & 0 & 0 & -2(\Omega_0 + \omega_{0R}) \end{pmatrix}, \quad (\text{C.2})$$

where  $n = \langle a^\dagger a \rangle$ ,  $\omega_{0R} = \frac{\omega_0}{2n+1}$ , and  $\Omega_0 = \frac{g^2}{\omega_0 - \omega}$ , and  $\Omega_n = \frac{g^2}{(\omega_0 - \omega)(2n+1)}$ . The eigenstates of the above Hamiltonian are

$$|\phi_1\rangle = |0, 0\rangle, \quad (\text{C.3})$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle + |0, 1\rangle), \quad (\text{C.4})$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle - |0, 1\rangle), \quad (\text{C.5})$$

$$|\phi_4\rangle = |1, 1\rangle, \quad (\text{C.6})$$

with corresponding eigenvalues:

$$E_1 = 2(\Omega_0 + \omega_{0R}) \quad (\text{C.7})$$

$$E_2 = \Omega_n \quad (\text{C.8})$$

$$E_3 = -\Omega_n \quad (\text{C.9})$$

$$E_4 = -2(\Omega_0 + \omega_{0R}). \quad (\text{C.10})$$

At a finite temperature, in the equilibrium state the density matrix  $\hat{\rho} = Z^{-1}e^{-\beta H_{tot}}$  of the system in the diagonal basis of the Hamiltonian is

$$\begin{aligned} \hat{\rho} &= Z^{-1}(e^{-\beta E_1}|\phi_1\rangle\langle\phi_1| + e^{-\beta E_2}|\phi_2\rangle\langle\phi_2| + e^{-\beta E_3}|\phi_3\rangle\langle\phi_3| \\ &+ e^{-\beta E_4}|\phi_4\rangle\langle\phi_4|) \end{aligned} \quad (\text{C.10})$$

$$\hat{\rho} = Z^{-1} \begin{pmatrix} e^{-2\beta(\Omega_0 + \omega_{0R})} & 0 & 0 & 0 \\ 0 & e^{-\beta\Omega_n} & 0 & 0 \\ 0 & 0 & e^{\beta\Omega_n} & 0 \\ 0 & 0 & 0 & e^{2\beta(\Omega_0 + \omega_{0R})} \end{pmatrix},$$

$$Z = 2 \cosh \beta 2(\Omega_0 + \omega_{0R}) + 2 \cosh \beta \Omega_n. \quad (\text{C.10})$$

Pauli operators  $\sigma_1^z$  and  $\sigma_2^z$  in the diagonal basis of Hamiltonian are written as:

$$\sigma_1^z = |\phi_1\rangle\langle\phi_1| + |\phi_3\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_3| - |\phi_4\rangle\langle\phi_4| \quad (\text{C.11})$$

$$\sigma_2^z = |\phi_1\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_2| - |\phi_2\rangle\langle\phi_3| - |\phi_4\rangle\langle\phi_4| \quad (\text{C.12})$$

Also the time evolution operators  $\exp(-iH_{tot}t)$  in the diagonal basis can be given as:

$$\begin{aligned} \exp(-iH_{tot}t) &= e^{-iE_1t}|\phi_1\rangle\langle\phi_1| + e^{-iE_2t}|\phi_2\rangle\langle\phi_2| \\ &+ e^{-iE_3t}|\phi_3\rangle\langle\phi_3| + e^{-iE_4t}|\phi_4\rangle\langle\phi_4|, \end{aligned} \quad (\text{C.12})$$

We calculate  $\sigma_1^z(t)$  as

$$\sigma_1^z(t) = e^{iH_{tot}t}\sigma_1^ze^{-iH_{tot}t} = |\phi_1\rangle\langle\phi_1| + e^{-2i\Omega_n t}|\phi_3\rangle\langle\phi_2| + e^{2i\Omega_n t}|\phi_2\rangle\langle\phi_3| - |\phi_4\rangle\langle\phi_4|. \quad (\text{C.12})$$

By successive application of operators in the sequence  $\sigma_1^z(t)\sigma_2^z\sigma_1^z(t)\sigma_2^z$  we get:

$$\sigma_1^z(t)\sigma_2^z\sigma_1^z(t)\sigma_2^z = (|\phi_1\rangle\langle\phi_1| - e^{4i\Omega_n t}|\phi_2\rangle\langle\phi_2| - e^{-4i\Omega_n t}|\phi_3\rangle\langle\phi_3| + |\phi_4\rangle\langle\phi_4|). \quad (\text{C.12})$$

We can calculate  $\rho\sigma_1^z(t)\sigma_2^z\sigma_1^z(t)\sigma_2^z$  as

$$\begin{aligned} \rho\sigma_1^z(t)\sigma_2^z\sigma_1^z(t)\sigma_2^z &= Z^{-1}(e^{-2\beta(\Omega_0+\omega_{0R})}|\phi_1\rangle\langle\phi_1| - e^{-\beta\Omega_n}e^{4i\Omega_n t}|\phi_2\rangle\langle\phi_2| \\ &- e^{\beta\Omega_n}e^{-4i\Omega_n t}|\phi_3\rangle\langle\phi_3| + e^{2\beta(\Omega_0+\omega_{0R})}|\phi_4\rangle\langle\phi_4|). \end{aligned} \quad (\text{C.12})$$

Further, we calculate the thermally averaged OTOC  $C_\rho(t) = 1 - \text{Re}(\text{Tr}\{\rho \sigma_1^z(t) \sigma_2^z \sigma_1^z(t) \sigma_2^z\})$

as

$$C_\rho(t) = 1 - \frac{2 \cosh 2\beta(\Omega_0 + \omega_{0R}) + 2 \cos 4\Omega_n t \cosh \beta\Omega_n}{Z}, \quad (\text{C.12})$$