Chapter 4

Delayed Output Feedback Sliding Mode Control of Uncertain Nonlinear Systems

This chapter introduces an output feedback sliding mode control with an artificial delay. The delayed output information-based sliding surface is outlined. An artificial delay and an exponential decay term are introduced in the sliding surface, which reflects these additional terms (artificial delay and exponential decay term) in the reduced system dynamics. With the aid of Lyapunov-Krasovskii, functional, locally input to state stability (ISS) for the reduced system dynamics and the closed-loop system are presented. Finally, the simulation results of the TORA (Translation Oscillator with Rotating Actuator) system are presented to validate the effectiveness of the proposed technique.

4.1 Introduction

State feedback control is simple and straightforward. But in practice, complete state information is not always available due to the expensive cost and unavailability of the sensors. To overcome such difficulty, many researchers stated output feedback stabilization in which one type is output feedback sliding mode control [21,30]. A series of papers in the area of output feedback sliding mode control are presented here. In [31], the authors reported an adaptive output feedback variable structure control for the nonlinear multi-input multi-output (MIMO) systems with matched and unmatched disturbances. Also, it relaxed the advanced upper bound knowledge of the disturbances and disturbances estimation error due to the employment of the adaptive mechanism.

The static output feedback sliding mode control for uncertain networked control systems (NCSs) under round-robin protocol and singularly perturbed systems (SPSs) under fast sampling is presented in [32], and [33] respectively. The event-triggered-based sliding mode control for the discrete-time Markov jump systems (MJSs) and the second-order nonlinear multi-agent systems is investigated in [34] and [35] respectively. The closed-loop system stability is confirmed by utilizing the Lyapunov functions.

We used sliding mode control due to its inherent properties like robustness, finite-time convergence, flexibility in gain tuning, and reduced dynamics. Edwards and Spurgeon have given the basic idea of the sliding mode control using only output information in [24]. This theory is further exploited in [12] by utilizing a static output feedback sliding mode control with the artificial delay for the uncertain linear plants. But the given stability analysis in terms of LMI is complex and not straightforward.

Emilia and the group presented a series of papers on delayed static output feedback for linear systems with less conservative results employing LMI [6,38]. In [39], the authors proposed a linear sliding surface-based sliding mode controller using output information for the descriptor systems. The discrete sliding mode control design for the time-delay systems based on the multi-rate output feedback is presented in [40]. Global decentralized stabilization of a class of interconnected time-varying delay systems using static output feedback sliding mode control is addressed in [41].

The motivation of the chapter is that the systems such as chain of integrators, oscillators, and inverted pendulum are not stabilized through only static output feedback but can be stabilized by applying small time-delay with static output feedback. In [14,17, 36], the authors have presented stabilization by utilizing delayed static output feedback with a known slight delay, but the robust stability analysis is missing.

The contribution of the stated approach is as follows:

(i) The delayed sliding surface with an extra exponential decay term is designed and guaranteed its finite-time convergence.

(ii) A delayed output feedback sliding mode control is constructed.

(iii) Sufficient conditions have been provided for the stability analysis of the closed-loop system and ensured its asymptotic stability.

(iv) Robustness against matched perturbation is provided.

(v) Efficient reduction of chattering phenomenon.

The proposed control method can be applied to systems where the output feedback gain F cannot stabilize the matrix $A_{11} - A_{12}FC_1$ (under the assumption that all the equations derived in this chapter are correct) due to the employment of delay in the sliding surface function as well as control design technique.

The organization of the chapter is as follows. Section 4.2 introduces the problem formulation. The design of the delayed sliding surface is reported in Section 4.3. Detailed stability analysis is described in Section 4.4. Section 4.5 presents the simulation results of the TORA system. Lastly, the chapter summarised in Section 4.6.

Definition 1 *A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be class $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$.*

Definition 2 *A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be class $\mathcal{KL}$ function if $\beta(\cdot, s)$ is of class $\mathcal{K}$ for any fixed $s > 0$ and $\beta(r, \cdot)$ is monotonically decreasing to zero for any fixed $r > 0$.*

Definition 3 *([42]) The system $\dot{z}(t) = Az(t) + A_h z(t) + Bw(t)$ is said to be locally ISS under external disturbances $(w(t))$ if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that, for any initial time t_0, any initial state $\|\phi\|_W \leq r_0$ and any measurable, locally essential bounded disturbance signal $w(t)$, the solution $z(t, t_0, \phi)$ exists for all $t > t_0$ and moreover it satisfies*

$$\|z(t, t_0, \phi)\| \leq \beta(\|\phi\|_W, t - t_0) + \gamma(|w_{|t_0,t|}|_\infty). \tag{4.1}$$

4.2 Problem Formulation

Consider a dynamical system as follows

$$\begin{cases} \dot{\Theta}(t) & = F(\Theta, \eta) + G(\Theta, \eta)u(t) \\ y(t) & = h(\Theta), \end{cases} \tag{4.2}$$

where $\Theta(t) \in \mathbb{D} \subset \mathbb{R}^n$ are the system states. $F : \mathbb{D} \times \mathbb{R} \longrightarrow \mathbb{R}^n$ and $G : \mathbb{D} \times \mathbb{R} \longrightarrow \mathbb{R}^n$ are locally Lipschitz and known continuous function in t, $y(t) \in \mathbb{R}^P$ is the output, $u(t) \in \mathbb{R}$ is the control input, $h : \mathbb{D} \longrightarrow \mathbb{R}^P$ is the output function and $\eta(t) \in \mathbb{R}$ describes the external matched disturbances.

Employing diffeomorphism [21, 22], one can transform the system (4.2) into the resulting standard form.

$$\begin{cases} \dot{z}_i(t) & = z_{i+1}(t), \ i = 1, \cdots, n-1 \\ \dot{z}_n(t) & = f(\Theta, z) + g(\Theta, z)u(t) + \rho(t), \end{cases} \tag{4.3}$$

where $f : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{R}$ and $g : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{R}$ are known smooth functions, and $\rho(t)$ denotes the external matched disturbances, which is supposed to be bounded and Lipschitz in t, i.e. $|\rho(t)| < \rho_1, \ |\dot{\rho}(t)| < \rho_0, \ \forall t \geq 0$.

Applying $u(t) := \dfrac{1}{g(\Theta, z)}(\vartheta(t) - f(\Theta, z))$ to system (4.3) yields the following perturbed chain of integrators

$$\begin{cases} \dot{z}_i(t) & = z_{i+1}(t), \ i = 1, \cdots, n-1 \\ \dot{z}_n(t) & = \vartheta(t) + \rho(t), \end{cases} \tag{4.4}$$

where $\vartheta(t)$ is the virtual control input to be designed. The stabilization problem of (4.3) about the origin is equivalent to stabilization problem of uncertain system (4.4).

Here, the primary purpose is to design a control input such that an uncertain dynamical system (4.4) is stable about the origin. To achieve that, we are employing a robust sliding mode control technique.

One can represent the system (4.4) in the regular form of linear time-invariant system

$$\dot{z}(t) = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{\text{A}} z(t) \ + \ \underbrace{\begin{bmatrix} 0_{(n-1)\times 1} \\ B_2 \end{bmatrix}}_{\text{B}} (\vartheta(t) + \rho(t)), \qquad y = \underbrace{\begin{bmatrix} 0_{p\times(n-p)} & C_2 \end{bmatrix}}_{\text{C}} z$$

where $z(t) \in \mathbb{R}^n, \vartheta(t) \in \mathbb{R}, y(t) \in \mathbb{R}^p$ with $1 < p < n$ belongs to the state, control, and output variables respectively. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{p\times n}$ are assumed to be known with $A_{11} \in \mathbb{R}^{(n-1)\times(n-1)}$, $B_2 \in \mathbb{R}$ and $C_2 \in \mathbb{R}^{p\times p}$. The matrix C_2 is a nonsingular matrix.

It has been assumed that the pair (A, B) and (A, C) is controllable and observable, respectively, the input and the output matrices B and C are of full rank. Moreover, it also assumed that $rank(CB) = 1$. Now, let's recall the concept of static output sliding mode control reported in [12, 24].

Consider a sliding surface $S(t)$ as

$$S(t) = \begin{bmatrix} F & I_m \end{bmatrix} C_2^{-1} y(t), \tag{4.5}$$

where $F \in \mathbb{R}^{1\times(p-1)}$ is the sliding surface parameter. The ideal sliding motion depends the choice of F. From (4.5) one can write

$$\begin{aligned} S(t) &= \begin{bmatrix} F & I \end{bmatrix} C_2^{-1} \begin{bmatrix} 0_{p\times(n-p)} & C_2 \end{bmatrix} z(t) = \begin{bmatrix} F & I \end{bmatrix} \begin{bmatrix} 0_{p\times(n-p)} & I_{p\times p} \end{bmatrix} z(t) \\ &= \begin{bmatrix} F & I \end{bmatrix} \begin{bmatrix} 0_{(p-1)\times(n-p)} & I_{p-1} & 0 \\ 0_{m\times(n-p)} & 0 & I \end{bmatrix} z(t) = \begin{bmatrix} F & I \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix} z(t) \\ &= \begin{bmatrix} FC_1 & I \end{bmatrix} z(t) \end{aligned} \tag{4.6}$$

where $C_1 = \begin{bmatrix} 0_{(p-1)\times(n-p)} & I_{(p-1)} \end{bmatrix}$.

It has been observed from (4.6) that the sliding surface can be rewritten as

$$S(t) = z_b(t) + FC_1 z_a(t), \tag{4.7}$$

where $S(t) \in \mathbb{R}$, $z(t) = \begin{bmatrix} z_a(t) & z_b(t) \end{bmatrix}^T \in \mathbb{R}^n$, $z_a(t) \in \mathbb{R}^{n-1}$ and $z_b(t) \in \mathbb{R}$.

The closed-loop dynamics can be expressed as

$$\begin{cases} \dot{z}_a(t) &= (A_{11} - A_{12}FC_1)z_a(t) + A_{12}S(t) \\ \dot{S}(t) &= (A_{21} + FC_1A_{11})z_a(t) - (A_{22} + FC_1A_{12})FC_1z_a(t) + (A_{22} + FC_1A_{12})S(t) \\ & \qquad + B_2\vartheta(t) + B_2\rho(t). \end{cases} \tag{4.8}$$

Remark 4 *It is noted that, from the definition of the sliding surface (4.7) one can compute the information of $z_b(t)$ from the knowledge of the output $C_1z_a(t)$ and the sliding surface $S(t)$.*

Now, define the control input $\vartheta(t)$ such that the ideal sliding motion occurs in finite-time.

$$\vartheta(t) = -(B_2)^{-1}\left\{-\nu(t) - (A_{22} + FC_1A_{12})FC_1z_a(t) + (A_{22} + FC_1A_{12})S(t)\right\}$$

where $\nu(t) := -\kappa \text{sign}(S)$ with $\kappa > \|B_2\|\rho_1$ a positive scalar.

The closed-loop system with the control input $\nu(t)$ can be written as

$$\begin{cases} \dot{z}_a(t) & = (A_{11} - A_{12}FC_1)z_a(t) + A_{12}S(t) \\ \dot{S}(t) & = (A_{21} + FC_1A_{11})z_a(t) + \nu(t) + B_2\rho(t) \end{cases} \tag{4.9}$$

When ideal sliding motion $S(t) = 0$ occurs, then the dynamics (4.9) becomes

$$\dot{z}_a(t) = (A_{11} - A_{12}FC_1)z_a(t)$$

where the matrix $(A_{11} - A_{12}FC_1)$ is Hurwitz. Therefore, the overall problem is now transformed into the design of output feedback control gain F. Moreover, for the design of output feedback gain F, various standard approaches are available in the literature [44].

The system (A_{11}, A_{12}, C_1) is assumed to be output feedback stabilizable if there exists feedback gain matrix F such that matrix $(A_{11} - A_{12}FC_1)$ is Hurwitz. It is already reported in [24] that the necessary condition for (A_{11}, A_{12}, C_1) to be stabilizable is that the invariant zeros of (A, B, C) lies in the left half-plane. Although, design of output feedback control gain F such that the matrix $(A_{11} - A_{12}FC_1)$ becomes Hurwitz is not always true.

To understand the theory more clearly, we have presented two numerical examples.

Example 1: Consider the following matrices

$$A_{11} = \begin{bmatrix} 0 & -2 \\ 1 & 0.1 \end{bmatrix}, A_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Now, substituting the matrices A_{11}, A_{12} and C_1 into (4.8) and designing F such that $\begin{bmatrix} 0 & -2-F \\ 1 & 0.1 \end{bmatrix}$ becomes Hurwitz is not always possible.

Example 2: Consider the triple integrator system with reduced dynamics matrices

$$A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Again, substitute the matrices A_{11}, A_{12} and C_1 into (4.8) and design F such that $\begin{bmatrix} 0 & 1 \\ 0 & -F \end{bmatrix}$ becomes Hurwitz is not always possible.

Therefore, from the given examples, we have shown that the output feedback gain F cannot be found to stabilize the matrix $A_{11} - A_{12}FC_1$. From here, one can conclude that the output feedback alone is not able to stabilize the systems.

To overcome these drawbacks, many authors selected a compensator to stabilize such types of plants, which increases the order of the controller, and also, the analysis gets complexed [45, 46]. To cope with such circumstances, we are employing artificial delay in the static output feedback such that the system is stabilizable without the aid of a compensator.

4.3 Design of Sliding Surface

Consider a sliding surface of the form (4.5) with additional artificial delay term and exponential decaying term. The idea and advantage of employing the artificial delay in the sliding surface are to stabilize the class of systems, which cannot be stabilized with only output feedback. Besides, the introduction of the exponential delay term provides the starting of sliding mode at $t = 0$. Hence, $S(t) = 0$ from the initial point onwards, and there is no reaching phase. Consequently, reduced dynamics results with two more terms, artificial delay term and exponential decaying term.

$$S(t) = \begin{bmatrix} F & I \end{bmatrix} C_2^{-1} y(t) + \begin{bmatrix} F_h & 0 \end{bmatrix} C_2^{-1} y(t-h) - w_2(t), \tag{4.10}$$

where

$$w_2(t) = e^{-\varrho(t-t_0)} \left\{ \begin{bmatrix} F & I \end{bmatrix} C_2^{-1} y(t_0) + \begin{bmatrix} F_h & 0 \end{bmatrix} C_2^{-1} y(t_0 - h) \right\}.$$

The sliding surface parameters are $F \in \mathbb{R}^{1\times(p-1)}$, and $F_h \in \mathbb{R}^{1\times(p-1)}$. The scalar $\varrho > 0$ is the convergence rate of the sliding surface, $y(t_0)$ and $y(t_0 - h)$ are initial values of the output state and delayed output state respectively. Here, h is a known artificial delay.

One can write (4.10) as follows

$$
\begin{aligned}
S(t) &= \begin{bmatrix} F & I \end{bmatrix} C_2^{-1} \begin{bmatrix} 0_{p\times(n-p)} & C_2 \end{bmatrix} z(t) + \begin{bmatrix} F_h & 0 \end{bmatrix} C_2^{-1} \begin{bmatrix} 0_{p\times(n-p)} & C_2 \end{bmatrix} z(t-h) - w_2(t) \\
&= \begin{bmatrix} F & I \end{bmatrix} \begin{bmatrix} 0_{p\times(n-p)} & I_{p\times p} \end{bmatrix} z(t) + \begin{bmatrix} F_h & 0 \end{bmatrix} \begin{bmatrix} 0_{p\times(n-p)} & I_{p\times p} \end{bmatrix} z(t-h) - w_2(t) \\
&= \begin{bmatrix} F & I \end{bmatrix} \begin{bmatrix} 0_{(p-1)\times(n-p)} & I_{p-1} & 0 \\ 0_{1\times(n-p)} & 0 & I \end{bmatrix} z(t) + \begin{bmatrix} F_h & 0 \end{bmatrix} \begin{bmatrix} 0_{(p-1)\times(n-p)} & I_{p-1} & 0 \\ 0_{1\times(n-p)} & 0 & I \end{bmatrix} z(t-h) \\
&\qquad - w_2(t) \\
&= \begin{bmatrix} F & I \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix} z(t) + \begin{bmatrix} F_h & 0 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix} z(t-h) - w_2(t) \\
&= \begin{bmatrix} FC_1 & I \end{bmatrix} z(t) + \begin{bmatrix} F_hC_1 & 0 \end{bmatrix} z(t-h) - w_2(t),
\end{aligned}
$$

where $C_1 = \begin{bmatrix} 0_{(p-1)\times(n-p)} & I_{(p-1)} \end{bmatrix}$ and after simplification

$$
w_2(t) = e^{-\varrho(t-t_0)} \left\{ z_b(t_0) + FC_1 z_a(t_0) + F_hC_1 z_a(t_0-h) \right\}.
$$

Therefore, it follows from the above expression that the sliding surface can be rewritten as

$$
S(t) = z_b(t) + FC_1 z_a(t) + F_hC_1 z_a(t-h) - w_2(t). \tag{4.11}
$$

Now, the closed-loop dynamics with additional exponential decay and delay term as

$$
\begin{aligned}
\dot{z}_a(t) &= (A_{11} - A_{12}FC_1) z_a(t) - A_{12}F_hC_1 z_a(t-h) + A_{12}(S + w_2(t)) \\
\dot{S}(t) &= (A_{21} + FC_1A_{11}) z_a(t) + F_hC_1A_{11} z_a(t-h) + (A_{22} + FC_1A_{12})S + F_hC_1A_{12}S(t-h) \\
&\quad + B_2(\vartheta(t) + \rho(t)) - (A_{22} + FC_1A_{12})FC_1 z_a(t) - F_hC_1A_{12}F_hC_1 z_a(t-2h) \\
&\quad - (FC_1A_{12}F_h + A_{22}F_h + F_hC_1A_{12}F)C_1 z_a(t-h) + (A22 + \varrho + FC_1A_{12}) w_2(t) \\
&\quad + F_hC_1A_{12} w_2(t-h).
\end{aligned} \tag{4.12}
$$

Remark 5 *It is noted that, from the definition of the sliding surface (4.11) one can compute the information of $z_b(t)$ from the knowledge of the output $C_1z_a(t)$, $C_1z_a(t-h)$, sliding surface $S(t)$ and the decaying exponential term $w_2(t)$.*

Remark 6 *It is important to note that delay, h is known and artificially inserted in the sliding surface to enhance the stability of the closed-loop system. Also, an exponential term is added to the sliding surface to remove the reaching phase.*

Remark 7 *The sliding mode dynamics given by (4.12) with $S(t) = 0$ is a type of retarded delay system, where the delay is known and can be utilized to stabilize the reduced dynamics.*

In the second equation of (4.12), there are many known terms ending with $S(t), C_1 z_a(t), C_1 z_a(t-h)$ and $C_1 z_a(t-2h)$. Therefore, one can design the following static output feedback control input

$$\begin{aligned}\vartheta(t) = -(B_2)^{-1}\{&(A_{22} + FC_1A_{12})S(t) + F_hC_1A_{12}S(t-h) - (A_{22} + FC_1A_{12})F(C_1z_a(t))\\ &-F_hC_1A_{12}F_h(C_1z_a(t-2h)) - (FC_1A_{12}F_h + A_{22}F_h + F_hC_1A_{12}F)(C_1z_a(t-h))\\ &+\Pi(t) + (A22 + \varrho + FC_1A_{12})w_2(t) + F_hC_1A_{12}w_2(t-h)\}\,. \end{aligned} \tag{4.13}$$

Therefore, the closed-loop system dynamics can be described as

$$\begin{cases} \dot{z}_a(t) &= \bar{A}z_a(t) + \bar{A}_h z_a(t-h) + \bar{B}w_2(t) + \bar{B}S(t) \\ \dot{S}(t) &= \bar{E}z_a(t) + \bar{E}_h z_a(t-h) - \Pi(t) + B_2\rho(t) \end{cases} \tag{4.14}$$

where $\bar{A} := (A_{11} - A_{12}FC_1),\ \bar{A}_h := -A_{12}F_hC_1,\ A_{12} = \bar{B}, \bar{E} := (A_{21} + FC_1A_{11}),\ \bar{E}_h := F_hC_1A_{11}$ and the control input $\Pi(t)$ is defined as

$$\Pi(t) = K_1|S(t)|^{\frac{1}{2}}\text{sign}(S) + \int_0^t K_2\text{sign}(S(\omega))d\omega \tag{4.15}$$

where $K_2 > \|B_2\|\rho_0 + \mu$, with μ a positive scalar and $K_1 > (\|\bar{E}\| + \|\bar{E}_h\|)|z_a(t)|_h + K_2$.

To compensate the disturbances and reduce the chattering, we are employing a continuous control named as super-twisting control.

Remark 8 *The signum function holds multivariable discontinuity at $S(t) = 0$. Although the first term of (4.15) is equal to zero at the point of discontinuity. Therefore, one can tell the first term is continuous. The second term is the integration of the signum function since discontinuity lies at a single point. However, its integration is continuous. Subsequently, the addition of two continuous functions is always continuous. For practical systems, we require continuous control and less influence of chattering. For this objective, we employed the super-twisting controller (4.15).*

Remark 9 *The super-twisting control is continuous, but during the implementation, $S(t)$ is not precisely zero; it has some minimal finite value. Due to the first term of the super-twisting control chattering phenomenon happens during the practical implementation, which is already reported by several researchers.*

4.4 Main Results

4.4.1 Locally ISS Stability of the Reduced System

Consider the linear system with fixed delay

$$\dot{z}_a(t) = \bar{A}z_a(t) + \bar{A}_h z_a(t-h) + \bar{B}w_2(t) \tag{4.16}$$

where $z_a(t) \in \mathbb{R}^{n-1}$ is the state. System (4.16) describes the reduced system dynamics with $S = 0$ in (4.14). Thus, the sliding surface (4.10) makes the sliding mode dynamics stabilizable by the delayed term $\bar{A}_h z_a(t-h)$. The term $w_2(t)$ in the system (4.16) is acting as external disturbances. Here the assumption is $w_2(t)$ belongs to the set $\mathcal{W} = \left\{w_2(t) | w_2^T(t)w_2(t) \leq \bar{w}_2^2(t)\right\}$. The Locally ISS stability is provided to validate the stability of the reduced system dynamics despite external disturbances.

Consider the resulting theorem based on the Lyapunov-Krasovskii functional $L_{V_a}(\delta) = \{z_a(t) | V(z_{1_t}, t) \leq \delta\}$

Theorem 1 *The system (4.16) is said to locally ISS under external disturbances, for given scalars* $h > 0$, $\delta > 0$ *and* $\alpha_1 > 0$, *if there exists a scalar* $\alpha_2 > 0$ *and matrices* $P > 0$, $Q_1 > 0$ *and* $R > 0$ *with appropriate dimensions, such that the resulting LMIs satisfy*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & 6e^{-\alpha_1 h}R & \Psi_{14} & \Psi_{15} \\ * & \Psi_{22} & 6e^{-\alpha_1 h}R & \Psi_{24} & T_3\bar{B} \\ * & * & -12e^{-\alpha_1 h}R & 0 & 0 \\ * & * & * & \Psi_{44} & T_2\bar{B} \\ * & * & * & * & -\alpha_2 I \end{bmatrix} < 0 \tag{4.17}$$

$$\alpha_2 \bar{w}_2^2(t) - \alpha_1 \delta \leq 0 \tag{4.18}$$

where

$$\Psi_{11} := \bar{A}^T P + P\bar{A} + Q_1 + \alpha_1 P - 4e^{-\alpha_1 h}R + Sym(T_1\bar{A})$$

$$\Psi_{12} := -2e^{-\alpha_1 h}R + P\bar{A}_h + T_1\bar{A}_h + \bar{A}^T T_3^T$$

$$\Psi_{14} := -T_1 + \bar{A}^T T_2^T, \quad \Psi_{15} := P\bar{B} + T_1\bar{B}$$

$$\Psi_{22} := -e^{-\alpha_1 h}Q_1 - 4e^{-\alpha_1 h}R + Sym(T_3\bar{A}_h)$$

$$\Psi_{24} := \bar{A}_h^T T_2^T - T_3, \quad \Psi_{44} := h^2 R - Sym(T_2).$$

Proof: Conisder the Lyapunov-Krasovskii functional $V_a(z_{1_t}, t)$, for simplicity we write $V_a = V_a(z_{1_t}, t)$

$$V_a = V_1 + V_2 + V_3, \tag{4.19}$$

where

$$V_1 := z_a^T(t) P z_a(t)$$

$$V_2 := \int_{t-h}^{t} e^{-\alpha_1(t-s)} z_a^T(t)(s) Q_1 z_a(s) ds$$

$$V_3 := h \int_{-h}^{0} \int_{t+\varphi}^{t} e^{-\alpha_1(t-s)} \dot{z}_a^T(s) R \dot{z}_a(s) ds d\varphi.$$

After differentiating the V_1, V_2 and V_3 with respect to time, we obtain

$$\begin{aligned} \dot{V}_1 = {} & z_a^T(t)(P\bar{A} + \bar{A}^T P) z_a(t) + z_a^T(t)(P\bar{A}_h) z_a(t-h) + z_a^T(t)(P\bar{B}) w_2(t) \\ & + w_2^T(t)(\bar{B}^T P) z_a(t) + z_a^T(t-h)(\bar{A}_h^T P) z_a(t) \end{aligned} \tag{4.20}$$

$$\begin{aligned} \dot{V}_2 = {} & z_a^T(t) Q_1 z_a(t) - e^{-\alpha_1 h} z_a^T(t-h) Q_1 z_a(t-h) \\ & - \alpha_1 \int_{t-h}^{t} e^{-\alpha_1(t-s)} z_a^T(s) Q_1 z_a(s) ds \end{aligned} \tag{4.21}$$

$$\begin{aligned} \dot{V}_3 = {} & h^2 \dot{z}_a^T(t) R \dot{z}_a(t) - h \int\limits_{t-h}^{t} e^{-\alpha_1(t-s)} \dot{z}_a^T(s) R \dot{z}_a(s) ds \\ & - \alpha_1 h \int_{-h}^{0} \int_{t+\varphi}^{t} e^{-\alpha_1(t-s)} \dot{z}_a^T(s) R \dot{z}_a(s) ds d\varphi. \end{aligned} \tag{4.22}$$

Applying Lemma 5 to the first integral term in (4.22)

$$-h \int\limits_{t-h}^{t} e^{-\alpha_1(t-s)} \dot{z}_a^T(s) R \dot{z}_a(s) ds \leq -e^{-\alpha_1 h} \Sigma^T R \Sigma - 3 e^{-\alpha_1 h} \Upsilon^T R \Upsilon$$

where

$\Sigma := z_a(t) - z_a(t-h), \Upsilon := \left(z_a(t) + z_a(t-h) - \frac{2}{h} \int\limits_{t-h}^{t} z_a(s) ds \right).$

Therefore, $\dot{V}_3$ becomes

$$\begin{aligned}
\dot{V}_3 \leq\ & h^2 \dot{z}_a^T(t) R \dot{z}_a(t) - e^{-\alpha_1 h}\left\{ z_a^T(t) R z_a(t) - z_a^T(t) R z_a(t-h) - z_a^T(t-h) R z_a(t) \right\} \\
& - 3e^{-\alpha_1 h}\Bigg\{ z_a^T(t) R z_a(t) + z_a^T(t) R z_a(t-h) - z_a^T(t) R \left(\frac{2}{h} \int\limits_{t-h}^{t} z_a(s) ds \right) \\
& + z_a^T(t-h) R z_a(t) + z_a^T(t-h) R z_a(t-h) - z_a^T(t-h) R \left(\frac{2}{h} \int\limits_{t-h}^{t} z_a(s) ds \right) \\
& - \left(\frac{2}{h} \int\limits_{t-h}^{t} z_a(s) ds \right)^T R z_a(t) - \left(\frac{2}{h} \int\limits_{t-h}^{t} z_a(s) ds \right)^T R z_a(t-h) \\
& + \left(\frac{2}{h} \int\limits_{t-h}^{t} z_a(s) ds \right)^T R \left(\frac{2}{h} \int\limits_{t-h}^{t} z_a(s) ds \right) \Bigg\} \\
& - e^{-\alpha_1 h} z_a^T(t-h) R z_a(t-h) - \alpha_1 h \int_{-h}^{0} \int_{t+\varphi}^{t} e^{-\alpha_1 (t-s)} \dot{z}_a^T(s) R \dot{z}_a(s) ds d\varphi.
\end{aligned} \tag{4.23}$$

The following equality satisfies for every matrices T_1, T_2 and T_3 with appropriate dimensions:

$$2\left\{ z_a^T(t) T_1 + \dot{z}_a^T(t) T_2 + z_a^T(t-h) T_3 \right\} \left\{ -\dot{z}_a(t) + \bar{A} z_a(t) + \bar{A}_h z_a(t-h) + \bar{B} w_2(t) \right\} = 0. \tag{4.24}$$

It follows from (4.19)-(4.24) that

$$\begin{aligned}
& \dot{V}_a + \alpha_1 V_a - \alpha_2 w_2^T(t) w_2(t) \leq \\
& \qquad z_a^T(t)(P\bar{A} + \bar{A}^T P) z_a(t) + z_a^T(t)(P\bar{A}_h) z_a(t-h) \\
& \qquad + z_a^T(t)(P\bar{B}) w_2(t) + w_2^T(t)(\bar{B}^T P) z_a(t) + z_a^T(t-h)(\bar{A}_h^T P) z_a(t) + z_a^T(t) Q_1 z_a(t) \\
& \qquad - e^{-\alpha_1 h}\left\{ z_a^T(t-h) Q_1 z_a(t-h) + z_a^T(t) R z_a(t) - z_a^T(t) R z_a(t-h) \right. \\
& \qquad \left. - z_a^T(t-h) R z_a(t) + z_a^T(t-h) R z_a(t-h) \right\} + h^2 \dot{z}_a^T(t) R \dot{z}_a(t)
\end{aligned}$$

$$
\begin{aligned}
&-3e^{-\alpha_1 h}\Bigg\{z_a^T(t)Rz_a(t)+z_a^T(t)Rz_a(t-h)-z_a^T(t)R\left(\frac{2}{h}\int\limits_{t-h}^{t}z_a(s)ds\right)\\
&+z_a^T(t-h)Rz_a(t)+z_a^T(t-h)Rz_a(t-h)-z_a^T(t-h)R\ \left(\frac{2}{h}\int\limits_{t-h}^{t}z_a(s)ds\right)\\
&-\left(\frac{2}{h}\int\limits_{t-h}^{t}z_a(s)ds\right)^T Rz_a(t)-\left(\frac{2}{h}\int\limits_{t-h}^{t}z_a(s)ds\right)^T Rz_a(t-h)\\
&+\left(\frac{2}{h}\int\limits_{t-h}^{t}z_a(s)ds\right)^T R\left(\frac{2}{h}\int\limits_{t-h}^{t}z_a(s)ds\right)\Bigg\}+\alpha_1 z_a^T(t)Pz_a(t)-\alpha_2 w_2^T(t)w_2(t)\\
&+2\left\{z_a^T(t)T_1+\dot{z}_a^T(t)T_2+z_a^T(t-h)T_3\right\}\left\{-\dot{z}_a(t)+\bar{A}z_a(t)+\bar{A}_h z_a(t-h)+\bar{B}w_2(t)\right\}.
\end{aligned}
\tag{4.25}
$$

Therefore, one can obtain LMI (4.26) from the above derivation.

$$
\begin{aligned}
&\dot{V}_a+\alpha_1 V_a-\alpha_2 w_2^T(t)w_2(t)\\
&\le\begin{bmatrix} z_a(t)\\ z_a(t-h)\\ \frac{1}{h}\int\limits_{t-h}^{t}z_a(s)ds\\ \dot{z}_a(t)\\ w_2(t)\end{bmatrix}^T
\begin{bmatrix}\Psi_{11} & \Psi_{12} & 6e^{-\alpha_1 h}R & \Psi_{14} & \Psi_{15}\\
* & \Psi_{22} & 6e^{-\alpha_1 h}R & \Psi_{24} & T_3\bar{B}\\
* & * & -12e^{-\alpha_1 h}R & 0 & 0\\
* & * & * & \Psi_{44} & T_2\bar{B}\\
* & * & * & * & -\alpha_2 I\end{bmatrix}
\begin{bmatrix} z_a(t)\\ z_a(t-h)\\ \frac{1}{h}\int\limits_{t-h}^{t}z_a(s)ds\\ \dot{z}_a(t)\\ w_2(t)\end{bmatrix}.
\end{aligned}
\tag{4.26}
$$

Thereafter, inequality (4.25) along with (4.18) provides

$$\dot{V}_a+\alpha_1(V_a-\delta)+\alpha_2(\bar{w}_2^2(t)-w_2^T(t)w_2(t))<0, \tag{4.27}$$

which ensures that $\dot{V}_a<0,\ \forall\phi\notin L_{V_a}(\delta)$ and $\forall w_2(t)\in\mathcal{W}$.

It can be conclude from the above discussion that

$$\dot{V}\le\delta,\ \forall\phi\in L_{V_a}(\delta)\ \text{ and }\ \forall w_2(t)\in\mathcal{W}. \tag{4.28}$$

From (4.19), we have Lyapunov-Krasovskii functional

$$z_a^T(t)Pz_a(t)\le V_a(z_{1_t},t), \tag{4.29}$$

and

$$\begin{aligned} V_a(\phi,0) &\le (\lambda_{\max}(P) + h\lambda_{\max}(Q_1))\,\|\phi\|_s^2 + \frac{1}{2}h^3\lambda_{\max}(R)\|\phi\|_s^2 \\ &\le \kappa_1\|\phi\|_W^2 \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} \kappa_1 &= \lambda_{\max}(P) + h\lambda_{\max}(Q_1)\|\phi\|_s^2 + \frac{1}{2}h^3\lambda_{\max}(R), \\ \|\phi\|_s &= \sup_{-h\le\theta\le 0}\|\phi(\theta)\|,\ \|\dot{\phi}\|_s = \sup_{-h\le\theta\le 0}\|\dot{\phi}(\theta)\| \\ \|\phi\|_W &= \sup_{-h\le\theta\le 0}\left\{\|\phi\|_s, \|\dot{\phi}\|_s\right\}. \end{aligned}$$

Consider $\mathcal{B}\left(0,\sqrt{\dfrac{\delta}{\kappa_1}}\right) = \left\{\phi : \|\phi\|_W \le \sqrt{\dfrac{\delta}{\kappa_1}}\right\}$. From (4.30) one can show that $V_a(\phi,0) \le \delta \implies \phi \in L_{V_a}(\delta),\ \forall\phi \in \mathcal{B}\left(0,\sqrt{\dfrac{\delta}{\kappa_1}}\right)$. This confirms that $\mathcal{B}\left(0,\sqrt{\dfrac{\delta}{\kappa_1}}\right) \subset L_{V_a}(\delta)$. Using inequalities (4.28) and (4.29) we get

$$z_a(t) \in \mathcal{E}(P,\delta),\ \forall\phi \in \mathcal{B}\left(0,\sqrt{\frac{\delta}{\kappa_1}}\right) \quad \text{and} \quad w_2(t) \in \mathcal{W}. \tag{4.31}$$

This validates, for all initial function $\phi \in \mathcal{B}\left(0,\sqrt{\dfrac{\delta}{\kappa_1}}\right)$, and for all disrturbances signal $w_2(t) \in \mathcal{W}$, the state trajectories remain within $\mathcal{E}(P,\delta)$.

Lastly, we assure that system (4.16) is locally ISS. Define

$$\dot{V}_a(z_{1_t},t) + \alpha_1 V_a(z_{1_t},t) = U(z_{1_t},t). \tag{4.32}$$

With the aid of inequality (4.27), it is obvious that $U(z_{1_t},t) \le \alpha_2\bar{w}_2^2(t)$. Without loss of generality, we assume that $t_0 = 0$. Motivated by the results of [47], we obtain

$$\int_0^t d(e^{\alpha_1 t}V_a(z_{1_t},t)) = \int_0^t (e^{\alpha_1 s})U(z_{1_s},s)ds, \tag{4.33}$$

which implies that

$$\begin{aligned} V_a(z_{1_t},t) = e^{-\alpha_1 t}V_a(\phi,0) + e^{-\alpha_1 t}\int_0^t e^{\alpha_1 s}U(z_{1_s},s)ds &\le e^{-\alpha_1 t}V_a(\phi,0) + e^{-\alpha_1 t}\int_0^t e^{\alpha_1 s}|U(z_{1_s},s)|ds \\ &\le e^{-\alpha_1 t}V_a(\phi,0) + \alpha_2\bar{w}_2^2(t)e^{-\alpha_1 t}\int_0^t e^{\alpha_1 s}ds = e^{-\alpha_1 t}V_a(\phi,0) + \frac{\alpha_2\bar{w}_2^2(t)}{\alpha_1}(1-e^{-\alpha_1 t}) \\ &\le e^{-\alpha_1 t}V_a(\phi,0) + \frac{\alpha_2}{\alpha_1}\bar{w}_2^2(t). \end{aligned} \tag{4.34}$$

Using this inequality along with (4.29) and (4.30)

$$\|z_a(t)\|^2 \leq \frac{\kappa_1}{\kappa_2}\|\phi\|_W^2 e^{-\alpha_1 t} + \frac{\alpha_2}{\alpha_1\kappa_2}\bar{w}_2^2(t), \tag{4.35}$$

where $\kappa_2 = \lambda_{\min}(P)$ and κ_1 is defined in (4.30).

It is easy to see from (4.35) that

$$\|z_a(t)\| \leq \sqrt{\frac{\kappa_1}{\kappa_2}} e^{\frac{-\alpha_1 t}{2}}\|\phi\|_W + \sqrt{\frac{\alpha_2}{\alpha_1\kappa_2}}|w_{[0,t]}|_\infty. \tag{4.36}$$

Moreover, considering $\beta(s,t) = \sqrt{\frac{\kappa_1}{\kappa_2}} e^{\frac{-\alpha_1 t}{2}} s$ and $\gamma(s) = \sqrt{\frac{\alpha_2}{\alpha_1\kappa_2}} s$, we obtain

$$\|z_a(t)\| \leq \beta(\|\phi\|_W, t) + \gamma(|w_{[0,t]}|_\infty). \tag{4.37}$$

Therefore, system (4.16) is locally ISS under external disturbance $w_2(t)$ via Definition 3.

4.4.2 Locally ISS Stability of the Closed-Loop System

In this section we have provided the local ISS for the overall closed-loop system (4.14). Besides, it is essential to show that the sliding motion $S(t) = 0$ is accomplished in finite-time. Define the Lyapunov-Krasovskii functional $L_V(\delta) = \{(z_a(t), S)|V(z_{1_t}, S, t) \leq \delta\}$ in the following theorem.

Theorem 2 *The system (4.14) is said to be locally ISS for given scalars* $h > 0$, $\delta > 0$, $\alpha_1 > 0$ *and matrices* F *and* F_h, *if there exists a scalar* $\alpha_2 > 0$ *and matrices* $P > 0, Q_1 > 0, Q_2 > 0$ *and* $R > 0$ *such that the following LMIs hold*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & 6e^{-\alpha_1 h}R & \Psi_{14} & \Psi_{15} & \Psi_{16} \\ * & \Psi_{22} & 6e^{-\alpha_1 h}R & \Psi_{24} & \Psi_{25} & T_3\bar{B} \\ * & * & -12e^{-\alpha_1 h}R & 0 & 0 & 0 \\ * & * & * & \Psi_{44} & \Psi_{45} & T_4\bar{B} \\ * & * & * & * & \Psi_{55} & T_2\bar{B} \\ * & * & * & * & * & -\alpha_2 I \end{bmatrix} < 0 \tag{4.38}$$

$$\alpha_2\bar{w}_2^2(t) - \alpha_1\delta \leq 0 \tag{4.39}$$

where

$$\begin{aligned}
\Psi_{11} &:= \bar{A}^T P + P\bar{A} + Q_1 + \alpha_1 P - 4e^{-\alpha_1 h} R + Sym(T_1 \bar{A}) \\
\Psi_{12} &:= -2e^{-\alpha_1 h} R + P\bar{A}_h + T_1 \bar{A}_h + \bar{A}^T T_3^T \\
\Psi_{14} &:= P\bar{B} + \bar{E}^T Q_2 + T_1 \bar{B} + \bar{A}^T T_4^T, \quad \Psi_{15} := -T_1 + \bar{A}^T T_2^T \\
\Psi_{16} &:= P\bar{B} + T_1 \bar{B}, \quad \Psi_{22} := -e^{-\alpha_1 h} Q_1 - 4e^{-\alpha_1 h} R, \\
\Psi_{24} &:= \bar{E}_h^T Q_2 + T_3 \bar{B} + \bar{A}_h^T T_4^T, \quad \Psi_{25} := \bar{A}_h^T T_2^T - T_3 \\
\Psi_{44} &:= \alpha_1 Q_2 + Sym(T_4 \bar{B}), \quad \Psi_{45} := \bar{B} T_2^T - T_4 \\
\Psi_{55} &:= h^2 R - Sym(T_2).
\end{aligned}$$

Proof: Consider the Lyapunov-Krasovskii functional

$$V = V_b + V_c, \tag{4.40}$$

where V_b and V_c are defined as

$$\begin{aligned}
V_b &= V_{b_1} + V_{b_2} + V_{b_3} \\
V_c &= S^T(t) Q_2 S(t).
\end{aligned} \tag{4.41}$$

The derivative of V_b

$$\dot{V}_b = \dot{V}_{b_1} + \dot{V}_{b_2} + \dot{V}_{b_3}, \tag{4.42}$$

where

$$\begin{aligned}
\dot{V}_{b_1} &= \dot{V}_1 + z_a^T(t) P\bar{B} S(t) + S^T(t) \bar{B}^T P z_a(t) \\
\dot{V}_{b_2} &= \dot{V}_2 \\
\dot{V}_{b_3} &= \dot{V}_3
\end{aligned}$$

where $\dot{V}_1, \dot{V}_2$ and $\dot{V}_3$ were defined in Theorem 1.

Thus, $\dot{V}_b$ is

$$\dot{V}_b = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + z_a^T(t) P\bar{B} S(t) + S^T(t) \bar{B}^T P z_a(t) \dot{V}_a + z_a^T(t) P\bar{B} S(t) + S^T(t) \bar{B}^T P z_a(t). \tag{4.43}$$

Also, the derivative of V_c satisfies the relation

$$\begin{aligned}
\dot{V}_c = {} & S^T(t) Q_2 \bar{E} z_a(t) + S^T(t) Q_2 \bar{E}_h z_a(t-h) + z_a^T(t) \bar{E}^T Q_2 S(t) + z_a^T(t-h) \bar{E}_h^T Q_2 S(t) \\
& - 2S^T Q_2 (\Pi(t) - B_2 \rho(t)).
\end{aligned} \tag{4.44}$$

Now, take the time derivative of (4.40) and substitute the $\dot{V}_b$ and $\dot{V}_c$ from (4.43) and (4.44) respectively.

$$\begin{aligned}\dot{V} = \dot{V}_a &+ z_a^T(t)P\bar{B}S(t) + S^T(t)\bar{B}^T P z_a(t) + S^T(t)Q_2\bar{E}z_a(t) + S^T(t)Q_2\bar{E}_h z_a(t-h)\\ &+ z_a^T(t)\bar{E}^T Q_2 S(t) + z_a^T(t-h)\bar{E}_h^T Q_2 S(t) - 2S^T(t)Q_2(\Pi(t) - B_2\rho(t)). \end{aligned} \tag{4.45}$$

The following equality is satisfied for every matrices T_1, T_2, T_3 and T_4 with appropriate dimensions:

$$\begin{aligned}&2\left\{z_a^T(t)T_1 + \dot{z}_a^T(t)T_2 + z_a^T(t-h)T_3 + S^T(t)T_4\right\}\\ &\quad\times\left\{-\dot{z}_a(t) + \bar{A}z_a(t) + \bar{A}_h z_a(t-h) + \bar{B}w_2(t) + \bar{B}S(t)\right\} = 0. \end{aligned} \tag{4.46}$$

It follows from (4.45)-(4.46) that

$$\begin{aligned}\dot{V} + \alpha_1 V - \alpha_2 w_2^T(t)w(t) \le{}& z_a^T(t)(P\bar{A} + \bar{A}^T P)z_a(t) + z_a^T(t)(P\bar{A}_h)z_a(t-h) + z_a^T(t)(P\bar{B})w_2(t)\\ &+ w_2^T(t)(\bar{B}^T P)z_a(t) + z_a^T(t-h)(\bar{A}_h^T P)z_a(t) + z_a^T(t)Q_1 z_a(t) - e^{-\alpha_1 h}z_a^T(t-h)Q_1 z_a(t-h)\\ &- e^{-\alpha_1 h}\left\{z_a^T(t)Rz_a(t) - z_a^T(t)Rz_a(t-h) - z_a^T(t-h)Rz_a(t) + z_a^T(t-h)Rz_a(t-h)\right\}\\ &- 3e^{-\alpha_1 h}\Bigg\{ z_a^T(t)Rz_a(t) + z_a^T(t)Rz_a(t-h) - z_a^T(t)R\left(\frac{2}{h}\int_{t-h}^{t} z_a(s)ds\right) + z_a^T(t-h)Rz_a(t)\\ &+ z_a^T(t-h)Rz_a(t-h) - z_a^T(t-h)R\left(\frac{2}{h}\int_{t-h}^{t} z_a(s)ds\right)\\ &- \left(\frac{2}{h}\int_{t-h}^{t} z_a(s)ds\right)^T Rz_a(t) - \left(\frac{2}{h}\int_{t-h}^{t} z_a(s)ds\right)^T Rz_a(t-h)\\ &+ \left(\frac{2}{h}\int_{t-h}^{t} z_a(s)ds\right)^T R\left(\frac{2}{h}\int_{t-h}^{t} z_a(s)ds\right)\Bigg\} + h^2\dot{z}_a^T(t)R\dot{z}_a(t)\\ &+ z_a^T(t)P\bar{B}S(t) + S^T(t)\bar{B}^T P z_a(t) + S^T(t)Q_2\bar{E}z_a(t) + S^T(t)Q_2\bar{E}_h z_a(t-h)\\ &+ z_a^T(t)\bar{E}^T Q_2 S(t) + z_a^T(t-h)\bar{E}_h^T Q_2 S(t) - 2S^T(t)Q_2(\Pi(t) - B_2\rho(t)) + \alpha_1 z_a^T(t)Pz_a(t)\\ &+ \alpha_1 S^T(t)Q_2 S(t) - \alpha_2 w_2^T(t)w_2(t) + 2\left\{z_a^T(t)T_1 + \dot{z}_a^T(t)T_2 + z_a^T(t-h)T_3 + S^T(t)T_4\right\}\\ &\quad\times\left\{-\dot{z}_a(t) + \bar{A}z_a(t) + \bar{A}_h z_a(t-h) + \bar{B}w_2(t) + \bar{B}S(t)\right\}. \end{aligned} \tag{4.47}$$

Therefore, one can obtained LMI (4.48) from the above derivation.

$$
\begin{aligned}
&\dot{V} + \alpha_1 V - \alpha_2 w_2^T(t) w(t) \\
&\leq \begin{bmatrix} z_a(t) \\ z_a(t)(t-h) \\ \frac{1}{h}\int\limits_{t-h}^{t} z_a(t)(s)ds \\ S(t) \\ \dot{z_a}(t) \\ w_2(t) \end{bmatrix}^T
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & 6e^{-\alpha_1 h}R & \Psi_{14} & \Psi_{15} & \Psi_{16} \\
* & \Psi_{22} & 6e^{-\alpha_1 h}R & \Psi_{24} & \Psi_{25} & T_3\bar{B} \\
* & * & -12e^{-\alpha_1 h}R & 0 & 0 & 0 \\
* & * & * & \Psi_{44} & \Psi_{45} & T_4\bar{B} \\
* & * & * & * & \Psi_{55} & T_2\bar{B} \\
* & * & * & * & * & -\alpha_2 I
\end{bmatrix}
\begin{bmatrix} z_a(t) \\ z_a(t)(t-h) \\ \frac{1}{h}\int\limits_{t-h}^{t} z_a(t)(s)ds \\ S(t) \\ \dot{z_a}(t) \\ w_2(t) \end{bmatrix}
\end{aligned}
\tag{4.48}
$$

From the definition of $\Pi(t)$ in (4.15) we can conclude that the term $S^T Q_2(\Pi(t) - B_2\rho(t))$ is negative. Therefore, without loss of generality similar to Theorem 1, it is easy to show that the closed-loop system (4.14) is locally ISS.

4.4.3 Existence of the Sliding Manifold in Finite-Time

In this chapter, we have employed super-twisting control (second-order sliding mode control) in which both $S(t) = 0$ and $\dot{S}(t) = 0$ are needed. Now, $S(t) = 0$ from the first moment, but one cannot comment that $\dot{S}(t) = 0$. Therefore reaching phase analysis is required.

An ideal sliding action is obtained at the surface $S(t) = 0$ in the domain

$$
\varPsi = \left\{ z_a(t), S) \in [t-h, t] \rightarrow \mathbb{R}^{n-m} \times \mathbb{R}^m : \, (\|\bar{E}\| + \|\bar{E}_h\|)|z_a(t)|_h < \mu - \varepsilon \right\}
$$

where ε is a small scalar which satisfies the relation $0 < \varepsilon < \mu$.

Suppose the resulting Lyapunov function $V_S = S^T(t) Q_2 S(t)$. By differentiating V_S with respect to time, one can get

$$
\dot{V}_S = 2S^T(t) Q_2 \left\{ \bar{E} z_a(t) + \bar{E}_h z_a(t-h) - K_1 |S(t)|^{\frac{1}{2}} \mathrm{sign}(S) - \int_0^t K_2 \mathrm{sign}(\omega) d\omega + B_2 \rho(t) \right\}.
$$

Now, rewrite the equation

$$
\begin{cases}
\dot{V}_S & = 2S^T(t) Q_2 \left(\bar{E} z_a(t) + \bar{E}_h z_a(t-h) - K_1 |S(t)|^{\frac{1}{2}} \mathrm{sign}(S) + \Omega \right) \\
\dot{\Omega} & = -K_2 \mathrm{sign}(S) + B_2 \dot{\rho}(t)
\end{cases}
$$

By assuming the upper bound on the first and second term, one can write the following inequality

$$\begin{cases} \dot{V}_S & \leq 2\|S(t)\|\|Q_2\| \Big((\|\bar{E}\| + \|\bar{E}_h\|)|z_a(t)|_h - K_1|S(t)|^{\frac{1}{2}}\text{sign}(S) + \Omega \Big) \\ \dot{\Omega} & = -K_2\text{sign}(S) + B_2\dot{\rho}(t) \end{cases} \tag{4.49}$$

If Theorem 2 is satisfied then $z_a(t)$ converges to the solution $z_a(t) = 0$. Select control gains $K_2 > \|B_2\|\rho_0 + \mu$, with μ a positive scalar and $K_1 > (\|\bar{E}\| + \|\bar{E}_h\|)|z_a(t)|_h + K_2$. This confirms that the domain Ψ has reached in finite-time and the following inequality satisfies.

$$\dot{V}_s \leq -\varpi\sqrt{V_s}.$$

For the detailed Lyapunov proof one can refer [48].

Remark 10 *It is important to note that in second-order sliding mode control (super-twisting control) from the first moment, the initial value of $S(t)$ and $\dot{S}(t)$ must be zero. One can easily conclude from (4.49) that initial value of Ω includes the initial value of the disturbance ρ. It is always difficult to predict the initial condition of disturbance ρ, except some particular cases where initially there is no disturbance, for example, fault detection problems where initially there is no fault means ρ. In this particular case, both $S(t)$ and $\dot{S}(t)$ are zero from the first moment $t \geq 0$ because $\Omega = 0$. However, in the case of nonzero initial disturbance, the sliding mode will start after some finite-time $t \geq \tau$ because of $\Omega \neq 0$, which can be designed as small as possible by selecting appropriate values of K_1, K_2 as given in [48]. Once the $S(t)$ and Ω are zero, then $\dot{S}(t)$ becomes zero and it remains zero forever despite disturbances.*

4.5 Simulation Results

In order to evaluate the performance of the proposed method, numerical simulation analysis is presented regarding the TORA system [49]. The system has a cart of mass M attached to a wall by a linear spring (constant k). The cart can swing without friction in the horizontal plane. On the cart, a rotating mass m is actuated by the DC motor. The mass is eccentric with a radius of eccentricity e and can be assumed to be a point mass fixed on a massless rotor. The rotating action of the mass m manages the swaying

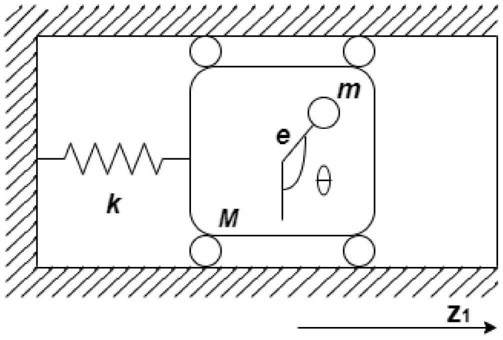

Figure 4.1: TORA System [49]

of the cart. The control torque employed to the rotating mass is expressed by $u(t)$. The dynamics of TORA system is

$$\begin{cases} \dot{z}_1(t) & = z_2(t) \\ \dot{z}_2(t) & = z_1 + z_3 + \dfrac{-z_1(t) + \epsilon z_4^2(t)\mathrm{sin} z_3(t)}{1 - \epsilon^2 \mathrm{cos}^2 z_3(t)} - \dfrac{\epsilon \mathrm{cos} z_3(t)(u(t) + \rho(t))}{1 - \epsilon^2 \mathrm{cos}^2 z_3(t)} \\ \dot{z}_3(t) & = z_4(t) \\ \dot{z}_4(t) & = \dfrac{\epsilon \mathrm{cos} z_3(t)(z_1(t) - \epsilon z_4^2(t)\mathrm{sin} z_3(t))}{1 - \epsilon^2 \mathrm{cos}^2 z_3(t)} + \dfrac{(u(t) + \rho(t))}{1 - \epsilon^2 \mathrm{cos}^2 z_3(t)} \end{cases} \tag{4.50}$$

Where $z_1(t)$ and $z_2(t)$ are the positions and velocity of the cart. The variables $z_3(t)$ and $z_4(t)$ are the angle and angular velocity of the rotor. The parameter ϵ depends on the eccentricity e and the masses M and m. u and y are the control input and output. $\rho(t) = 2.5\mathrm{sin}(t)$ is external matched Lipschitz disturbance.

By diffeomorphism, the following variables are defined

$$\begin{aligned} x_1(t) &= z_1(t) + \epsilon \mathrm{sin} z_3(t) \\ x_2(t) &= z_2(t) + \epsilon z_4(t) \mathrm{cos} z_3(t) \\ x_3(t) &= z_3(t) \\ x_4(t) &= z_4(t). \end{aligned}$$

Then, system (4.50) is equivalently represented as

$$\begin{cases} \dot{x}_1(t) & = x_2(t) \\ \dot{x}_2(t) & = x_3(t) \\ \dot{x}_3(t) & = x_4(t) \\ \dot{x}_4(t) & = f(x) + g(x)(u(t) + \rho(t)), \end{cases} \tag{4.51}$$

where

$$
\begin{aligned}
x(t) &= \mathrm{col}\left\{x_1(t), x_2(t), x_3(t), x_4(t)\right\} \\
f(x) &= \frac{\epsilon \mathrm{cos} x_3(t)(x_1(t) - \epsilon \mathrm{sin} x_3(t) - \epsilon x_4^2(t) \mathrm{sin} x_3(t))}{1 - \epsilon^2 \mathrm{cos}^2 x_3(t)} \\
g(x) &= \frac{1}{1 - \epsilon^2 \mathrm{cos}^2 x_3(t)}.
\end{aligned}
$$

Applying $u(t) = \frac{1}{g(x)}\left[-f(x) + \nu(t)\right]$ to system (4.51) yields the following perturbed fourth-order integrator

$$
\begin{cases}
\dot{x}(t) & = Ax(t) + B(\nu(t) + \rho_1(t)) \\
y(t) & = Cx(t)
\end{cases}
\tag{4.52}
$$

where $\nu(t)$ is the virtual control input to be designed, and $\rho_1(t) = \frac{\rho(t)}{1 - \epsilon^2 \mathrm{cos}^2 x_3(t)}$.

Define $x(t) = \begin{bmatrix} x_a(t) & x_b(t) \end{bmatrix}^T \in \mathbb{R}^4,\ x_a(t) \in \mathbb{R}^3,\ x_b(t) \in \mathbb{R}$, and matrices A, B, C are

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0_{3\times 1} & I_3 \end{bmatrix} = \begin{bmatrix} 0_{3\times 1} & C_2 \end{bmatrix}.
$$

It is necessary to remark that the controller is precisely described for $-\pi/2 < x_3 < \pi/2$, because of the controller denominator which is ϵ $\mathrm{cos} x_3(t)$. After this, one can represent the resulting linear system in the generalized regular form as mentioned in the problem formulation section.

Currently, choose the sliding surface as defined in (4.10)

$$
S(t) = \begin{bmatrix} F & I_m \end{bmatrix} C_2^{-1} y(t) + \begin{bmatrix} F_h & 0_m \end{bmatrix} C_2^{-1} y(t-h) - w_2(t)
$$

where $C_2 = I_3$.

The control input as defined in (4.13)

$$
\begin{aligned}
\nu(t) = -(B_2)^{-1} \{ & (A_{22} + FC_1A_{12})S(t) + F_hC_1A_{12}S(t-h) - (A_{22} + FC_1A_{12})F(C_1x_a(t)) \\
& - F_hC_1A_{12}F_h(C_1x_a(t-2h)) - (FC_1A_{12}F_h + A_{22}F_h + F_hC_1A_{12}F)(C_1x_a(t-h)) \\
& + \Pi(t) + (A22 + \varrho + FC_1A_{12})w_2(t) + F_hC_1A_{12}w_2(t-h) \} .
\end{aligned}
\tag{4.53}
$$

where $C_1 = \begin{bmatrix} 0_{2\times 1} & I_2 \end{bmatrix} \in \mathbb{R}^{2\times 3}$, and $w_2(t)$ is defined in (4.10). The continuous control $\Pi(t)$ is defined as (4.15)

$$\Pi(t) = K_1 |S(t)|^{\frac{1}{2}} \mathrm{sign}(S) + \int_0^t K_2 \mathrm{sign}(S(\omega)) d\omega, \tag{4.54}$$

where K_1 and K_2 are defined earlier in (4.15).

Remark 11 *It is important to note that the control input (4.53) has the knowledge of the state $C_1 x_a(t)$, $S(t)$ and its delayed terms.*

In order to explain the advantage and benefits of the proposed method, we have compared the proposed results with that obtained in [12] Seuret et al. The control law stated in [12] is

$$\begin{aligned} \nu(t) = -(B_2)^{-1} \{ & (A_{22} + F C_1 A_{12}) S(t) + F_h C_1 A_{12} S(t-h) - (A_{22} + F C_1 A_{12}) F (C_1 x_a) \\ & - F_h C_1 A_{12} F_h (C_1 x_a(t-2h)) - (F C_1 A_{12} F_h + A_{22} F_h + F_h C_1 A_{12} F)(C_1 x_a(t-h)) \\ & + \Pi(t) \} . \end{aligned} \tag{4.55}$$

where

$$\begin{aligned} S(t) &= \begin{bmatrix} F & I_m \end{bmatrix} C_2^{-1} y(t) + \begin{bmatrix} F_h & 0_m \end{bmatrix} C_2^{-1} y(t-h) \\ \Pi(t) &= K \mathrm{sign}(S). \end{aligned}$$

The tuning parameters of the indicated method are listed in Table 4.1, which are chosen according to their corresponding works [175]. The control parameters of the proposed strategy (which satisfy conditions of Theorem 2) are also listed in Table 4.1. The initial value of states is chosen as $z(0) = \begin{bmatrix} 0.4 & 0 & 0.2 & 0 \end{bmatrix}$.

Table 4.1: Tuning parameters for different approaches

Approach	Control law	Tuning Parameters
Proposed Method	Eq. (4.53)	$F = \begin{bmatrix} 1.02 & 1.747 \end{bmatrix}$, $F_h = \begin{bmatrix} 0.1 & 1 \end{bmatrix}$, $h = 0.01$, $K_1 = 20,\ K_2 = 9,\ \epsilon \in (0,1),\ \varrho = 20.$
Seuret et al. R[12]	Eq. (4.55)	$F = \begin{bmatrix} 1.02 & 1.747 \end{bmatrix}$, $F_h = \begin{bmatrix} 0.1 & 1 \end{bmatrix}$, $h = 0.01$, $K = 9,\ \epsilon \in (0,1).$

Fig. 4.2 presents the state trajectories convergence of the TORA system in which $z_1(t)$ and $z_2(t)$ are the translation position and velocity of the cart respectively. $z_3(t)$ and $z_4(t)$ are the rotor's angular position and velocity, respectively. A sliding surface for the TORA system is presented in Fig. 4.3. Fig. 4.4 defines the control input $u(t)$ of the TORA system.

The simulation results of Fig. 4.2 validate that the states are converging to zero and reaching the steady-state fastly, with the proposed method in comparison to the results in [12]. Fig. 4.3 presents that the sliding surface is achieved at the beginning and ensures robustness. Whereas in [12] the sliding surface is obtained later, and robustness is ensured when the trajectory reached the sliding surface. Fig. 4.4 illustrates that the control input of the proposed method is continuous with a more negligible chattering effect, unlike [12], where control is discontinuous and full of chattering phenomenon. It is noteworthy to mention that the proposed control is super-twisting control, which is also helpful for practical systems.

Therefore, from the TORA system's simulation results, one can easily conclude that the results obtained from the proposed approach are far better than the results obtained from [12].

4.6 Summary

This chapter suggests a delayed output feedback sliding mode control. An artificial delay and an exponential decay term are introduced in the sliding surface, which yields an appearance of delayed and exponential decay terms in the reduced system dynamics. Therefore, the reduced system dynamics become a retarded time-delay system with external disturbances. The locally ISS for the reduced system dynamics and the closed-loop system are presented through Lyapunov-Krasovskii functional with the aid of LMIs. The simulation results of the TORA system are illustrated to describe the efficacy of the proposed approach.

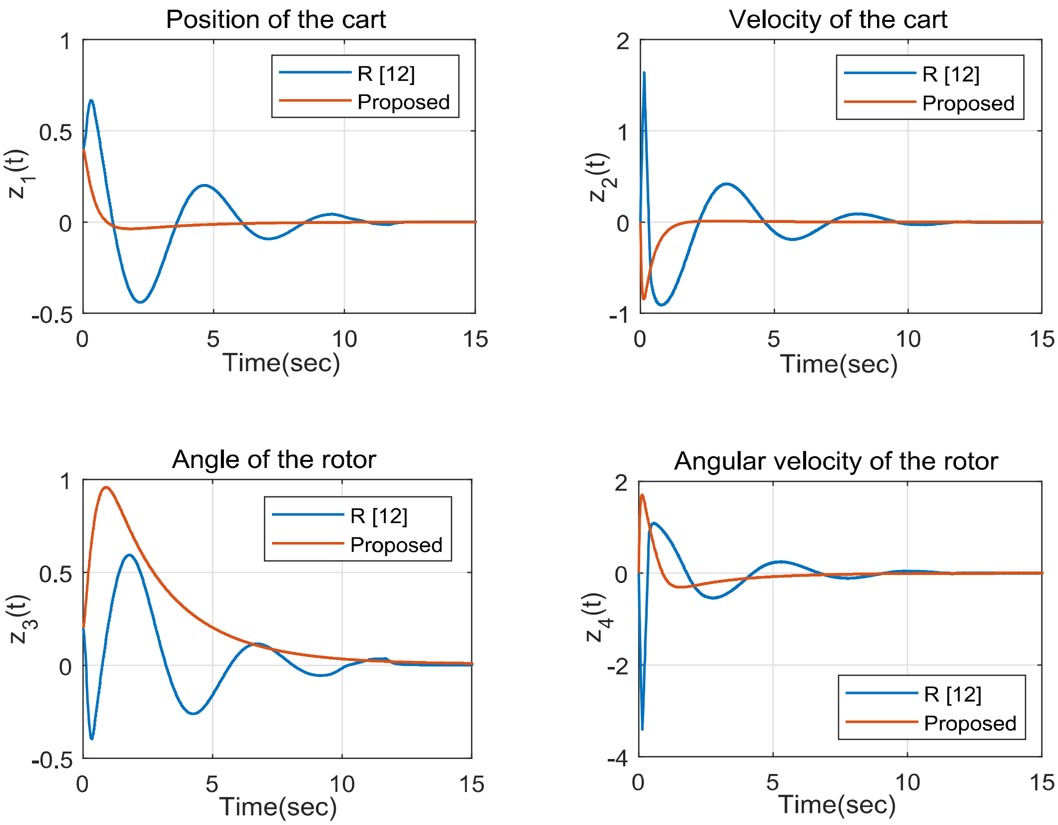

Figure 4.2: A convergence of state trajectories of the TORA system

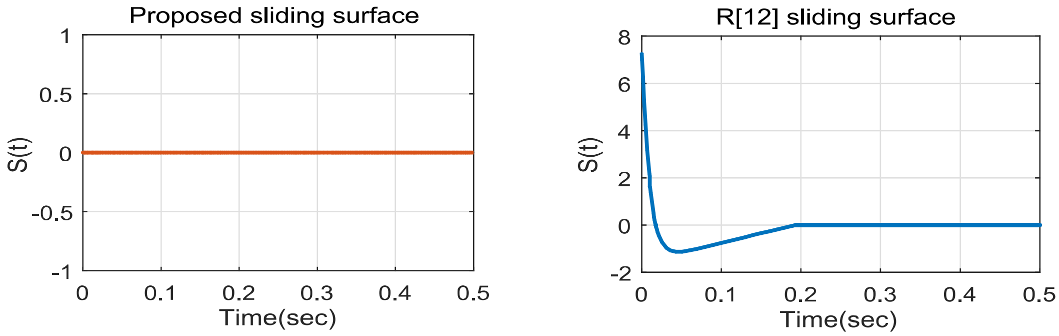

Figure 4.3: Sliding surface of the TORA system

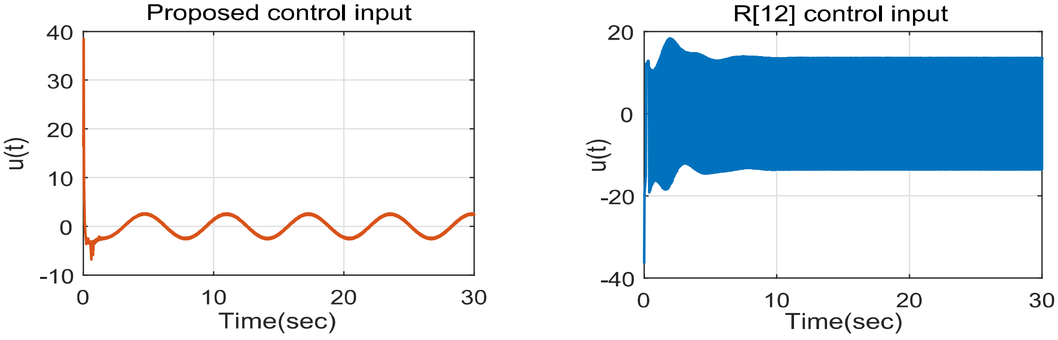

Figure 4.4: Control input of the TORA system