

Chapter 7

Representability of fuzzy biorders and fuzzy weak orders

7.1 Introduction

The idea of providing a representation for a binary relation \mathcal{R} between two non empty sets A and X was formulated by Guttman[46] in 1944, by proposing two functions $f : A \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ such that

$$a\mathcal{R}x \Leftrightarrow f(a) > g(x),$$

for each $a \in A$ and $x \in X$. In 1969, Ducamp and Falmagne[36] showed that if A and X both are finite, then the existence of such type of functions f and g for a binary relation \mathcal{R} between A and X is equivalent to the following condition:

$$a\mathcal{R}x \text{ and } b\mathcal{R}y \Rightarrow a\mathcal{R}y \text{ or } b\mathcal{R}x, \tag{7.1}$$

for each $a, b \in A$ and $x, y \in X$.

A relation satisfying the condition (7.1) (called as Ferrers condition[79, 91]) is said to be a biorder. It has been proved by Doignon et al.[34] that the condition (7.1) on \mathcal{R} is equivalent to the condition

$$\mathcal{R}\mathcal{R}^d\mathcal{R} \subseteq \mathcal{R}, \tag{7.2}$$

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where \mathcal{R}^d denotes the dual of \mathcal{R} . They also studied representability of biorders and interval orders. Representations of different types of ordering have been studied by several authors (see e.g., [19, 82, 86]).

The fuzzy analogues of the Ferrers conditions (7.1) and (7.2), which are given as follows:

$$\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}, \quad \text{for each } a, b \in A \text{ and } x, y \in X \quad (7.3)$$

and

$$\mathcal{R} \circ_T \mathcal{R}^d \circ_T \mathcal{R} \subseteq \mathcal{R} \quad (7.4)$$

are no more equivalent in general. A condition under which (7.3) and (7.4) are equivalent has been provided by Fodor[41]. Keeping this in view, in literature, there are two different definitions of fuzzy biorders, corresponding to these two non-equivalent Ferrers conditions (also called T -Ferrers conditions)[41, 97]. Comparative studies of T -Ferrers relations, fuzzy biorders and fuzzy interval orders have been carried out by several authors (see e.g., [28, 30]). Baets and Walle[10] had introduced the notion of T -fuzzy interval orders and studied two particular types of T -fuzzy interval orders: Weak and Strong fuzzy interval orders.

The study of fuzzy weak orders with respect to a left continuous t-norm T and their representability by the residual implication operator associated with T (called T -representable fuzzy weak orders) has been done by several authors (see e.g., [11, 98]). Characterizations for a T_M -representable (also called Gödel representable) fuzzy weak orders and for the fuzzy relation which can be written as the union or intersection of a finite family of T_M -representable (or Gödel representable) fuzzy weak orders have been obtained by Baets et al.[11]. Characterizations for a T_P -representable fuzzy weak order and for finite intersections of fuzzy weak orders with respect to any left continuous t-norm T have been obtained by Sali et al.[98].

The representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the α -cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al.[1] and Induráin et al.[52].

In this chapter, we have studied representability of both fuzzy biorders and fuzzy weak orders. It is observed that union of a finite family of fuzzy weak orders

with respect to T is a fuzzy quasi-transitive relation with respect to T . In the last theorem, we have obtained a characterization for a T_L -representable fuzzy weak order.

Now we recall some definitions and results which will be used throughout the chapter.

Definition 7.1. [123] Let \mathcal{R} be a fuzzy relation between A and X . Then for $\alpha \in [0, 1]$, the α -cut \mathcal{R}_α is given by

$$\mathcal{R}_\alpha = \{(a, x) \in A \times X : \mathcal{R}(a, x) \geq \alpha\}.$$

Note that each \mathcal{R}_α is a binary relation between A and X .

Definition 7.2. [84] Let \mathcal{R} be a fuzzy relation on A . Then its *strict part* $P_{\mathcal{R}}$ is the fuzzy relation on A given by:

$$P_{\mathcal{R}}(a, b) = \begin{cases} \mathcal{R}(a, b), & \text{if } \mathcal{R}(a, b) > \mathcal{R}(b, a) \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7.3. [11] Let T be a t-norm. Then a fuzzy relation \mathcal{R} on A is called a *fuzzy weak order* with respect to T if it is:

1. strongly S_M -complete;
2. T -transitive.

Definition 7.4. [11] Let T be a t-norm. Then a fuzzy relation \mathcal{R} on A is called a *fuzzy quasi order* with respect to T if it is:

1. reflexive;
2. T -transitive.

Definition 7.5. [11] Let T be a t-norm and S be its dual t-conorm. Then a fuzzy relation \mathcal{R} on A is called a *fuzzy quasi-transitive* relation with respect to T if it is:

1. strongly S_M -complete;
2. negatively S -transitive.

Definition 7.6. [34] Let \mathcal{R} and \mathcal{Q} be two binary relations between A and X , X and Y respectively. Then the *composition* $\mathcal{R}\mathcal{Q}$ is the binary relation between A and Y given by

$$\mathcal{R}\mathcal{Q} = \{(a, y) : \text{there exists } x \in X \text{ such that } (a, x) \in \mathcal{R} \text{ and } (x, y) \in \mathcal{Q}\}.$$

Definition 7.7. [61] Let \mathcal{R} and \mathcal{Q} be two fuzzy relations between A and X , X and Y respectively and T be a t-norm. Then the *fuzzy composition* $\mathcal{R}o_T\mathcal{Q}$ with respect to T is the fuzzy relation between A and Y given by

$$(\mathcal{R}o_T\mathcal{Q})(a, y) = \sup_{x \in X} T\{\mathcal{R}(a, x), \mathcal{Q}(x, y)\}, \text{ for each } (a, y) \in A \times Y.$$

Definition 7.8. [34, 79, 91] A binary relation \mathcal{R} between A and X is said to satisfy the *Ferrers property* if

$$(a, x) \in \mathcal{R} \text{ and } (b, y) \in \mathcal{R} \Rightarrow (a, y) \in \mathcal{R} \text{ or } (b, x) \in \mathcal{R}, \quad (7.5)$$

for each $a, b \in A$ and $x, y \in X$. Equivalently,

$$\mathcal{R}\mathcal{R}^d\mathcal{R} \subseteq \mathcal{R}. \quad (7.6)$$

Definition 7.9. [34] A binary relation \mathcal{R} between A and X is said to be a *biorder* if \mathcal{R} satisfies the Ferrers property.

7.2 Fuzzy biorder and its representability

Corresponding to (7.5) and (7.6), in case of fuzzy relations on A we have the following:

Definition 7.10. [42] Let T be a t-norm and S be a t-conorm. Then a fuzzy relation \mathcal{R} on A is said to satisfy *type1 $T-S$ Ferrers property* if $T\{\mathcal{R}(a, b), \mathcal{R}(c, d)\} \leq S\{\mathcal{R}(a, d), \mathcal{R}(c, b)\}$, for each $a, b, c, d \in A$. In case T is a t-norm and S is the corresponding t-conorm (or dual t-conorm), then the type1 $T-S$ Ferrers property is simply called type1 T Ferrers property.

Definition 7.11. [31] A fuzzy relation \mathcal{R} on A is said to satisfy *type2 T Ferrers property* if $\mathcal{R}o_T\mathcal{R}^d o_T\mathcal{R} \subseteq \mathcal{R}$.

Now we state the following important result in this context, given by Fodor[42], which shows under which condition, the two types of Ferrers properties turn out to be equivalent.

Proposition 7.12. [41] *The following statements are equivalent for a fuzzy relation on A :*

1. *A type1 T Ferrers relation is also type2 T Ferrers.*
2. *A type2 T Ferrers relation is also type1 T Ferrers.*
3. *T is rotational invariant(i.e., $T(x, y) \leq z \Leftrightarrow T(x, 1 - z) \leq (1 - y)$, for each $x, y, z \in [0, 1]$).*

From the above proposition, it is clear that type1 T Ferrers property and type2 T Ferrers property are not equivalent in general. Keeping this in view, in literature, there are two different definitions of fuzzy biorders, one corresponding to type1 T_M Ferrers property of a fuzzy relation between A and X and another corresponding to type2 T Ferrers property of a fuzzy relation on A (cf.[41, 97]), which we have called here as type1 T_M biorder and type2 T biorder respectively.

Definition 7.13. [97] A fuzzy relation \mathcal{R} between A and X is said to be *type1 T_M biorder* if $\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}$, for every $a, b \in A$ and $x, y \in X$.

Definition 7.14. [41] A fuzzy relation \mathcal{R} on A is said to satisfy *type2 T biorders* if $\mathcal{R} \circ_T \mathcal{R}^d \circ_T \mathcal{R} \subseteq \mathcal{R}$.

Proposition 7.15. *Let \mathcal{R} be a fuzzy relation between A and X . Then \mathcal{R} is a type1 T_M biorder iff each \mathcal{R}_α is a biorder between A and X .*

Proof. Let \mathcal{R} be a type1 T_M biorder between A and X . Then

$$\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\},$$

for each $a, b \in A$ and $x, y \in X$.

We have to show that \mathcal{R}_α is a biorder between A and X , for each $\alpha \in [0, 1]$. Assume the contrary. Let for some $\alpha \in [0, 1]$, there exist $a, b \in A$ and $x, y \in X$

such that

$$\begin{aligned}
& a\mathcal{R}_\alpha x, b\mathcal{R}_\alpha y, a(\mathcal{R}_\alpha)^c y, b(\mathcal{R}_\alpha)^c x \\
\Rightarrow & \mathcal{R}(a, x) \geq \alpha, \mathcal{R}(b, y) \geq \alpha, \mathcal{R}(a, y) < \alpha, \mathcal{R}(b, x) < \alpha \\
\Rightarrow & \min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \geq \alpha > \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\},
\end{aligned}$$

which is a contradiction.

Conversely, assume that each \mathcal{R}_α is a biorder between A and X , for $\alpha \in [0, 1]$. We have to show that \mathcal{R} is a type1 T_M biorder between A and X i.e,

$$\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}$$

for each $a, b \in A$ and $x, y \in X$. For this, we need to consider the following cases:

Case 1: If $\mathcal{R}(a, x) = 0$ or $\mathcal{R}(b, y) = 0$. Then the above inequality is obviously satisfied.

Case 2: If $\mathcal{R}(a, x) = \beta \neq 0$ and $\mathcal{R}(b, y) = \gamma \neq 0$. Set $\delta = \min\{\beta, \gamma\}$. Then

$$\begin{aligned}
& \mathcal{R}(a, x) \geq \delta \text{ and } \mathcal{R}(b, y) \geq \delta \\
\Rightarrow & a\mathcal{R}_\delta x \text{ and } b\mathcal{R}_\delta y \\
\Rightarrow & a\mathcal{R}_\delta y \text{ or } b\mathcal{R}_\delta x \quad (\text{Since } \mathcal{R}_\delta \text{ is a biorder}) \\
\Rightarrow & \mathcal{R}(a, y) \geq \delta \text{ or } \mathcal{R}(b, x) \geq \delta \\
\Rightarrow & \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\} \geq \delta = \min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\}.
\end{aligned}$$

Thus \mathcal{R} is a type1 T_M biorder. □

The following Example 7.1 shows that Proposition 7.15 is not true if we replace type1 T_M biorder with type2 T_M biorder or type2 T_{nM} biorder(which is equivalent to type1 T_{nM} biorder using Proposition 7.12, as T_{nM} is the largest rotational invariant t-norm[28]).

Example 7.1. Consider the following fuzzy relation \mathcal{R} between A and X , where $A = X = \{a, b\}$ as follows:

\mathcal{R}	a	b
a	1	0.5
b	0.5	0.6

1. It can be checked easily that \mathcal{R} is a type2 T_M biorder.
2. It can also be verified that \mathcal{R} is a type1 T_{nM} biorder. Further, since T_{nM} is the largest rotational invariant t -norm(cf.[28]) and hence in view of Proposition 7.12, it is also a type2 T_{nM} biorder.

Now if we take $\alpha = 0.6$, then $(a, a) \in \mathcal{R}_\alpha$ and $(b, b) \in \mathcal{R}_\alpha$, but (a, b) and (b, a) both do not belong to \mathcal{R}_α , which implies that $\mathcal{R}_{0.6}$ is not a biorder.

The representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the α -cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al.[1] and Induráin et al.[52]. Motivated by these facts and keeping in view the Proposition 7.15, **from now onwards we mean fuzzy biorders in the sense of Definition 7.13** i.e., a fuzzy relation \mathcal{R} between A and X will be called a fuzzy biorder relation if $\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}$, for each $a, b \in A$ and $x, y \in X$.

In [41], It has been shown that if \mathcal{R} is a fuzzy relation on A , then \mathcal{R} is a fuzzy biorder if and only if \mathcal{R}^d is a fuzzy biorder. In the following proposition, we show that this result also holds good if we take a fuzzy relation between A and X .

Proposition 7.16. *Let \mathcal{R} be a fuzzy relation between A and X . Then \mathcal{R} is a fuzzy biorder between A and X iff \mathcal{R}^d is a fuzzy biorder between X and A .*

Proof. Let \mathcal{R} be a fuzzy biorder between A and X . So, we have

$$\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}, \quad (7.7)$$

for each $a, b \in A$ and $x, y \in X$. Now,

$$\begin{aligned} \min\{\mathcal{R}^d(x, a), \mathcal{R}^d(y, b)\} &= \min\{1 - \mathcal{R}(a, x), 1 - \mathcal{R}(b, y)\} \\ &= 1 - \max\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \\ &\leq 1 - \min\{\mathcal{R}(a, y), \mathcal{R}(b, x)\} \quad (\text{Since } \mathcal{R} \text{ is a fuzzy biorder}) \\ &= \max\{1 - \mathcal{R}(a, y), 1 - \mathcal{R}(b, x)\} \\ &= \max\{\mathcal{R}^d(y, a), \mathcal{R}^d(x, b)\}, \end{aligned}$$

for each $a, b \in A$ and $x, y \in X$. Therefore \mathcal{R}^d is a fuzzy biorder between X and A .

Conversely, assume that \mathcal{R}^d is a fuzzy biorder between X and A . So, we have

$$\min\{\mathcal{R}^d(x, a), \mathcal{R}^d(y, b)\} \leq \max\{\mathcal{R}^d(x, b), \mathcal{R}^d(y, a)\}, \quad (7.8)$$

for each $a, b \in A$ and $x, y \in X$.

Now,

$$\begin{aligned} \min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} &= \min\{1 - \mathcal{R}^d(x, a), 1 - \mathcal{R}^d(y, b)\} \\ &= 1 - \max\{\mathcal{R}^d(x, a), \mathcal{R}^d(y, b)\} \\ &\leq 1 - \min\{\mathcal{R}^d(x, b), \mathcal{R}^d(y, a)\} \quad (\text{Since } \mathcal{R}^d \text{ is a fuzzy biorder}) \\ &= \max\{1 - \mathcal{R}^d(x, b), 1 - \mathcal{R}^d(y, a)\} \\ &= \max\{\mathcal{R}(b, x), \mathcal{R}(a, y)\}, \end{aligned}$$

for each $a, b \in A$ and $x, y \in X$. Therefore \mathcal{R} is a fuzzy biorder between A and X . \square

Union and intersection of fuzzy biorders need not be a fuzzy biorder. This is exhibited through the following examples.

Example 7.2. Let \mathcal{R}_1 and \mathcal{R}_2 be two fuzzy relations between A and X , where $A = X = \{a, b\}$ and are given as follows:

\mathcal{R}_1	a	b
a	0.7	0.7
b	0.3	0.8

and

\mathcal{R}_2	a	b
a	0.6	0.3
b	0.6	0.8

It is easy to verify that \mathcal{R}_1 and \mathcal{R}_2 both are fuzzy biorder between A and X , but $\mathcal{R}_1 \cap \mathcal{R}_2$ which is given as follows:

$\mathcal{R}_1 \cap \mathcal{R}_2$	a	b
a	0.6	0.3
b	0.3	0.8

is not a fuzzy biorder between A and X as

$$\min\{\mathcal{R}(a, a), \mathcal{R}(b, b)\} = 0.6 > \max\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} = 0.3,$$

where $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$.

Example 7.3. Let \mathcal{R}_1 and \mathcal{R}_2 be two fuzzy relations between A and X , where $A = X = \{a, b\}$ and are given as follows:

\mathcal{R}_1	a	b
a	0.7	0.5
b	0.4	0.4

and

\mathcal{R}_2	a	b
a	0.6	0.6
b	0.3	0.8

It is easy to verify that \mathcal{R}_1 and \mathcal{R}_2 both are fuzzy biorder between A and X , but $\mathcal{R}_1 \cup \mathcal{R}_2$ which is given as follows:

$\mathcal{R}_1 \cup \mathcal{R}_2$	a	b
a	0.7	0.6
b	0.4	0.8

is not a fuzzy biorder between A and X as

$$\min\{\mathcal{R}(a, a), \mathcal{R}(b, b)\} = 0.7 > \max\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} = 0.6,$$

where $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.

Definition 7.17. [61, 97] Let T be a left continuous t-norm. Then the *residual implication operator* I_T associated with T is defined as $I_T(x, y) = \sup\{z \in [0, 1] : T(x, z) \leq y\}$, for each $(x, y) \in [0, 1] \times [0, 1]$. For example,

1. If $T = T_P$, then the residual implication operator I_{T_P} associated with T_P is given by

$$I_{T_P}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise,} \end{cases}$$

for each $(x, y) \in [0, 1] \times [0, 1]$.

2. If $T = T_L$, then the residual implication operator I_{T_L} associated with T_L is given by

$$I_{T_L}(x, y) = \min\{1, 1 - x + y\} = \begin{cases} 1, & \text{if } x \leq y \\ 1 - x + y, & \text{otherwise,} \end{cases}$$

for each $(x, y) \in [0, 1] \times [0, 1]$.

Definition 7.18. [34] Let \mathcal{R} be a binary relation between A and X and for any $a \in A$ and $x \in X$, $a\mathcal{R} = \{y \in X : a\mathcal{R}y\}$ and $\mathcal{R}x = \{c \in A : c\mathcal{R}x\}$. Then the binary relations \mathcal{R}_A on A and \mathcal{R}_X on X are defined as follows:

$$\begin{aligned} a\mathcal{R}_A b & \text{ if } b\mathcal{R} \subseteq a\mathcal{R} \\ x\mathcal{R}_X y & \text{ if } \mathcal{R}x \subseteq \mathcal{R}y. \end{aligned}$$

Now, we prove the following:

Proposition 7.19. *Let \mathcal{R} be a fuzzy biorder on A . Then*

1. *If \mathcal{R} is reflexive on A , then \mathcal{R} is strongly S_M -complete and negatively S_M -transitive on A .*
2. *If \mathcal{R} is irreflexive on A , then \mathcal{R} is T_M -asymmetric and T_M -transitive on A .*

Proof. 1. Since T_M has no zero divisors, it follows from Theorem 3 in [32] that \mathcal{R} is strongly S_M -complete on A . Now we prove that \mathcal{R} is negatively S_M -transitive as follows:

Since $\min\{\mathcal{R}(a, b), \mathcal{R}(c, c)\} \leq \max\{\mathcal{R}(a, c), \mathcal{R}(c, b)\}$, for each $a, b, c \in A$ and $\mathcal{R}(a', a') = 1$, for each $a' \in A$, so $\mathcal{R}(a, b) \leq \max\{\mathcal{R}(a, c), \mathcal{R}(c, b)\}$, for each $a, b, c \in A$.

2. We show that $\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} = 0$, for each $a, b \in A$. Since

$$\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} \leq \max\{\mathcal{R}(b, b), \mathcal{R}(a, a)\},$$

for each $a, b \in A$ and by the irreflexivity of \mathcal{R} , $\mathcal{R}(a', a') = 0$, for each $a' \in A$, so we get $\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} \leq 0$ and hence $\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} = 0$, for each $a, b \in A$. Now it follows from Lemma 5.1(i) in [33] that \mathcal{R} is T_M -transitive on A .

□

Definition 7.20. [34] A binary relation \mathcal{R} between A and X is said to be *representable* with respect to \leq (resp., $<$) if there exist two mappings $u : A \rightarrow \mathbb{R}$ and $v : X \rightarrow \mathbb{R}$ such that $(a, x) \in \mathcal{R}$ iff $u(a) \leq v(x)$ (resp., $u(a) < v(x)$), for each $a \in A$ and $x \in X$.

In the case of biorders, we have the following result related to its representability.

Proposition 7.21. [34] *Let \mathcal{R} be a binary relation between A and X , where A and X both are countable. Then the following statements are equivalent:*

1. \mathcal{R} is a biorder.
2. \mathcal{R} is representable with respect to \leq .
3. \mathcal{R} is representable with respect to $<$.

Now we prove the fuzzy analogue of this proposition.

Proposition 7.22. *Let \mathcal{R} be a fuzzy relation between A and X , where A and X both are countable. Then the following statements are equivalent:*

1. \mathcal{R} is a fuzzy biorder.
2. Each \mathcal{R}_α , $\alpha \in [0, 1]$ is representable with respect to \leq .
3. Each \mathcal{R}_α , $\alpha \in [0, 1]$ is representable with respect to $<$.

Proof. (1) \Rightarrow (2) Let \mathcal{R} be a fuzzy biorder between A and X , where A and X both are countable. Then in view of Proposition 7.15, each \mathcal{R}_α , $\alpha \in [0, 1]$ is a biorder between A and X , where A and X both are countable. Therefore from the Proposition 7.21, we get a representation of \mathcal{R}_α with respect to \leq , for each $\alpha \in [0, 1]$.

(2) \Rightarrow (3) Let each \mathcal{R}_α , $\alpha \in [0, 1]$ be representable with respect to \leq . Then from

the Proposition 7.21, each \mathcal{R}_α , $\alpha \in [0, 1]$ is representable with respect to $<$.

(3) \Rightarrow (1) Let each \mathcal{R}_α , $\alpha \in [0, 1]$ be representable with respect to $<$. Then from the Proposition 7.21, each \mathcal{R}_α , $\alpha \in [0, 1]$ is a biorder between A and X . Finally from the Proposition 7.15, we get \mathcal{R} is a fuzzy biorder. \square

Definition 7.23. [34] Let \mathcal{R} be a binary relation between A and X . A subset $M^* \subseteq A \cup X$ is said to be *widely dense* if for each $a \in A$ and $x \in X$, $a\mathcal{R}^c x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in A$, $m^*\mathcal{R}^c x$ and $m^*\mathcal{R}_{Aa}$ or $m^* \in X$, $x\mathcal{R}_X m^*$ and $a\mathcal{R}^c m^*$.

Proposition 7.24. [34] Let \mathcal{R} be a binary relation between A and X . Then the following statements are equivalent:

1. \mathcal{R} is a biorder with a countable widely dense subset M^* .
2. \mathcal{R} is representable with respect to \leq .

Now we give the fuzzy analogue of the this proposition.

Proposition 7.25. Let \mathcal{R} be a fuzzy relation between A and X . Then the following statements are equivalent:

1. \mathcal{R} is a fuzzy biorder and for each $\alpha \in [0, 1]$, there exists a countable subset $M_\alpha^* \subseteq A \cup X$ such that for each $a \in A$ and $x \in X$, $a(\mathcal{R}_\alpha)^c x$ implies that there exists an element $m^* \in M_\alpha^*$ such that either $m^* \in A$, $m^*(\mathcal{R}_\alpha)^c x$ and $m^*(\mathcal{R}_\alpha)_{Aa}$ or $m^* \in X$, $x(\mathcal{R}_\alpha)_X m^*$ and $a(\mathcal{R}_\alpha)^c m^*$.
2. Each \mathcal{R}_α , $\alpha \in [0, 1]$ is representable with respect to \leq .

The proof follows from Proposition 7.15 and 7.24.

Definition 7.26. [34] Let \mathcal{R} be a binary relation between A and X . A subset M^* of $A \cup X$ is said to be *strictly dense* if for all $a \in A$ and $x \in X$, $a\mathcal{R}x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in X$, $a\mathcal{R}m^*$ and $m^*\mathcal{R}_X x$ or $m^* \in A$, $a\mathcal{R}_A m^*$ and $m^*\mathcal{R}x$.

Proposition 7.27. [34] Let \mathcal{R} be a binary relation between A and X . Then the following statements are equivalent:

1. \mathcal{R} is a biorder with a countably strictly dense subset M^* .

2. \mathcal{R} is representable with respect to $<$.

Now we give the fuzzy analogue of this proposition.

Proposition 7.28. *Let \mathcal{R} be a fuzzy relation between A and X . Then the following statements are equivalent:*

1. \mathcal{R} is a fuzzy biorder and for each $\alpha \in [0, 1]$, there exists a countable subset $M_\alpha^* \subseteq A \cup X$ such that for each $a \in A$ and $x \in X$, $a\mathcal{R}_\alpha x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in X$, $a\mathcal{R}_\alpha m^*$ and $m^*(\mathcal{R}_\alpha)_X x$ or $m^* \in A$, $a(\mathcal{R}_\alpha)_A m^*$ and $m^*\mathcal{R}_\alpha x$.

2. Each \mathcal{R}_α , $\alpha \in [0, 1]$ is representable with respect to $<$.

The proof follows from Proposition 7.15 and 7.27.

Definition 7.29. [14] A binary relation \mathcal{R} on A is said to be an *interval order* if:

1. it is asymmetric i.e, if $(a, b) \in \mathcal{R}$, then $(b, a) \notin \mathcal{R}$;
2. it satisfies the Ferrers property i.e, $(a, b) \in \mathcal{R}$ and $(c, d) \in \mathcal{R} \Rightarrow (a, d) \in \mathcal{R}$ or $(c, b) \in \mathcal{R}$.

Now we prove the following result which is a corollary of Proposition 7.15:

Corollary 7.30. *Let \mathcal{R} be a fuzzy relation on A . Then \mathcal{R} is an irreflexive fuzzy biorder on A iff for each $\alpha \in (0, 1]$, \mathcal{R}_α is an interval order on A .*

The proof follows from Proposition 7.15, Proposition 7.19 and the facts that a fuzzy relation is T_M -asymmetric if and only if its α -cuts are asymmetric for every $\alpha \in (0, 1]$ and every T_M -asymmetric fuzzy relation is irreflexive.

Proposition 7.31. [34] *Let \mathcal{R} be a binary relation on A . Then the following statements are equivalent:*

1. \mathcal{R} is an interval order with a countable widely dense subset M^* .
2. There exist two mappings $u : A \rightarrow \mathbb{R}$ and $r : A \rightarrow \mathbb{R}_0^+$ such that for $a, b \in A$, $a\mathcal{R}b \Leftrightarrow u(a) + r(a) \leq u(b)$.

Proposition 7.32. [34] *Let \mathcal{R} be a binary relation on A . Then the following statements are equivalent:*

1. \mathcal{R} is an interval order with a countable strictly dense subset M^* .
2. There exist two mappings $u : A \rightarrow \mathbb{R}$ and $r : A \rightarrow \mathbb{R}^+$ such that for all $a, b \in A$, $a\mathcal{R}b$ iff $u(a) + r(a) < u(b)$.

We now give the following two propositions:

Proposition 7.33. *Let \mathcal{R} be a fuzzy relation on A . Then the following statements are equivalent:*

1. \mathcal{R} is an irreflexive fuzzy biorder such that for each $\alpha \in (0, 1]$, there exists a countable widely dense subset M_α^* .
2. For each $\alpha \in (0, 1]$, there exist two mappings $u_\alpha : A \rightarrow \mathbb{R}$ and $r_\alpha : A \rightarrow \mathbb{R}_0^+$ such that for $a, b \in A$, $a\mathcal{R}_\alpha b \Leftrightarrow u_\alpha(a) + r_\alpha(a) \leq u_\alpha(b)$.

The proof follows from Corollary 7.30 and Proposition 7.31.

Proposition 7.34. *Let \mathcal{R} be a fuzzy relation on A . Then the following statements are equivalent:*

1. \mathcal{R} is an irreflexive fuzzy biorder such that for each $\alpha \in (0, 1]$, there exists a countable strictly dense subset M_α^* .
2. For each $\alpha \in (0, 1]$, there exist two mappings $u_\alpha : A \rightarrow \mathbb{R}$ and $r_\alpha : A \rightarrow \mathbb{R}^+$ such that for $a, b \in A$, $a\mathcal{R}_\alpha b \Leftrightarrow u_\alpha(a) + r_\alpha(a) < u_\alpha(b)$.

The proof follows from Corollary 7.30 and Proposition 7.32.

7.3 Representability of fuzzy weak orders using the residual implication operator

Definition 7.35. [98] *Let T be a left continuous t-norm. A fuzzy relation \mathcal{R} on a finite set A is called T -representable, if there exists a mapping $f : A \rightarrow [0, 1]$ such*

that $\mathcal{R}(a, b) = \mathcal{R}_{T,f}(a, b)$, for each $(a, b) \in A \times A$, where $\mathcal{R}_{T,f}$ is the fuzzy relation defined by $\mathcal{R}_{T,f}(a, b) = I_T(f(b), f(a))$, for each $(a, b) \in A \times A$ and I_T denotes the residual implication operator associated with T .

Theorem 7.36. [97] *A T -representable fuzzy relation on a finite set A is a fuzzy weak order on A with respect to T .*

Theorem 7.37. [98] *Let \mathcal{R} be a fuzzy relation on a finite set A and T be a left continuous t -norm. Then \mathcal{R} is a fuzzy quasi order with respect to T iff it is intersection of a finite family of fuzzy weak orders with respect to T .*

Remark 7.38. Let \mathcal{R} be a T -transitive fuzzy relation on A . Then the following property holds for every $a, b, c \in A$:

$$(\mathcal{R}(a, b) = 1 \text{ and } \mathcal{R}(b, c) = 1) \Rightarrow (\mathcal{R}(a, c) = 1 \text{ and } \mathcal{R}(c, a) \leq \mathcal{R}(c, b)).$$

The proof is trivial.

Baets et al.[11] had proved the following result in case of T_M . These authors have also given an example to show that the converse of the following result does not hold good in that case.

Proposition 7.39. [11] *If a fuzzy relation \mathcal{R} on A is negatively S_M -transitive, then its strict part $P_{\mathcal{R}}$ is T_M -transitive.*

Here we show that none of the implications hold good in case of T_P , through counter examples.

Example 7.4. *Let \mathcal{R} be a fuzzy relation on $A = \{a, b, c\}$ whose matrix representation is as follows:*

\mathcal{R}	a	b	c
a	0	0.84	0.6
b	0.8	0	0.7
c	0.7	0.6	0

Then its strict part $P_{\mathcal{R}}$ is as follows:

$P_{\mathcal{R}}$	a	b	c
a	0	0.84	0
b	0	0	0.7
c	0.7	0	0

Now it is easy to verify that \mathcal{R} is negatively S_P -transitive but $P_{\mathcal{R}}$ is not T_P -transitive as $P_{\mathcal{R}}(a, b) \cdot P_{\mathcal{R}}(b, c) = 0.84 \times 0.7 = 0.58 > P_{\mathcal{R}}(a, c) = 0$.

Example 7.5. Let \mathcal{R} be a fuzzy relation on $A = \{a, b, c\}$ whose matrix representation is as follows:

\mathcal{R}	a	b	c
a	1	0.5	0.9
b	1	1	0.6
c	1	1	1

Then its strict part $P_{\mathcal{R}}$ is as follows:

$P_{\mathcal{R}}$	a	b	c
a	0	0	0
b	1	0	0
c	1	1	0

Now it is easy to verify that $P_{\mathcal{R}}$ is T_P -transitive but \mathcal{R} is not negatively S_P -transitive as $\mathcal{R}(a, b) + \mathcal{R}(b, c) - \mathcal{R}(a, b) \cdot \mathcal{R}(b, c) = 0.5 + 0.6 - 0.3 = 0.8 < \mathcal{R}(a, c) = 0.9$.

Proposition 7.40. If \mathcal{R} is a fuzzy relation on A which is strongly S_M -complete. It satisfies, for each $a, b, c \in A$

$$P_{\mathcal{R}}(a, b) = 1 \text{ and } P_{\mathcal{R}}(b, c) = 1 \Rightarrow \mathcal{R}(c, a) \leq S(\mathcal{R}(c, b), \mathcal{R}(b, a)),$$

if and only if \mathcal{R} is negatively S -transitive.

Proof. To show that \mathcal{R} is negatively S -transitive, we have to show that

$$S(\mathcal{R}(a, b), \mathcal{R}(b, c)) \geq \mathcal{R}(a, c) \quad (7.9)$$

for each $a, b, c \in A$. Assume the contrary. Let

$$\mathcal{R}(a, c) > S(\mathcal{R}(a, b), \mathcal{R}(b, c)) \quad (7.10)$$

for some $a, b, c \in A$. Then from (7.10) and since \mathcal{R} is strongly S_M -complete, we get $\mathcal{R}(b, a) = 1 > \mathcal{R}(a, b)$ and $\mathcal{R}(c, b) = 1 > \mathcal{R}(b, c)$. This implies that $P_{\mathcal{R}}(b, a) = 1$

and $P_{\mathcal{R}}(c, b) = 1$. So by our assumption, we must have

$$\mathcal{R}(a, c) \leq S(\mathcal{R}(a, b), \mathcal{R}(b, c)),$$

which contradicts (7.10).

Conversely, if \mathcal{R} is negatively S -transitive, then by its definition itself: $\mathcal{R}(c, a) \leq S(\mathcal{R}(c, b), \mathcal{R}(b, a))$, for each $a, b, c \in A$. \square

Proposition 7.41. *The union \mathcal{R} of any finite family $\{\mathcal{R}_i\}_{i=1}^n$ of fuzzy weak orders on A with respect to T is a fuzzy quasi-transitive relation on A with respect to T .*

Proof. Obviously, \mathcal{R} is strongly S_M -complete. Let $a, b, c \in A$ be such that $P_{\mathcal{R}}(a, b) = 1$ and $P_{\mathcal{R}}(b, c) = 1$. Now $P_{\mathcal{R}}(a, b) = 1$ implies that $1 > \mathcal{R}(b, a)$ and $P_{\mathcal{R}}(b, c) = 1$ implies that $1 > \mathcal{R}(c, b)$. Then for each $i \in \{1, 2, \dots, n\}$, $\mathcal{R}_i(b, a) < 1$ and since \mathcal{R}_i is strongly S_M -complete so $\mathcal{R}_i(a, b) = 1$. Similarly, $\mathcal{R}(c, b) < 1$ implies that $\mathcal{R}_i(b, c) = 1$. Since each \mathcal{R}_i is a fuzzy weak order with respect to T , so by Remark 7.38, we have

$$\mathcal{R}_i(c, a) \leq \mathcal{R}_i(c, b), \quad \text{for each } i \in \{1, 2, \dots, n\} \quad (7.11)$$

Now, $\mathcal{R}(c, a) = \max_i \mathcal{R}_i(c, a) = \mathcal{R}_t(c, a)$, for some $t \in \{1, 2, \dots, n\}$, then

$$\begin{aligned} S(\mathcal{R}(c, b), \mathcal{R}(b, a)) &\geq S_M(\mathcal{R}(c, b), \mathcal{R}(b, a)) \\ &\geq \mathcal{R}(c, b) \\ &\geq \mathcal{R}_t(c, b) \\ &\geq \mathcal{R}_t(c, a) = \mathcal{R}(c, a) \end{aligned}$$

Therefore, by Proposition 7.40, \mathcal{R} is negatively S -transitive. Thus, $\mathcal{R} = \bigcup_i \mathcal{R}_i$ is strongly S_M -complete as well as negatively S -transitive and hence it is a fuzzy quasi-transitive relation with respect to T . \square

Characterizations for fuzzy weak orders with respect to T_M and T_P on a finite set which are T_M -representable and T_P -representable have been respectively obtained by Baets et al.[11] and Sali et al.[98]. In the following theorem, we have obtained a characterization for fuzzy weak orders with respect to T_L on a finite set which are T_L representable.

Theorem 7.42. *A fuzzy weak order \mathcal{R} with respect to T_L on a finite set A is T_L -representable if and only if*

$$\mathcal{R}(a, b) < 1 \text{ and } \mathcal{R}(b, c) < 1 \Rightarrow \mathcal{R}(a, c) = \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1 \quad (7.12)$$

holds for each $a, b, c \in A$.

To prove the above theorem, we need to prove the following lemma, the proof of which is on the similar lines as that of Lemma 7 in [98].

Lemma 7.43. *Let \mathcal{R} be a reflexive fuzzy relation on a finite set A satisfying (7.12). Then there exists $c \in A$ such that $\mathcal{R}(c, a) = 1$ for each $a \in A$.*

Proof. Assume the contrary, i.e, for each $c \in A$, there exists $a_c \in A$ such that $\mathcal{R}(c, a_c) < 1$. Now define an oriented graph $\vec{G} = (V, E)$, where $V = A$ and there is an arc from a to b iff $\mathcal{R}(a, b) < 1$. By our assumption the out-degree of each node is atleast one, so there is a directed cycle C in \vec{G} , which is obtained by taking connected nodes in the cyclic order. Let the nodes of C be $\{a_1, a_2, \dots, a_n\}$. Then by using (7.12), we have

$$\begin{aligned} 1 = n - (n - 1) &> \mathcal{R}(a_1, a_2) + \mathcal{R}(a_2, a_3) + \dots + \mathcal{R}(a_{n-1}, a_n) + \mathcal{R}(a_n, a_1) - (n - 1) \\ &= \mathcal{R}(a_1, a_1) \end{aligned}$$

which contradicts the reflexivity of \mathcal{R} . □

Proof of the theorem. Let \mathcal{R} be a fuzzy weak order with respect to T_L which is T_L -representable. Then for each $a, b \in A$, $\mathcal{R}(a, b) = \mathcal{R}_{T_L, f}(a, b) = I_{T_L}(f(b), f(a))$ for some mapping $f : A \rightarrow [0, 1]$. If $\mathcal{R}(a, b) < 1$ and $\mathcal{R}(b, c) < 1$, for some $a, b, c \in A$, then $\mathcal{R}(a, b) = 1 - f(b) + f(a)$ and $\mathcal{R}(b, c) = 1 - f(c) + f(b)$ such that $f(b) > f(a)$ and $f(c) > f(b)$. This implies that $f(c) > f(a)$ and hence $\mathcal{R}(a, c) = 1 - f(c) + f(a) = \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1$.

Conversely, let \mathcal{R} be a fuzzy weak order with respect to T_L satisfying (7.12). Then \mathcal{R} is reflexive and so by the previous lemma, there exists $c \in A$ such that $\mathcal{R}(c, a) = 1$, for each $a \in A$. Now define the mapping $f : A \rightarrow [0, 1]$ by $f(a) =$

$\mathcal{R}(a, c)$, for each $a \in A$. We show that

$$\mathcal{R}(a, b) = \mathcal{R}_{T_L, f}(a, b) = I_{T_L}(f(b), f(a)) = \begin{cases} 1, & \text{if } f(b) \leq f(a) \\ 1 - f(b) + f(a), & \text{otherwise,} \end{cases}$$

To prove this we need to consider the following cases:

Case 1: If $f(a) \geq f(b)$ (i.e., $\mathcal{R}(a, c) \geq \mathcal{R}(b, c)$). In this case $I_{T_L}(f(b), f(a)) = 1$, so we have to show that $\mathcal{R}(a, b) = 1$. Assume the contrary that $\mathcal{R}(a, b) < 1$. If $\mathcal{R}(b, c) < 1$, then by (7.12),

$$\begin{aligned} \mathcal{R}(a, c) &= \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1 \\ &\leq \mathcal{R}(a, b) + \mathcal{R}(a, c) - 1 \\ &< \mathcal{R}(a, c) \quad (\text{since } \mathcal{R}(a, b) < 1) \end{aligned}$$

which is a contradiction. Next, if $\mathcal{R}(b, c) = 1$, then $\mathcal{R}(a, c) = 1$ and in view of the previous lemma, $\mathcal{R}(c, b) = 1$. So by the T_L -transitivity of \mathcal{R} , we have

$$\begin{aligned} T_L(\mathcal{R}(a, c), \mathcal{R}(c, b)) &\leq \mathcal{R}(a, b) \\ 1 = \max\{0, \mathcal{R}(a, c) + \mathcal{R}(c, b) - 1\} &\leq \mathcal{R}(a, b) < 1 \end{aligned}$$

which is again a contradiction. Hence in this case we are done.

Case 2: If $f(a) < f(b)$ (i.e., $\mathcal{R}(a, c) < \mathcal{R}(b, c)$). In this case $I_{T_L}(f(b), f(a)) = 1 - f(b) + f(a)$, so we have to show that $\mathcal{R}(a, b) = 1 - f(b) + f(a)$. Let $\mathcal{R}(b, c) < 1$. Now by the T_L -transitivity of \mathcal{R} , we have $T_L(\mathcal{R}(a, b), \mathcal{R}(b, c)) = \max\{0, \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1\} \leq \mathcal{R}(a, c)$. If $\max\{0, \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1\} = \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1$, then $\mathcal{R}(a, b) + \mathcal{R}(b, c) - 1 \leq \mathcal{R}(a, c) < \mathcal{R}(b, c)$ which implies that $\mathcal{R}(a, b) < 1$. If $\max\{0, \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1\} = 0$, then $\mathcal{R}(a, b) \leq 1 - \mathcal{R}(b, c)$ and hence $0 < \mathcal{R}(a, b) < 1$ (since $0 < \mathcal{R}(b, c) < 1$). Now by using (7.12), we have $\mathcal{R}(a, c) = \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1$ which implies that $\mathcal{R}(a, b) = 1 - \mathcal{R}(b, c) + \mathcal{R}(a, c) = 1 - f(b) + f(a)$. Next, if $\mathcal{R}(b, c) = 1$, then by the T_L -transitivity of \mathcal{R} , we have $T_L(\mathcal{R}(a, b), \mathcal{R}(b, c)) = \max\{0, \mathcal{R}(a, b) + \mathcal{R}(b, c) - 1\} \leq \mathcal{R}(a, c)$ which implies that $\mathcal{R}(a, b) \leq \mathcal{R}(a, c)$. Again by using T_L -transitivity of \mathcal{R} , we have $T_L(\mathcal{R}(a, c), \mathcal{R}(c, b)) = \max\{0, \mathcal{R}(a, c) + \mathcal{R}(c, b) - 1\} \leq \mathcal{R}(a, b)$ which implies that $\mathcal{R}(a, c) \leq \mathcal{R}(a, b)$, since $\mathcal{R}(c, b) = 1$ using the previous lemma. Hence $\mathcal{R}(a, b) = \mathcal{R}(a, c) = 1 - \mathcal{R}(b, c) + \mathcal{R}(a, c) = 1 - f(b) + f(a)$. This proves the theorem.

7.4 Conclusion

In this chapter, representability of fuzzy biorders in terms of their α -cuts and fuzzy weak orders using residual implication operators, have been studied. Further, we have shown that union of a finite family of fuzzy weak orders with respect to a t-norm T is fuzzy quasi-transitive with respect to T and counter examples have been produced to show that unions and intersections of fuzzy biorders need not be fuzzy biorder. In the last theorem, we have also obtained a characterization for a T_L -representable fuzzy weak orders.