Chapter 2

Two-Dimensional Shifted Legendre Polynomial Collocation Method for Electromagnetic Waves in Dielectric Media via Almost Operational Matrices

2.1 Introduction

In recent years there has been a growing interest in the areas of fractional calculus (see for instant [82], [83], [84], [85], [86]). Fractional derivative has application in physics, mathematics, engineering and applied science as physical sciences phenomena in area like Damping law [87], Diffusion process [88], Electrochemistry [89], Arterial sciences [90], the theory of Ultra-slow processes [91], etc. Now a days, we can find applications and models involving fractional derivative in Probability, Astrophysics, Anomalous diffusion, Chemical physics, Finance, Robust control, Electromagnetism, Optic and signal processing, Seismic analysis, Viscoelasticity, Acoustics, Biology,etc. In this chapter, we have discussed a fractional partial differential equations which arises from electromagnetic waves in dielectric media (EMWDM) (see [92]).

As we known that the dielectric relaxation in solids, which is described by the complex frequency dependent dielectric sensitivity $\chi^{\sim}(\omega) = \chi'(\omega) - i\chi''(\omega)$, establish the universal power law dependence:

$$\begin{aligned} \chi'(0) - \chi'(\omega) \sim \omega^m, \chi''(\omega) \sim \omega^m, \omega << \omega_p, \\ \chi'(0) \sim \omega^{n-1}, \chi''(\omega) \sim \omega^{n-1}, \omega >> \omega_p, \end{aligned} \tag{2.1}$$

where, $\chi'(0)$ is the static polarization, $0 < n, m < 1$ and ω_p is the loss peak frequency.

As a result, we obtained fractional partial differential equations (FPDEs) for EMWDM using the Maxwell equation. Such a power law dependence in the frequency domain results in the connection between the electric field **E** and the polarization density **B** expressed as a weakly singular Liouville integral and, as a result, the field equations take a form of FPDEs (see [92]):

$$({}_0D_t^{\alpha}\mathbf{E})(t,r) - \lambda_1({}_0D_t^{\beta}\mathbf{E})(t,r) + \lambda_2(\text{grad div}\mathbf{E}(t,r) - \nabla^2\mathbf{E}(t,r)) = -\mu\lambda_2\frac{\partial j(t,r)}{\partial t}, \tag{2.2}$$

$$({}_0D_t^{\alpha}\mathbf{B})(t,r) - \lambda_1({}_0D_t^{\beta}\mathbf{B})(t,r) - \lambda_2\nabla^2\mathbf{B}(t,r) = \mu\lambda_2\text{curlj}(t,r). \tag{2.3}$$

Here the constant coefficient λ_1 and λ_2 depend on the frequency independent properties of a medium, μ is the magnetic constant; $1 \leq \beta < \alpha < 3$. Note that such a form allows simultaneous consideration of both systems, before and after the peak frequency and the conversion between them. The last aspect can be perspectively important due to its generality for the consideration of non-trivial transitions, which occurs in modern composite materials formed by ferroelectric nanoparticles in a polymer matrix [93].

For the simplicity of a pure mathematical example, we take $Eq.$(2.3), assuming $\mathbf{B} = (0, u(t,x), 0), \mu\lambda_2\text{curl}j = (0, f(t,x), 0)$. Then, $Eq.$(2.3) will be reduced to as follows:

$$({}_0D_t^{\alpha}u)(t,x) - \lambda_1({}_0D_t^{\beta}u)(t,x) - \lambda_2\nabla^2 u(t,x) = f(t,x), \tag{2.4}$$

with initial condition

$$u(t,x) = 0, \forall\ t \leqslant 0, \quad u(t,x) \neq 0, \forall\ t > 0, 0 < x < 1.$$

Here both fractional derivative existent in $Eq.(2.4)$ are defined in the Riemann-Liouville derivative sense and λ_1, λ_2 and $f(t,x)$ are known. In this chapter, our aim is to find unknown function $u(t,x)$. The scalar $Eq.(2.4)$ can be considered as a generalization of Szabo equation [94], which describes lossy propagation of acoustical waves in media with power law attenuation. It has more restricted form in comparison with $Eq.(2.4)$, since the case $\alpha = 2$ was considered and extensively studied [95]- [96] only. The differential equations with derivatives of non integer order proposed for describing the EMWDM solve numerically by the Grunwald-Lenikov discretization scheme [97].

Since, we converted our problem $Eq.(2.4)$ into a fractional partial integro-differential equations (FPIDEs) then this FPIDEs (with $1 \leq \beta < \alpha < 3$, in our case) is said to be Volterra type. Although the theory of Volterra FPIDEs has undergone rapid development during last four decades. So, it remains wide open for further progress.

Mostly, we used orthogonal function as Legendre polynomials [98], sine-cosine function [99], block pulse function [36], function [100], Chebyshev [101], Laguerre polynomials [102], etc. The main aim of using an orthogonal basis is that our proposed problem reduces to a system of linear or nonlinear algebraic equations. This can be done by the solution of the proposed problem using the operational matrices (see for instant [103], [104], [105], [106], [107], [108],). An important property of operational matrix method is that it simplifies the problem and also speedup the computation.

2.1.1 Function approximation

Let $\{\Phi_{00}(t,x), \Phi_{01}(t,x), ..., \Phi_{0M}(t,x), ..., \Phi_{N0}(t,x), \Phi_{N1}(t,x), ..., \Phi_{NM}(t,x)\} \subset L^2(\Omega)$ be the set of two-dimensional shifted Legendre polynomials (TDSLPs) defined in $Eq.(1.5)$ then we have

$$X_{NM} = span\left\{\Phi_{00}, \Phi_{01}, ..., \Phi_{0N}, \Phi_{10}, \Phi_{11}, ..., \Phi_{1N}, ..., \Phi_{N0}, \Phi_{N1}, ..., \Phi_{NM}\right\}.$$

Let $f(t,x)$ be an arbitrary function in $L^2(\Omega)$. Since X_{NM} is finite dimensional vector space so $f(t,x)$ has a unique best approximation as follows:

$$f(t,x) \approx \sum_{n=0}^{N}\sum_{m=0}^{M} f_{nm}\Phi_{nm}(t,x) = F^T\Phi(t,x) = \Phi(t,x)^T F,$$

where, F and $\Phi(t,x)$ are $(N+1)^2 \times 1$ vector given by

$$F = [f_{00}, f_{01}, ..., f_{0M}, f_{10}, f_{11}, ..., f_{1M}, ..., f_{N0}, f_{N1}, ..., f_{NM}]^T,$$

$$\Phi = [\Phi_{00}, \Phi_{01}, ..., \Phi_{0M}, \Phi_{10}, \Phi_{11}, ..., \Phi_{1M}, ..., \Phi_{N0}, \Phi_{N1}, ..., \Phi_{NM}]^T,$$

and

$$\Phi_{nm}(t,x) = \Phi_n(t,x)\Phi_m(t,x).$$

2D shifted Legendre function coefficients f_{nm} are obtained by

$$f_{nm} = \frac{\langle f(t,x), \Phi_{nm}(t,x)\rangle}{\|\Phi_{nm}(t,x)\|_2^2}.$$

Theorem 2.1.1 *Let $f_N(t,x) = F_N^T\Phi(t,x)$ be the TDSLPs expansion of real sufficiently smooth function $f(t,x)$ in $L^2(\Omega)$, where*

$$F_N = [f_{00}, f_{01}, ..., f_{0M}, f_{10}, f_{11}, ..., f_{1M}, ..., f_{N0}, f_{N1}, ..., f_{NM}]^T,$$

and

$$f_{nm} = (2n+1)(2m+1)\int_0^1\int_0^1 f(t,x)\Phi_{nm}(t,x)dtdx,$$

then there is a real number δ such that

$$\|f(t,x) - f_N(t,x)\|_2 \leq \frac{\delta}{(N+1)!2^{2N+1}}.$$

Proof See [103].

Remark 4 *By theorem 2.1.1 we conclude as follows:*

$$\|f(t,x) - f_N(t,x)\|_2 \to 0 \quad as \quad N \to \infty.$$

2.2 Operational matrices

2.2.1 Operational matrix of differentiation

The relation between one dimensional shifted Legendre polynomial and their derivatives can be described as follows:

$$[\Phi_0(x)\Phi_1(x)\Phi_2(x)...\Phi_N(x)]' = [\Phi_0(x)\Phi_1(x)\Phi_2(x)...\Phi_N(x)]D_L$$

Let $\Phi(t,x)$ be TDSLPs defined in $Eq.$(1.5) then

$$\begin{aligned}\frac{\partial}{\partial x}\Phi(t,x) &= \frac{\partial}{\partial x}(\Phi(t)\otimes\Phi(x))\\ &= \Phi(t)\otimes\frac{\partial}{\partial x}\Phi(x)\\ &= (I\Phi(t))\otimes(D_L\Phi(x))\\ &= (I\otimes D_L)(\Phi(t)\otimes\Phi(x))\\ &= D_{L,x}\Phi(t,x),\end{aligned}$$

so,

$$\frac{\partial}{\partial x}\Phi(t,x) = D_{L,x}\Phi(t,x), \tag{2.5}$$

where, $D_{L,x} = I \otimes D_L$,

again,

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(t,x) &= \frac{\partial}{\partial t}(\Phi(t)\otimes\Phi(x))\\ &= (D_L\Phi(t))\otimes(I\Phi(x))\\ &= (D_L\otimes I)(\Phi(t)\otimes\Phi(x))\\ &= D_{L,t}\Phi(t,x),\end{aligned}$$

where, $\quad D_{L,t} = D_L \otimes I$,

so,

$$\frac{\partial}{\partial t}\Phi(t,x) = D_{L,t}\Phi(t,x), \tag{2.6}$$

where, D_L and I are $(N+1)\times(N+1)$ matrices.

When N is even,

$$D_L = \begin{pmatrix} 0 & 2 & 0 & 2 & \dots & 2 & 0\\ 0 & 0 & 6 & 0 & \dots & 0 & 6\\ 0 & 0 & 0 & 2 & \dots & 10 & 6\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 4N-6 & 0\\ 0 & 0 & 0 & 0 & \dots & 0 & 4N-2\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

when N is odd,

$$D_L = \begin{pmatrix} 0 & 2 & 0 & 2 & \dots & 2 & 0 \\ 0 & 0 & 6 & 0 & \dots & 6 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 5 \\ 0 & 0 & 0 & 0 & \dots & 14 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 4N-6 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 4N-2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

we call D_L, Legendre operational matrix differentiation. Trivially $\Phi_n^k(x) = \Phi_n(x)D_L^k$ for all positive integer k, where $\Phi_n^k(x)$ is the k^{th} derivative of $\Phi_n(x)$. Since, in this chapter, we deals with two variable functions, then the above mentioned matrix must be extended to product of two matrices. So $\Phi^{ij}(t,x) = \Phi(t,x)D^i_{L,t}D^j_{L,x}$, where $i, j = 0, 1, 2, ..., N$.

2.2.2 Almost operational matrix of integration with singularity for variable t

Let $\Phi(t,x)$ be TDSLPs defined in $Eq.(1.5)$ then,we get

$$\int_0^t \frac{\Phi(s,x)}{(t-s)^{\{\alpha\}}}ds = \int_0^t \frac{\Phi(s)\otimes\Phi(x)}{(t-s)^{\{\alpha\}}}ds$$

$$= \int_0^t \frac{\Phi(s)}{(t-s)^{\{\alpha\}}}ds \otimes \Phi(x)$$

$$\approx (Q_\alpha\Phi(t)) \otimes (I\Phi(x))$$

$$= (Q_\alpha \otimes I)(\Phi(t)\otimes\Phi(x))$$

$$= Q_{\alpha,L}\Phi(t,x),$$

so,

$$\int_0^t \frac{\Phi(s,x)}{(t-s)^{\{\alpha\}}}ds \approx Q_{\alpha,L}\Phi(t,x), \tag{2.7}$$

where, $Q_{\alpha,L} = Q_\alpha \otimes I$.

Similarly, we find operational matrix as follows:

$$\begin{aligned}
\int_0^t \frac{\Phi(s,x)}{(t-s)^{\{\beta\}}}ds &= \int_0^t \frac{\Phi(s)\otimes\Phi(x)}{(t-s)^{\{\beta\}}}ds \\
&= \int_0^t \frac{\Phi(s)}{(t-s)^{\{\beta\}}}ds \otimes \Phi(x) \\
&\approx (Q_\beta\Phi(t)) \otimes (I\Phi(x)) \\
&= (Q_\beta \otimes I)(\Phi(t)\otimes\Phi(x)) \\
&= Q_{\beta,L}\Phi(t,x),
\end{aligned}$$

so,

$$\int_0^t \frac{\Phi(s,x)}{(t-s)^{\{\beta\}}}ds \approx Q_{\beta,L}\Phi(t,x), \tag{2.8}$$

where, $Q_{\beta,L} = Q_\beta \otimes I$.

2.3 Method of the solution

Using $Eq.(1.17)$ in $Eq.(2.5)$, we get

$$\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{u(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{u(s,x)}{(t-s)^{\{\beta\}}}ds$$
$$-\lambda_2(u_{tt}(t,x)+u_{xx}(t,x)) = f(t,x). \tag{2.9}$$

To find the solution of converted fractional partial integro-differential equations (FPIDEs) $Eq.(2.9)$ by using the collocation method, we will first obtain matrix form of FPIDEs. In this way, we suppose the approximation of $u(t,x)$ as follows

$$u(t,x) \approx \sum_{n=0}^{N}\sum_{m=0}^{M} C_{n,m}\Phi_{n,m}(t,x) = C^T\Phi(t,x), \tag{2.10}$$

where, $C_{n,m}$ is unknown for $n = 0,1,2,...,N$, $m = 0,1,...,M$.

Now, partially differentiate twice $Eq.(2.10)$ with respect to t as follows:

$$u_{tt}(t,x) \approx C^T(D_{L,t})^2\Phi(t,x), \tag{2.11}$$

similarly,

$$u_{xx}(t,x) \approx C^T(D_{L,x})^2\Phi(t,x), \tag{2.12}$$

$$(\frac{\partial}{\partial t})^{[\alpha]+1}\Phi(t,x) \approx D_{L,\alpha}\Phi(t,x), \tag{2.13}$$

$$(\frac{\partial}{\partial t})^{[\beta]+1}\Phi(t,x) \approx D_{L,\beta}\Phi(t,x). \tag{2.14}$$

Grouping $Eq.(2.9)$ and $Eq.(2.10)$ as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{C^T\Phi(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{C^T\Phi(s,x)}{(t-s)^{\{\beta\}}}ds \\ -\lambda_2(u_{tt}(t,x)+u_{xx}(t,x)) = f(t,x). \tag{2.15}$$

Using $Eqs.(2.11) - (2.12)$ in $Eq.(2.15)$, we get

$$\frac{C^T}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{\Phi(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{\Phi(s,x)}{(t-s)^{\{\beta\}}}ds$$
$$-\lambda_2(C^T(D_{L,t})^2\Phi(t,x) + C^T(D_{L,x})^2\Phi(t,x)) = F^T\Phi(t,x), \tag{2.16}$$

grouping $Eqs.(2.7) - (2.8)$ and $Eq.(2.16)$ as follows

$$\frac{C^T}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1} Q_{\alpha,L}\Phi(t,x) - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1} Q_{\beta,L}\Phi(t,x)$$
$$-\lambda_2\left(C^T(D_{L,t})^2 + C^T(D_{L,x})^2\right)\Phi(t,x) = F^T\Phi(t,x),$$

or,

$$\frac{C^T}{\Gamma(1-\{\alpha\})}Q_{\alpha,L}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\Phi(t,x) - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}Q_{\beta,L}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\Phi(t,x)$$
$$-\lambda_2 C^T((D_{L,t})^2 + (D_{L,x})^2)\Phi(t,x) = F^T\Phi(t,x). \tag{2.17}$$

Grouping $Eqs.(2.13) - (2.14)$ and $Eq.(2.17)$ as follows:

$$\left(\frac{C^T}{\Gamma(1-\{\alpha\})}Q_{\alpha,L}D_{L,\alpha} - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}Q_{\beta,L}D_{L,\beta} - \lambda_2 C^T(D_{L,t})^2 - \lambda_2 C^T(D_{L,x})^2\right)$$
$$\Phi(t,x) = F^T\Phi(t,x),$$

or,

$$C^T\left(\frac{1}{\Gamma(1-\{\alpha\})}Q_{\alpha,L}D_{L,\alpha} - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}Q_{\beta,L}D_{L,\beta} - \lambda_2(D_{L,t})^2 - \lambda_2(D_{L,x})^2\right) = F^T,$$

so,

$$C^T = F^T\left[\frac{1}{\Gamma(1-\{\alpha\})}Q_{\alpha,L}D_{L,\alpha} - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}Q_{\beta,L}D_{L,\beta} - \lambda_2(D_{L,t})^2 - \lambda_2(D_{L,x})^2\right]^{-1}. \tag{2.18}$$

Grouping $Eq.$(2.10) and $Eq.$(2.18), we get

$$u(t,x) = F^T \left[\frac{1}{\Gamma(1-\{\alpha\})} Q_{\alpha,L} D_{L,\alpha} - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})} Q_{\beta,L} D_{L,\beta} - \lambda_2 (D_{L,t})^2 - \lambda_2 (D_{L,x})^2 \right]^{-1} \Phi(t,x). \tag{2.19}$$

Now, we use the collocation method for solving $Eq.$(2.19). For this, we suppose $t = \{t_i\}_{i=0}^{N} = \frac{i}{N}$ and $x = \{x_j\}_{j=0}^{N} = \frac{j}{N}$ are the set of $(N+1)$ nodes. We substitute these nodes in $Eq.$(2.19) and then we find the numerical solution of $Eq.$(2.9) and hence $Eq.$(2.5) solved numerically.

2.4 Convergence Analysis

Theorem 2.4.1 *Let* $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)_N$, $\tau \geq 0$ *be the approximation of* $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)$ *and assume that the mixed derivative of* $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)$ *is bounded by a constant* K*.i.e.* $\mid \frac{\partial^{\tau+4} u(t,x)}{\partial t^2 \partial x^{\tau+2}} \mid < K$*, then we have the following upper bound of error:*

$$\left\| \left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right) - \left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)_N \right\|_{L^2}^2 < \frac{K^2 \wp^2}{65536},$$

where, $\wp$ *is a polygamma function* ,$\wp = F_3(-3/2+N)$.

Proof Let $\left(\frac{\partial^\tau u}{\partial x^\tau}\right)_N = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{nm}\Phi_{nm}(t,x)$,
after truncating upto N level, we get

$$\left(\frac{\partial^\tau u}{\partial x^\tau}\right)_N \approx \sum_{n=0}^{N}\sum_{m=0}^{N} a_{nm}\Phi_{nm}(t,x).$$

Now,

$$\left(\frac{\partial^\tau u}{\partial x^\tau}\right) - \left(\frac{\partial^\tau u}{\partial x^\tau}\right)_N = \sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty} a_{nm}\Phi_{nm}(t,x),$$

taking L^2 norm both side, we get

$$\| \left(\frac{\partial^\tau u}{\partial x^\tau}\right) - \left(\frac{\partial^\tau u}{\partial x^\tau}\right)_N \|^2 = \int_0^1\int_0^1 \left(\left(\frac{\partial^\tau u}{\partial x^\tau}\right) - \left(\frac{\partial^\tau u}{\partial x^\tau}\right)_N\right)^2 dtdx,$$

or,

$$\| \left(\frac{\partial^\tau u}{\partial x^\tau}\right) - \left(\frac{\partial^\tau u}{\partial x^\tau}\right)_N \|^2 = \int_0^1\int_0^1 \left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty} a_{nm}\Phi_{nm}(t,x)\right)^2 dtdx,$$

or,

$$\| \left(\frac{\partial^\tau u}{\partial x^\tau}\right) - \left(\frac{\partial^\tau u}{\partial x^\tau}\right)_N \|^2 = \sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty} \frac{a_{nm}^2}{(2n+1)(2m+1)}, \tag{2.20}$$

where,

$$\begin{aligned} a_{nm} &= (2n+1)(2m+1)\int_0^1\int_0^1 \frac{\partial^\tau u(t,x)}{\partial x^\tau}\Phi_{nm}(t,x)dtdx,\\ &= (2n+1)(2m+1)\int_0^1\int_0^1 \frac{\partial^\tau u(t,x)}{\partial x^\tau}P_n(t)P_m(x)dtdx,\\ &= (2n+1)(2m+1)\int_0^1\int_0^1 \frac{\partial^\tau u(t,x)}{\partial x^\tau}L_n(2t-1)L_m(2x-1)dtdx. \end{aligned}$$

Let $2t-1=s$ then, we get

$$\begin{aligned} a_{nm} &= (2n+1)(2m+1)\int_0^1 L_m(2x-1)\int_{-1}^1 \frac{\partial^\tau u(\frac{s+1}{2},x)}{\partial x^\tau}L_n(s)dsdx,\\ &= \frac{(2n+1)(2m+1)}{2(2n+1)}\int_0^1 L_m(2x-1)\int_{-1}^1 \frac{\partial^\tau u(\frac{s+1}{2},x)}{\partial x^\tau}(L_{n+1}(s)-L_{n-1}(s))dsdx,\\ &= -\frac{2m+1}{4}\int_0^1 L_m(2x-1)\int_{-1}^1 \frac{\partial^{\tau+1} u(\frac{s+1}{2},x)}{\partial s\partial x^\tau}(L_{n+1}(s)-L_{n-1}(s))dsdx,\\ &= -\frac{2m+1}{4}\int_0^1 L_m(2x-1)\int_{-1}^1 \frac{\partial^{\tau+1} u(\frac{s+1}{2},x)}{\partial s\partial x^\tau}(\frac{L_{n+2}(s)-L_n(s)}{2n+3} - \frac{L_{n-1}(s)-L_{n-2}(s)}{2n-1})'dsdx,\\ &= \frac{2m+1}{8}\int_0^1 L_m(2x-1)\int_{-1}^1 \frac{\partial^{\tau+2} u(\frac{s+1}{2},x)}{\partial s^2\partial x^\tau}(\frac{L_{n+2}(s)-L_n}{2n+3} - \frac{L_{n-1}(s)-L_{n-2}(s)}{2n-1})dsdx. \end{aligned}$$

Let $2x-1=r$, then similar processing as done above, we get,

$$a_{nm}=\frac{1}{64}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{\tau+4}u(\frac{s+1}{2},\frac{r+1}{2})}{\partial s^2\partial r^{\tau+2}}\sigma_n(s)\sigma_m(t)dsdr,$$

where,

$$\sigma_n(s)=\frac{L_{n+2}(s)-L_n(s)}{2n+3}-\frac{L_n(s)-L_{n-2}(s)}{2n-1},$$

and,

$$\sigma_m(s)=\frac{L_{m+2}(s)-L_m(s)}{2m+3}-\frac{L_m(s)-L_{n-2}(s)}{2m-1}.$$

Now,

$$\begin{aligned}
\mid a_{nm}\mid^2 &= \frac{1}{4096}\mid\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{\tau+4}u(\frac{s+1}{2},\frac{r+1}{2})}{\partial s^2\partial r^{\tau+2}}\sigma_n(s)\sigma_m(r)dsdr\mid^2,\\
\leq \frac{1}{4096}\int_{-1}^{1}\int_{-1}^{1}\mid\frac{\partial^{\tau+4}u(\frac{s+1}{2},\frac{r+1}{2})}{\partial s^2\partial r^{\tau+2}}\mid^2 dsdr &\int_{-1}^{1}\int_{-1}^{1}\mid\sigma_n(s)\sigma_m(r)\mid^2 dsdr\\
&\leq\frac{4K^2}{4096)}\int_{-1}^{1}\int_{-1}^{1}\mid\sigma_n(s)\sigma_m(r)\mid^2 dsdr,
\end{aligned}$$

where, $\mid\frac{\partial^{\tau+4}u(\frac{s+1}{2},\frac{r+1}{2})}{\partial s^2\partial r^{\tau+2}}\mid< K$,

so,

$$\mid a_{nm}\mid^2\leq\frac{K^2}{1024}\int_{-1}^{1}\mid\sigma_n(s)\mid^2 ds\int_{-1}^{1}\mid\sigma_m(r)\mid^2 dr. \tag{2.21}$$

Now,

$$\int_{-1}^{1} \mid \sigma_n(s) \mid^2 ds = \int_{-1}^{1} \left(\frac{L_{n+2}(s) - L_n(s)}{2n+3} - \frac{(L_n(s) - L_{n-2}(s)}{2n-1} \right)^2 ds$$
$$= \int_{-1}^{1} (\frac{(2n-1)L_{n+2}(s) - (4n+2)L_n(s) + (2n-3)L_{n-2}(s)}{(2n+3)(2n-1)})^2 ds,$$
$$< \int_{-1}^{1} (\frac{(2n-1)^2 L_{n+2}^2(s) + (4n+2)^2 L_n^2(s) + (2n+3)^2 L_{n-2}^2(s)}{(2n+3)^2(2n-1)^2}) ds,$$
$$< \frac{12(2n+3)^2}{(2n+3)^2(2n-1)^2(2n-3)}.$$

Thus ,we get

$$\int_{-1}^{1} \mid \sigma_n(s) \mid^2 ds < \frac{12}{(2n-1)^2(2n-3)}. \tag{2.22}$$

Similarly,

$$\int_{-1}^{1} \mid \sigma_m(r) \mid^2 dr < \frac{12}{(2m-1)^2(2m-3)}. \tag{2.23}$$

Grouping $Eq.(2.21)$, $Eq.(2.22)$ and $Eq.(2.23)$,we get

$$\mid a_{nm} \mid^2 < \frac{9K^2}{64(2n-3)^4(2m-3)^4}, \tag{2.24}$$

$$\frac{\mid a_{nm} \mid^2}{(2n+1)(2m+1)} < \frac{9K^2}{64(2n-3)^4(2m-3)^4},$$

which implies,

$$\sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{\mid a_{nm} \mid^2}{(2n+1)(2m+1)} < \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{9K^2}{64(2n-3)^4(2m-3)^4}, \tag{2.25}$$

so,

$$\sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{\mid a_{nm} \mid^2}{(2n+1)(2m+1)} < \frac{9K^2}{64} \frac{1}{9216} \left(F_3(\frac{-1}{2} + N) \right)^2. \tag{2.26}$$

$Eqs.$(2.20) and (2.26) together implies:

$$\parallel \left(\frac{\partial u^{\tau}(t,x)}{\partial x^{\tau}}\right) - \left(\frac{\partial u^{\tau}(t,x)}{\partial x^{\tau}}\right)_N \parallel_{L^2}^2 < \frac{9K^2}{64}\frac{1}{9216}\left(F_3(\frac{-1}{2}+N)\right)^2,$$

i.e.

$$\parallel \left(\frac{\partial u^{\tau}(t,x)}{\partial x^{\tau}}\right) - \left(\frac{\partial u^{\tau}(t,x)}{\partial x^{\tau}}\right)_N \parallel_{L^2}^2 < \frac{K^2}{65536}\wp^2,$$

where, $\wp = F_3\left(\frac{-1}{2}+N\right)$.

Remark 5 *By theorem* (2.5.1)*, we conclude that*

$$\parallel \left(\frac{\partial u^{\tau}(t,x)}{\partial x^{\tau}}\right) - \left(\frac{\partial u^{\tau}(t,x)}{\partial x^{\tau}}\right)_N \parallel_{L^2}^2 \to 0, \quad as \quad N \to \infty.$$

Lemma 2.4.2 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $(u_{xx})_N(t,x)$ be the approximation of $u_{xx}(t,x)$. Assume that the mixed second derivative of $u(t,x)$ is bounded by a constant K_1.*
i.e.$|(\frac{\partial u^6(t,x)}{\partial t^2 \partial x^4})| < K_1$, then ,we have the following upper bound of error :

$$\parallel u_{xx} - (u_{xx})_N \parallel < \frac{K_1^2\wp^2}{65536},$$

where, $\wp = F_3(\frac{-1}{2}+N)$.

Proof Proof is similar to theorem 2.5.1 after taking $\tau = 2$.

Remark 6 *By lemma* (2.5.2)*, we conclude that*

$$\parallel u_{xx} - (u_{xx})_N \parallel_{L^2}^2 \to 0, \quad as \quad N \to \infty.$$

Lemma 2.4.3 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $(u_{tt})_N(t,x)$ be the approximation of $u_{tt}(t,x)$. Assume that the mixed second derivative of $u(t,x)$ is bounded by a constant K_2. i.e.$|(\frac{\partial u^6(t,x)}{\partial t^2 \partial x^4})| < K_2$, then ,we have the following upper bound of error :*

$$\parallel u_{tt} - (u_{tt})_N \parallel_{L^2}^2 < \frac{K_2^2\wp^2}{65536},$$

where, $\wp = F_3(\frac{-1}{2} + N)$.

Proof Similar proof as theorem 2.5.1.

Remark 7 *By lemma* 2.5.3*, we conclude that*

$$\| u_{tt} - (u_{tt})_N \|_{L^2}^2 \to 0, \quad as \quad N \to \infty.$$

2.5 Error Analysis

2.5.1 Error bound

Let $u_N(t,x)$, $(u_{tt})_N(t,x)$ and $(u_{xx})_N(t,x)$ be the N^{th} approximation of $u(t,x)$, $u_{tt}(t,x)$ and $u_{xx}(t,x)$ respectively, $e_N = u(t,x) - u_N(t,x)$ be the bounded error and then substituting these approximation in $Eq.$(2.9), we get

$$\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{u_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{u_N(s,x)}{(t-s)^{\{\beta\}}}ds$$
$$-\lambda_2\left((u_{tt})_N(t,x) + (u_{xx})_N(t,x)\right) = f_N(t,x). \tag{2.27}$$

Subtracting $Eq.$(2.27) from $Eq.$(2.9) as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}$$
$$\int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\beta\}}}ds = \lambda_2(u_{tt}(t,x) - (u_{tt})_N(t,x) \tag{2.28}$$
$$+u_{xx}(t,x) - (u_{xx})_N(t,x)) + f(t,x) - f_N(t,x).$$

Taking L^2-norm both side in $Eq.$(2.28) as follows:

$$\|\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}$$
$$\int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\beta\}}}ds\| = \|\lambda_2(u_{tt}(t,x) - (u_{tt})_N(t,x)$$
$$+u_{xx}(t,x) - (u_{xx})_N(t,x)) + f(t,x) - f_N(t,x)\|,$$

or,

$$|\frac{1}{\Gamma(1-\{\alpha\})}|\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{\|u(s,x)-u_N(s,x)\|}{(t-s)^{\{\alpha\}}}ds-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{\|u(s,x)-u_N(s,x)\|}{(t-s)^{\{\beta\}}}ds \leq |\lambda_2|(\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| + \|u_{xx}(t,x)-(u_{xx})_N(t,x)\|)+\|f(t,x)-f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{1}{(t-s)^{\{\alpha\}}}ds-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{1}{(t-s)^{\{\beta\}}}ds) \leq |\lambda_2|(\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| + \|u_{xx}(t,x)-(u_{xx})_N(t,x)\|)+\|f(t,x)-f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}(\frac{t^{1-\{\alpha\}}}{\{\alpha\}-1})-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}(\frac{t^{1-\{\beta\}}}{\{\beta\}-1}) \leq |\lambda_2|(\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| + \|u_{xx}(t,x)-(u_{xx})_N(t,x)\|)+\|f(t,x)-f_N(t,x)\|, \quad (2.29)$$

since, $1 \leq \beta < \alpha < 3$ then

$$[\alpha]=\begin{cases} 1 \ , & if \ \ 1 \leq \alpha < 2, \\ 2 \ , & if \ \ 2 \leq \alpha < 3 \end{cases} \quad (2.30)$$

and also,

$$[\beta]=\begin{cases} 1 \ , & if \ \ 1 \leq \beta < 2, \\ 2 \ , & if \ \ 2 \leq \beta < 3. \end{cases} \quad (2.31)$$

After grouping the $Eqs.(2.29)-(2.31)$, we get three cases for the error bound as follows:

Case 1. If $1 < \alpha < 2$ and $1 \leq \beta < 2$ then $Eq.(2.29)$ as follows:

$$\|u(t,x) - u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}| \left(\frac{\partial}{\partial t}\right)^2 (\frac{t^{1-\{\alpha\}}}{\{\alpha\}-1}) - |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}| \left(\frac{\partial}{\partial t}\right)^2 (\frac{t^{1-\{\beta\}}}{\{\beta\}-1})$$
$$\leq |\lambda_2|(\|u_{tt}(t,x) - (u_{tt})_N(t,x)\| + \|u_{xx}(t,x) - (u_{xx})_N(t,x)\|) + \|f(t,x) - f_N(t,x)\|,$$

or,

$$\|u(t,x) - u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|(-\{\alpha\})t^{-1-\{\alpha\}} - |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)(-\{\beta\})t^{-1-\{\beta\}})$$
$$\leq |\lambda_2|\|u_{tt}(t,x) - (u_{tt})_N(t,x)\| + |\lambda_2|\|u_{xx}(t,x) - (u_{xx})_N(t,x)\| + \|f(t,x) - f_N(t,x)\|,$$

or,

$$\|u(t,x) - u_N(t,x)\| \leq \frac{\|f(t,x) - f_N(t,x)\| + |\lambda_2|\|u_{tt}(t,x) - (u_{tt})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(-\{\alpha\})t^{-1-\{\alpha\}} + |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)(\{\beta\})t^{-1-\{\beta\}})}$$
$$+ \frac{|\lambda_2|\|u_{xx}(t,x) - (u_{xx})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(-\{\alpha\})t^{-1-\{\alpha\}} + |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)(\{\beta\})t^{-1-\{\beta\}})}. \tag{2.32}$$

Since,

$$(|\frac{1}{\Gamma(1-\{\alpha\})}|(-\{\alpha\})t^{-1-\{\alpha\}} + |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)(\{\beta\})t^{-1-\{\beta\}}) \neq 0 \quad \forall\ 0 < t < 1.$$

Grouping remark 4, remark 6 remark 7 and $Eq.(2.32)$ with $N \to \infty$,

$$\|(u(t,x) - u_N(t,x))\| \leq 0 \quad as \quad N \to \infty.$$

Since, $\|u(t,x) - u_N(t,x)\| \geq 0$ always $\forall\ t$ and x,

so, $\|u(t,x) - u_N(t,x)\| = 0\ \forall\ t$ and x,

thus,

$$e_N \to 0 \quad as \quad N \to \infty.$$

Case 2. If $2 < \alpha < 3$ and $2 \leq \beta < 3$ then $Eq.(2.29)$ as follows:

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|(\frac{\partial}{\partial t})^3(\frac{t^{1-\{\alpha\}}}{\{\alpha\}-1})-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|$$
$$(\frac{\partial}{\partial t})^3(\frac{t^{1-\{\beta\}}}{\{\beta\}-1})\leq|\lambda_2|(\|u_{xx}(x,y)-(u_{xx})_N(x,y)\|$$
$$+\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|)+\|f(t,x)-f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)$$
$$\{\beta\}(1+\{\beta\})t^{-2-\{\beta\}})\leq|\lambda_2|\|u_{tt}(t,x)-(u_{tt})_N(t,x)\|$$
$$+|\lambda_2|\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|+\|f(t,x)-f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\|\leq\frac{\|f(t,x)-f_N(t,x)\|+|\lambda_2|\|u_{tt}(t,x)-(u_{tt})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)\{\beta\}(1+\{\beta\})t^{-2-\{\beta\}})}$$
$$+\frac{|\lambda_2|\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)\{\beta\}(1+\{\beta\})t^{-2-\{\beta\}})}. \tag{2.33}$$

Since,

$$(|\frac{1}{\Gamma(1-\{\alpha\})}|\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)\{\beta\}(1+\{\beta\})t^{-2-\{\beta\}})\neq 0\ \forall\, 0<t<1.$$

Grouping remark 4, remark 6, remark 7 and $Eq.$(2.33) with $N\rightarrow\infty$, we get

$$\|(u(t,x)-u_N(t,x))\|\leq 0 \quad as \quad N\rightarrow\infty.$$

Since $\|u(t,x)-u_N(t,x)\|\geq 0$ always $\forall$ t and x,

so, $\|u(t,x)-u_N(t,x)\|=0$ $\forall$ t and x,

thus,

$$e_N\rightarrow 0 \quad as \quad N\rightarrow\infty.$$

Case 3. If $2 \leq \alpha < 3$ and $1 \leq \beta < 2$ then $Eq.$(2.29) as follows:

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|(\frac{\partial}{\partial t})^3(\frac{t^{1-\{\alpha\}}}{\{\alpha\}-1})-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}| (\frac{\partial}{\partial t})^2(\frac{t^{1-\{\beta\}}}{\{\beta\}-1}) \leq |\lambda_2|(\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| +\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|)+\|f(t,x)-f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|(\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}| (-\{\beta\})t^{-1-\{\beta\}}) \leq |\lambda_2|\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| +|\lambda_2|\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|+\|f(t,x)-f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\| \leq \frac{\|f(t,x)-f_N(t,x)\|+|\lambda_2|\|u_{tt}(t,x)-(u_{tt})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}+|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|(\{\beta\})t^{-1-\{\beta\}})} + \frac{|\lambda_2|\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}+|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|(\{\beta\})t^{-1-\{\beta\}})}. \tag{2.34}$$

Since,

$$(|\frac{1}{\Gamma(1-\{\alpha\})}|(\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}+|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|(\{\beta\})t^{-1-\{\beta\}}) \neq 0 \ \ \forall\, 0<t<1.$$

Grouping remark 4, remark 6, remark 7 and $Eq.$(2.34) with $N \to \infty$, we get

$$\|(u(t,x)-u_N(t,x))\| \leq 0 \quad as \quad N \to \infty.$$

Since $\|u(t,x)-u_N(t,x)\| \geq 0$ always $\forall\ t$ and x,

so, $\|u(t,x)-u_N(t,x)\| = 0\ \forall\ t$ and x, thus,

$$e_N \to 0 \quad as \quad N \to \infty.$$

2.5.2 Numerical error

Let numerical approximate solution of $Eq.(2.9)$ as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{u_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1}\int_0^t \frac{u_N(s,x)}{(t-s)^{\{\beta\}}}ds$$
$$-\lambda_2((u_{tt})_N(t,x) + (u_{xx})_N(t,x)) + R(t,x) = f(t,x), \tag{2.35}$$

where, $R(t,x)$ is the residual of approximation.

Grouping $Eqs.(2.9)$ and (2.35) as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{(u-u_N)(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1}$$
$$\int_0^t \frac{(u-u_N)(s,x)}{(t-s)^{\{\beta\}}}ds - \lambda_2((u_{tt}-(u_{tt})_N)(t,x) + (u_{xx}-(u_{xx})_N)(t,x)) = R(t,x), \tag{2.36}$$

$$R(t,x) = \frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{e_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1}$$
$$\int_0^t \frac{(e_N(s,x)}{(t-s)^{\{\beta\}}}ds - \lambda_2((e_{tt})_N)(t,x) + (e_{xx})_N)(t,x)). \tag{2.37}$$

Finally, we can applied the proposed technique to approximate $e_N(t,x)$ in the $Eq.(2.37)$.

2.6 Numerical examples

In this section, we presents the implementation of our proposed numerical scheme and investigate its accuracy and simplicity by applying it on numerical examples (2.1-2.6) with known analytical solution. In all the error figures (2.1-2.6) the truncation is done at level $N = 2$ and $M = 2$.

Also, we calculated the computational order of the numerical collocation method (CONCM) with following formula [110]:

$$\frac{log(\frac{E_1}{E_2})}{log(\frac{h_1}{h_2})}.$$

For calculating the CONCM in space, E_1 and E_2 are $l_2 - norm$ errors corresponding to spatial step size h_1 and h_2 respectively.

Example 2.1 Consider the following FPDE for $\alpha = \frac{4}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x), \\ u(t,0) = 0, \\ f(t,x) = \frac{x}{t^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} - \frac{x}{4t^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right)}. \end{cases}$$

For these conditions there is known analytical solution, $u(t,x) = tx$. The numerical solution is $u(t,x) = C^T\Phi(t,x)$,, where, C^T is given by

$$C^T = [0.24999 \;\; 0.24999 \;\; -1.9165 \times 10^{-16} \;\; 0.24999 \;\; 0.24999 \;\; 5.5341 \times 10^{-17} \\ -1.3157 \times 10^{-15} \;\; -4.3127 \times 10^{-17} \;\; 1.0438 \times 10^{-16}].$$

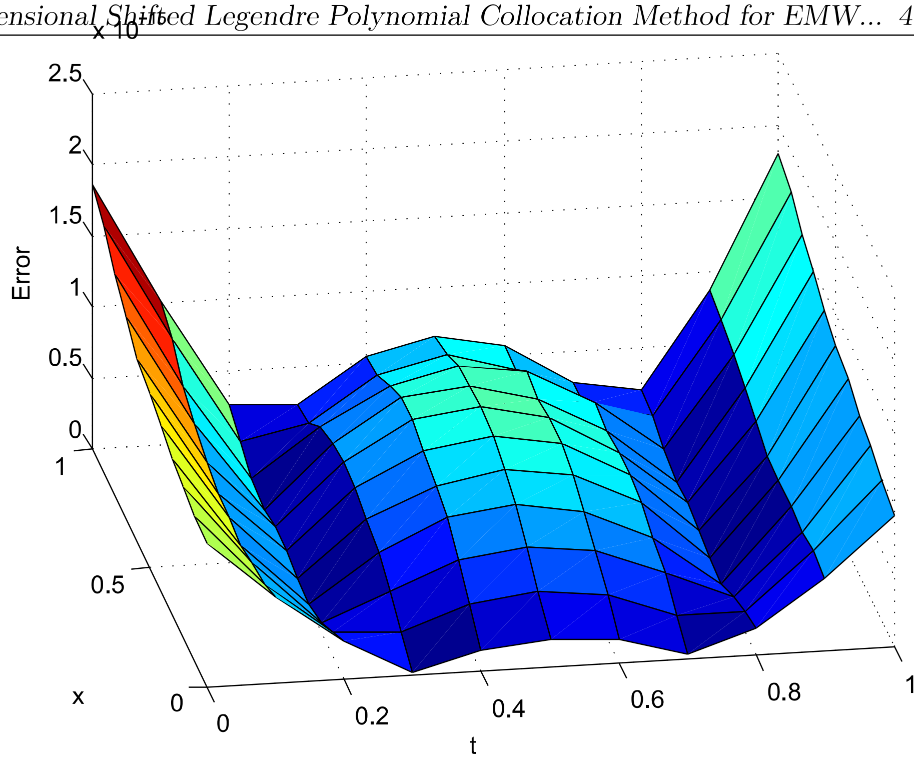

Figure 2.1: Absolute error graph for Example 2.1 at $N = 2, M = 2$.

Example 2.2 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{9}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x), \\ u(t,0) = 0, \\ f(t,x) = \frac{-x}{3t^{\frac{4}{3}}\Gamma\left(\frac{2}{3}\right)} - \frac{x}{16t^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right)}. \end{cases}$$

For these conditions there is known analytical solution $u(t,x) = tx$. The numerical solution for is $u(t,x) = C^T\Phi(t,x)$, where, C^T is given by

$$C^T = [0.24999 \;\; 0.24999 \;\; -9.9267\times10^{-15} \;\; 0.24999 \;\; 0.24999 \;\; -2.8321\times10^{-16} \;\; 2.8105\times10^{-15} \;\; 2.8978\times10^{-15} \;\; 4.0637\times10^{-16}].$$

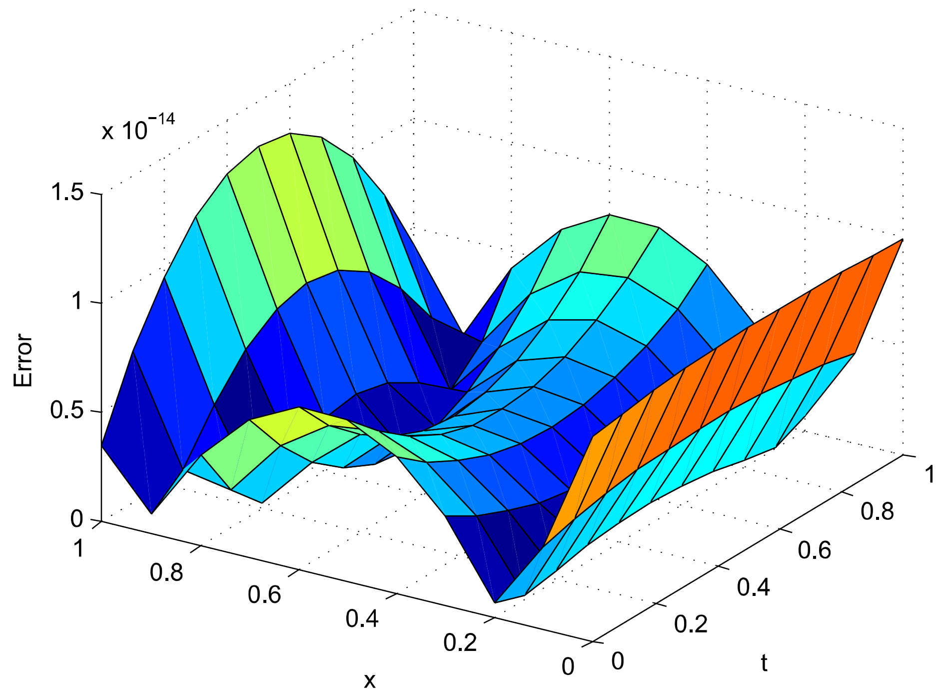

Figure 2.2: Absolute error graph for Example 2.2 at $N = 2, M = 2$.

Example 2.3 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x), \\ u(t,0) = 0, \\ f(t,x) = \frac{-x}{3t^{\frac{4}{3}}\Gamma\left(\frac{2}{3}\right)} - \frac{x}{4t^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right)}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx.$$

The numerical solution is

$$u(t,x) = C^T\Phi(t,x),$$

where, C^T is given by

$$C^T = [0.24999 \;\; 0.24999 \;\; -1.3533 \times 10^{-14} \;\; 0.24999 \;\; 0.24999 \;\; 6.5635 \times 10^{-17} \\ -3.8669 \times 10^{-15} \;\; -3.2937 \times 10^{-15} \;\; -2.1111 \times 10^{-16}].$$

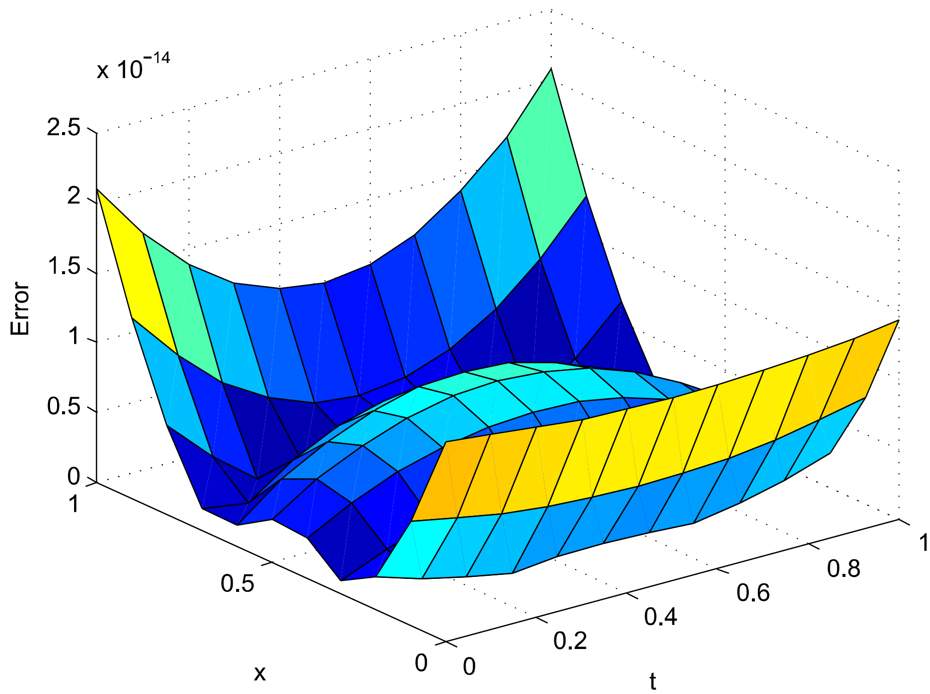

Figure 2.3: Absolute error graph for Example 2.3 at $N = 2, M = 2$.

Example 2.4 Consider the following FPDE for $\alpha = \frac{4}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x), \\ u(t,0) = 0, \\ f(t,x) = \frac{1}{5}[2(1-t)t + 2x(1-x)] + \frac{1}{4}[\frac{16\sqrt{t}x(1-x)}{5\sqrt{\pi}} + \frac{(4t-1)(1-x)x}{5\sqrt{\pi}\sqrt{t}}] + \frac{9x(x-1)}{4t^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} + \frac{(3t-4)x(x-1)}{12t^{\frac{4}{3}}\Gamma\left(\frac{2}{3}\right)}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx(t-1)(x-1)).$$

The numerical solution is

$$u(t,x) = C^T \Phi(t,x),$$

where, C^T is given by

$$C^T = [0.0278 \;\; 5.3049\times10^{-15} \;\; -0.0278 \;\; -1.1569\times10^{-15} \;\; -3.8171\times10^{-16} \;\; 8.1498\times10^{-17} \;\; -0.0278 \;\; 6.0421\times10^{-16} \;\; 0.0278].$$

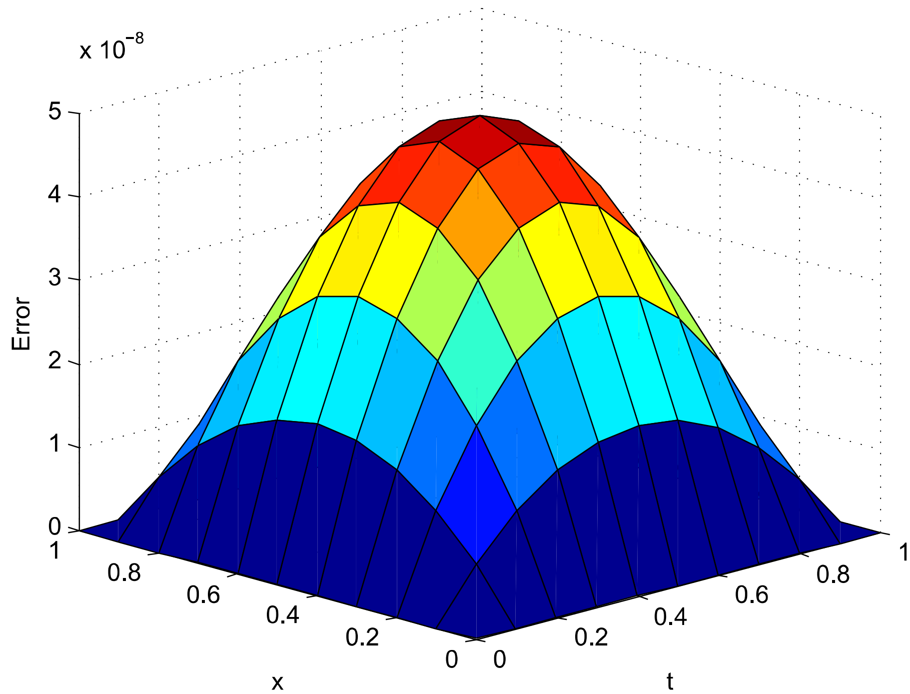

Figure 2.4: Absolute error graph for Example 2.4 at $N = 2, M = 2$.

Example 2.5 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{9}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x), \\ u(t,0) = 0, \\ f(t,x) = \frac{1}{5}[2(1-t)t + 2x(1-x)] + \frac{1}{4}[\frac{-64t^{\frac{3}{4}}x(1-x)}{33\Gamma\frac{3}{4}} - \frac{(8t-11)(1-x)x}{11t^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right)}] + \frac{9t^{\frac{2}{3}}x(x-1)}{4\Gamma\left(\frac{2}{3}\right)} + \frac{(3t-4)x(x-1)}{4t^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx(t-1)(x-1)).$$

The numerical solution is

$$u(t,x) = C^T\Phi(t,x),$$

where, C^T is given by

$$C^T = [0.0278 \;\; 1.6782\times10^{-14} \;\; -0.0278 \;\; 1.1944\times10^{-15} \;\; 1.1799\times10^{-15} \;\; 1.4851\times10^{-16} \;\; -0.0278 \;\; -5.2232\times10^{-16} \;\; -0.0278].$$

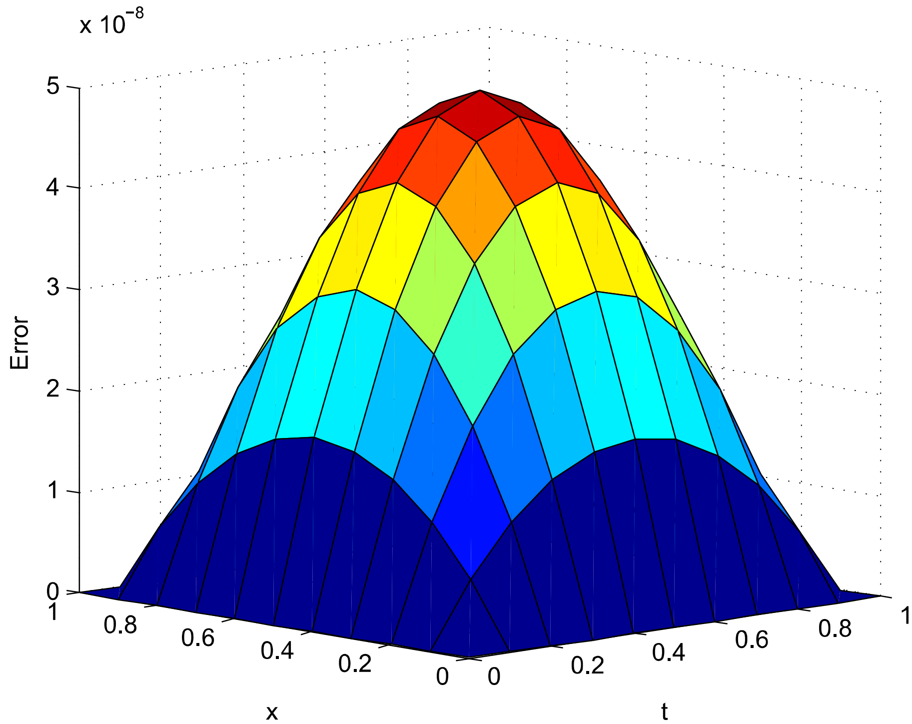

Figure 2.5: Absolute error graph for Example 2.5 at $N = 2, M = 2$.

Example 2.6 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x), \\ u(t,0) = 0, \\ f(t,x) = \frac{1}{5}[2(1-t)t + 2x(1-x)] + \frac{1}{4}[\frac{-64t^{\frac{3}{4}}x(1-x)}{33\Gamma\left(\frac{3}{4}\right)} - \frac{(8t-11)(1-x)x}{11t^{\frac{1}{4}}}] + \frac{9x(x-1)}{4t^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} + \frac{(3t-4)x(x-1)}{12t^{\frac{4}{3}}\Gamma\frac{2}{3}}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx(t-1)(x-1)).$$

The numerical solution for is

$$u(t,x) = C^T \Phi(t,x),$$

where, C^T is given by

$$C^T = [0.0278 \;\; 1.8209\times10^{-13} \;\; -0.0278 \;\; -1.5203\times10^{-15} \;\; -1.4796\times10^{-15} \\ -1.077\times10^{-16} \;\; -0.0278 \;\; 3.7497\times10^{-15} \;\; 0.0278].$$

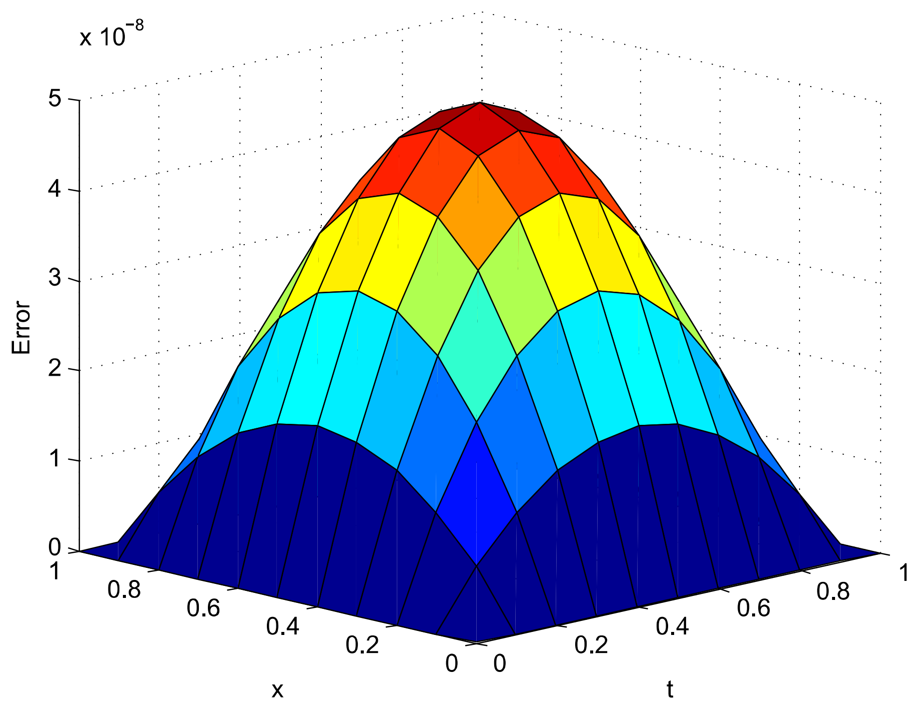

Figure 2.6: Absolute error graph for Example 2.6 at $N = 2, M = 2$.

Table 2.1: Error for $N = 2$ and $M = 2$.

(t, x)	Example 2.1	Example 2.2	Example 2.3
$(0.1, 0.1)$	5.97e-16	4.14e-15	6.85e-15
$(0.2, 0.2)$	5.70e-17	3.50e-16	6.20e-16
$(0.3, 0.3)$	3.66e-16	2.15e-15	4.18e-15
$(0.4, 0.4)$	6.35e-16	3.42e-15	7.33e-15
$(0.5, 0.5)$	7.95e-16	3.64e-15	8.65e-15
$(0.6, 0.6)$	7.29e-16	2.98e-15	7.87e-15
$(0.7, 0.7)$	4.44e-16	1.58e-15	4.82e-15
$(0.8, 0.8)$	1.17e-16	2.70e-15	8.40e-16
$(0.9, 0.9)$	8.32e-16	2.21e-15	9.23e-15

Table 2.2: Normed error for l_2-norm and l_∞-norm at $N = 2$ and $M = 2$.

Norm	Example 2.1	Example 2.2	Example 2.3
l_2-norm	2.69e-15	1.29e-14	3.17e-14
l_∞-norm	1.81e-15	9.33e-15	2.08e-15

Table 2.3: Error for $N = 2$ and $M = 2$.

(t, x)	Example 2.4	Example 2.5	Example 2.6
$(0.1, 0.1)$	6.48e-9	6.57e-9	6.44e-9
$(0.2, 0.2)$	2.05e-8	2.12e-8	2.15e-8
$(0.3, 0.3)$	3.53e-8	3.63e-8	3.52e-8
$(0.4, 0.4)$	4.61e-8	4.51e-8	4.61e-8
$(0.5, 0.5)$	5.00e-8	4.99e-8	5.00e-8
$(0.6, 0.6)$	4.61e-8	4.72e-8	4.68e-8
$(0.7, 0.7)$	3.53e-8	3.41e-8	3.53e-8
$(0.8, 0.8)$	2.05e-8	2.12e-8	2.05e-8
$(0.9, 0.9)$	6.48e-9	6.57e-9	6.45e-9

Table 2.4: Normed error for l_2-norm and l_∞-norm at $N = 2$ and $M = 2$.

Norm	Example 2.4	Example 2.5	Example 2.6
l_2-norm	1.17e-7	1.01e-7	1.01e-7
l_∞-norm	6.48e-8	4.99e-8	5.00e-8

Table 2.5: CONCM for examples 2.1-2.6 with $h_1 = \frac{1}{3}$ and $h_2 = \frac{1}{4}$.

Examples	α	β	CONCM
1	$\frac{4}{3}$	$\frac{5}{4}$	5.97364
2	$\frac{7}{3}$	$\frac{9}{4}$	5.59450
3	$\frac{7}{3}$	$\frac{5}{4}$	6.40291
4	$\frac{4}{3}$	$\frac{5}{4}$	6.59447
5	$\frac{7}{3}$	$\frac{9}{4}$	5.59453
6	$\frac{7}{3}$	$\frac{5}{4}$	5.59450

2.7 Conclusion

In this chapter, we presents a new almost operational matrix collocation method based on two-dimensional shifted Legendre polynomial approximation process for solving fractional partial differential equations subject to the initial condition. The scheme is based on the transformation of fractional partial differential equations into a equivalent weak singular fractional partial integro-differential equations by using partial Riemann Liouville derivative operator. Moreover, we studied the convergence analysis, error analysis and computational order of convergence for the proposed method when $1 \leq \beta < \alpha < 3$. Finally from the convergence analysis, error analysis, examples, computational order of convergence and table 2.2-2.5, it should be noted that this is the first almost operational matrix collocation method approach for which the accuracy can be justified both theoretically and numerically for $1 \leq \beta < \alpha < 3$.

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