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It is certified that the work contained in this thesis titled "***Numerical Approximations of Some Fractional Partial Differential Equations***" by ***Swati Yadav*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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DECLARATION BY THE CANDIDATE

I, ***Swati Yadav***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Dr. Rajesh Kumar Pandey*** from ***July, 2016*** to ***December, 2020*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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To

My Beloved Parents

Mrs. Nirmala Yadav

&

Mr. Ram Prakash Yadav

ACKNOWLEDGEMENTS

I take this opportunity to express my deep sense of gratitude to all who helped me directly or indirectly during this thesis work. Though I cannot express my gratitude in words for the support, love, and affection I got throughout this journey from so many people, it is a mere attempt to name a few.

First of all, I would like to express my sincere thanks to my respected supervisor, ***Dr. Rajesh K. Pandey, Associate Professor, Department of Mathematical Sciences, IIT(BHU), Varanasi***, for being a great mentor and the best adviser I could ever have. His advice, encouragement, and critics are the sources of innovative ideas, inspiration, and causes behind the successful completion of this thesis work. The confidence shown on me by him was the most significant source of inspiration for me. It has been a privilege working with him for several years.

I express my cordial thanks to ***Prof. T. Som,*** the Head of Department for his supports throughout my research work. I am highly obliged to all the Department of Mathematical Sciences faculty members for their support and encouragement. I express my sincere thanks to my RPEC members ***Dr. Rajeev*** and ***Dr. Kishor P. Sarawadekar, Department of Electronics*** for their constant moral supports, suggestions, and encouragement. I express my gratitude to the Director, Registrars, Deans, Heads, and Student Alumni of the Indian Institute of Technology (BHU) Varanasi.

I would like to extend my thanks to my seniors ***Dr. Kamlesh K. Pandey, Dr. Shiva Sharma,*** and ***Dr. Seema Mishra*** for guiding me. A special thanks to ***Mr. Prashant K. Pandey, Ms. Pratima Tiwari, Mr. Sandeep, Ms. Deeksha Singh,*** and ***Ms. Farheen Sultana*** for their cooperation. I thank to my colleagues ***Mr. Anil K. Shukla, Mr. Abhishek Singh, Mr. Anup, Mr. Avinash Dixit, Mr. Om Namah Shivay, Mr. Pankaj Gautam, Mr. Rahul Maurya, Mr. Rakesh, Mr. Sumit,*** and ***Ms. Vinita*** and all the research scholars of the department for their help, support, and motivation.

Further, I feel very thankful to my close friends ***Dr. Neera Singh, Dr. Gargi Srivastava, Dr. Surbhi Saraswat, Ms. Manushi Gupta, Ms. Anuwedita Singh, Ms. Adiba Khan, Mr. Ashutosh Jalan,*** and ***Ms. Krati Gupta*** for their continued encouragement, stimulating help, and criticism, which gave me physical and moral strength throughout my career and the present research.

I am also grateful to my Institute, IIT(BHU), for providing necessary resources throughout my research. Also, the versatile qualities I developed here are going to remain with me throughout my life. I express my thanks to all non-teaching staff members of the department for their supports. I gratefully acknowledge the Council of Scientific and Industrial Research (CSIR), India, for providing the fellowships in the form of Junior Research Fellowship and Senior Research fellowship.

I express my sincere and cordial gratitude to my mother ***Mrs. Nirmala Yadav***, and my father ***Mr. Ram Prakash Yadav***, who gave me the power and brain to work out on this research and their help at every level. Words are insufficient to express my profound sense of gratitude to my siblings ***Mrs. Sarita Yadav, Ms. Shalini Yadav, Mr. Satyam Yadav,*** and ***Mr. Shivam Yadav*** who always stood by my decisions and provided all kinds of supports and moral. It was their love, care, and patience, which encouraged me to move on. I pay my special thanks to brother-in-law ***Mr. Amit Yadav*** and sister-in-law ***Mrs. Pragati Yadav***, and special love to my sweet niece ***Anika*** and nephew ***Advik***.

This acknowledgement would be incomplete if the name of great visionary ***Pt. Madan Mohan Malaviya*** is not mentioned, who made this divine centre of knowledge. Deepest regards to him.

Above all, I praise and thank God, the Almighty, for His shower of blessings throughout my research work, which made everything possible.

Swati Yadav

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Abbreviations

ABC	**A**tangana-**B**aleanu **C**aputo
ADE	**A**dvection-**D**iffusion **E**quation
CO	**C**onvergence **O**rder
FADE	**F**ractional **A**dvection-**D**iffusion **E**quation
FDM	**F**inite **D**ifference **M**ethod
FEA	**F**inite **E**lement **A**nalysis
FEM	**F**inite **E**lement **M**ethod
FPDE	**F**ractional **P**artial **D**ifferential **E**quation
GFADE	**G**eneralized **F**ractional **A**dvection-**D**iffusion **E**quation
GFD	**G**eneralized **F**ractional **D**erivative
GTFTE	**G**eneralized **T**ime-**F**ractional **T**elegraph **E**quation
LTE	**L**ocal **T**runcation **E**rror
MAE	**M**aximum **A**bsolute **E**rror
PDE	**P**artial **D**ifferential **E**quation
TE	**T**elegraph **E**quation
TFTE	**T**ime-**F**ractional **T**elegraph **E**quation

Symbols

u	solute concentration
a	average fluid velocity
v	dispersion coefficient
V	kinematic viscosity
x	independent variable in space direction
t	independent variable in time direction
$H^k(\omega)$	Sobolev space
$H_0^k(\omega)$	closure in $H^k(\omega)$ of the space $C_c^\infty(\omega)$ of infinitely differentiable compactly supported functions
O	Big O
α	fractional order (non-integer)
E_α	Mittag-Leffler function of one parameter
$E_{\alpha,\beta}$	Mittag-Leffler function of two parameters
$M(\alpha)$	normalization function
h	space step-size
τ	time step-size
$M,\ N$	number of subintervals
$z(t)$	scale function
$w(t)$	weight function
f	source function or unknown function
I_{a+}^{α}	Riemann-Liouville fractional integral of order $\alpha > 0$
${}_a^R D_t^{\alpha}$	Riemann-Liouville fractional derivative of order α

${}_{a}^{C}D_{t}^{\alpha}$	Caputo fractional derivative of order α
D^{m}	derivative of integer order m
${}^{CFC}_{\ \ \ 0}D_{t}^{\alpha}$	Caputo-Fabrizio derivative of Caputo type of order α
${}^{ABC}_{\ \ \ 0}D_{t}^{\alpha}$	Atangana-Baleanu derivative of Caputo type of order α
${}_{a}I_{t,(z,w)}^{\alpha}$	generalized fractional integral of order α
${}_{a}D_{t,(z,w)}^{m}$	generalized derivative of integer order m
${}_{a}D_{t,(z,w)}^{\alpha}$	generalized fractional derivative of order α

PREFACE

The classical derivative is local operator, but fractional derivatives are non-local. Hence, the latter are good enough to provide a mechanism to incorporate the hereditary properties of many physical phenomena. Fractional partial differential equations (FPDEs) are partial differential equations that involve fractional derivatives. FPDEs are widely used in fluid mechanics, optical fibers, plasma physics, mathematical biology, viscoelasticity, etc. In this thesis, the fractional advection-diffusion equation (FADE), fractional telegraph equation, and fractional Burgers equations are considered and numerical methods are investigated to solve them.

Fractional generalization of the diffusion equation was introduced to describe the anomalous kinetics of a simple dynamical system with chaotic motion. FADE is used in groundwater hydrology to model the movement of passive tracers carried by fluid flow in a porous medium. FADE, in which time derivative is of fractional order, arises from power-law particle residence time distribution and describes particle motion with memory in time. Another considered linear PDE is telegraph equation which is used in the fields of signal analysis, diffusion process of chemicals, heat transfer theory in thermodynamics, wave propagation, transportation charged particles, etc. Some processes in nature are governed by time-fractional telegraph equations, such as the anomalous diffusion processes in blood flow experiments.

A Burgers equation is a non-linear PDE whose solution gives a travelling wave containing front sharpening. This equation describes non-linear propagation with diffusion effects and is a mathematical model of traffic flow. The generalization of the Burgers equation describes the non-linear phenomena more accurately. Fractional Burgers equation models the unidirectional propagation of non-linear acoustic waves through a pipe filled with gas.

This thesis mainly focuses on the FPDEs generated by time-fractional derivatives such as Atangana-Baleanu derivative and generalized fractional derivative. The researchers and scientists investigated many definitions of fractional derivative, like Caputo, Grünwald-Letnikov, Hadamard, Riemann-Liouville, Riesz, etc. These definitions have shown significant roles in fractional calculus but still have some limitations. They contain a singular kernel, which is a restriction in modelling the behaviour of viscoelastic material, electromagnetic systems, etc. The generalizations of fractional derivatives and integrals are studied by many authors and in different ways. In 2012, Agrawal proposed the definition of generalized fractional operators using scale function and weight function. With the scale function, the assumed domain can be changed, and the weight function is used to extend the kernel in operators. With the help of them, a model can be made more flexible. In 2015, Caputo and Fabrizio proposed a new definition for the fractional derivative in term of exponential kernel to solve the problems of non-local dynamical system. Atangana and Baleanu modified the Caputo-Fabrizio derivative using the Mittag-Leffler function as the kernel, which is non-singular and non-local. The benefit of a non-singular kernel is that it does not lead the addition of artificial singularity into the models. Also, it describes the two different waiting times distribution, which is an ideal waiting time distribution, as noted in the biological phenomena like in the spread of cancer.

As the presence of non-integer order derivatives makes the differential equations too complex to be solved analytically, and hence the numerical methods becomes important to study such problems. It is challenging to provide more accurate numerical results, especially for the newly proposed fractional derivatives. This research work aims to develop some high order numerical approximations for Atangana-Baleanu and generalized fractional derivatives. The finite difference method (FDM) is used to maintain simplicity and provide the proficiency of the scheme. For achieving a high order of convergence, Taylor series expansion is used, followed by FDM. The scope of the thesis is to form the numerical schemes for FPDEs like the FADE, fractional telegraph equation, and fractional Burgers equation; and to study the stability of proposed methods; to provide the experimental analysis to support theoretical statements.