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It is certified that the work contained in this thesis titled "*Numerical Approximations of Some Fractional Partial Differential Equations*" by *Swati Yadav* has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy, and SOTA for the award of Ph.D. degree.

  
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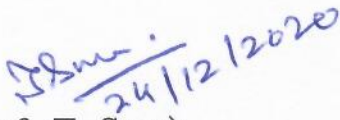
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*To*  
*My Beloved Parents*  
*Mrs. Nirmala Yadav*  
*&*  
*Mr. Ram Prakash Yadav*



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---

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# Contents

<b>List of Figures</b>	<b>xv</b>
<b>List of Tables</b>	<b>xvii</b>
<b>Abbreviations</b>	<b>xix</b>
<b>Symbols</b>	<b>xxi</b>
<b>Preface</b>	<b>xxiii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Background . . . . .	2
1.1.1 Fractional calculus . . . . .	2
1.1.2 Fractional partial differential equation . . . . .	2
1.1.2.1 Fractional advection-diffusion equation . . . . .	3
1.1.2.2 Fractional telegraph equation . . . . .	4
1.1.2.3 Fractional Burgers equation . . . . .	5
1.2 Definitions of Fractional Derivatives . . . . .	5
1.2.1 Basic definitions . . . . .	6
1.2.2 Atangana-Baleanu derivative . . . . .	7
1.2.3 Generalized fractional derivative . . . . .	8
1.3 Literature Review . . . . .	9
1.3.1 Literature review on fractional partial differential equations . .	10
1.3.1.1 Literature review on fractional advection-diffusion equation . . . . .	11
1.3.1.2 Literature review on fractional Burgers equation . . .	12
1.3.1.3 Literature review on fractional telegraph equation . .	13
1.4 Challenges and Motivation . . . . .	14
1.5 Problem Statement and Thesis Objective . . . . .	16
1.6 Outline of the Thesis . . . . .	17

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<b>2</b>	<b>Numerical Approximations of Atangana-Baleanu Caputo Derivative and Its Application</b>	<b>19</b>
2.1	Introduction . . . . .	19
2.2	Numerical Approximation to ABC Derivative . . . . .	20
2.2.1	Method 1: Discretization using FDM . . . . .	21
2.2.2	Method 2: Discretization using Taylor expansion . . . . .	22
2.3	Application: Solution of Fractional Advection-Diffusion Equation . .	25
2.4	Numerical Examples . . . . .	29
2.5	Conclusion . . . . .	40
<b>3</b>	<b>Numerical Approximation of Fractional Burgers Equation with Atangana-Baleanu Caputo Derivative</b>	<b>41</b>
3.1	Introduction . . . . .	42
3.2	Formation of Numerical Scheme . . . . .	43
3.2.1	Stability analysis of numerical scheme . . . . .	47
3.3	Numerical Examples . . . . .	50
3.4	Conclusion . . . . .	60
<b>4</b>	<b>High-Order Approximation for Generalized Fractional Derivative and Its Application</b>	<b>61</b>
4.1	Introduction . . . . .	62
4.2	Main Result . . . . .	63
4.2.1	Formation of numerical scheme . . . . .	64
4.2.2	Stability of the numerical scheme . . . . .	72
4.3	Numerical Examples . . . . .	74
4.3.1	Effect of scale and weight functions on the solution . . . . .	81
4.4	Conclusion . . . . .	84
<b>5</b>	<b>Finite Difference Scheme for a Generalized Fractional Time-Derivatives Telegraph Equation</b>	<b>87</b>
5.1	Introduction . . . . .	88
5.2	Finite Difference Scheme . . . . .	89
5.3	Stability and Convergence Analysis . . . . .	92
5.4	Numerical Examples . . . . .	95
5.4.1	Influence of scale function on the solution of GTFTE . . . . .	99
5.4.2	Influence of weight function on the solution of GTFTE . . . . .	101
5.5	Conclusion . . . . .	106
<b>6</b>	<b>Conclusion and Future Work</b>	<b>107</b>
6.1	Conclusion . . . . .	107
6.2	Future Work . . . . .	109

<i>Contents</i>	xiii
<b>Bibliography</b>	<b>111</b>
<b>List of Publications</b>	<b>139</b>



# List of Figures

2.1	Comparison of the numerical solution obtained by Method 1 to the analytical solution of Example 2.4.2. . . . .	32
2.2	Comparison of the numerical solution obtained by Method 1 to the analytical solution of Example 2.4.3. . . . .	32
3.1	A comparison between the solution obtained numerically by the scheme to the exact solution of Example 3.3.1. . . . .	50
3.2	Numerical solution of Example 3.3.2 with fixed time $t = 1$ . . . . .	51
3.3	Numerical solution of Example 3.3.2 with fixed time $t = 5$ . . . . .	52
3.4	Numerical solution of Example 3.3.3 with fixed time $t = 1$ . . . . .	53
3.5	Numerical solution of Example 3.3.3 with fixed time $t = 5$ . . . . .	53
4.1	Comparison of the exact and the numerical solutions of Example 4.3.2. . . . .	78
4.2	Comparison of the exact and the numerical solutions of Example 4.3.3. . . . .	80
4.3	The numerical solutions of Example 4.3.4 with different advection coefficient $a(x, t)$ . . . . .	82
4.4	The numerical solutions of Example 4.3.5 with different values of $\alpha$ . . . . .	83
4.5	The numerical solutions of Example 4.3.5 with different weight functions. . . . .	84
4.6	The numerical solutions of Example 4.3.5 with different scale functions. . . . .	85
5.1	The analytical solution of Example 5.4.1. . . . .	96
5.2	The numerical solutions of Example 5.4.1 for different parameters. . . . .	97
5.3	The analytical solution of Example 5.4.2. . . . .	100
5.4	The numerical solutions of Example 5.4.2 for different parameters. . . . .	101
5.5	The numerical solutions of Example 5.4.1 for different parameters and scale functions. . . . .	103
5.6	The numerical solutions of Example 5.4.1 for different parameters and weight functions. . . . .	105



# List of Tables

2.1	MAE and CO of Example 2.4.1 with different values of $\tau = \frac{1}{n}$ by Method 1. . . . .	30
2.2	MAE and CO of Example 2.4.1 with different values of $\tau = \frac{1}{n}$ by Method 2. . . . .	31
2.3	MAE and CO of Example 2.4.2 with different values of $\tau = \frac{1}{n}$ , and fixed $h = \frac{1}{1000}$ by Method 1. . . . .	33
2.4	MAE and CO of Example 2.4.2 with different values of $\tau = \frac{1}{n}$ , and $h = \frac{1}{m}$ by Method 1. . . . .	34
2.5	MAE and CO of Example 2.4.2 with different values of $\tau = \frac{1}{n}$ , and fixed $h = \frac{1}{1000}$ by Method 2. . . . .	35
2.6	MAE and CO of Example 2.4.3 with different values of $\tau = \frac{1}{n}$ , and fixed $h = \frac{1}{1000}$ by Method 1. . . . .	36
2.7	MAE and CO of Example 2.4.3 with different values of $h = \frac{1}{m}$ , and fixed $\tau = \frac{1}{500}$ by Method 1. . . . .	37
2.8	MAE and CO of Example 2.4.3 with different values of $\tau = \frac{1}{n}$ , and fixed $h = \frac{1}{1000}$ by Method 2. . . . .	38
2.9	MAE and CO of Example 2.4.3 with different values of $h = \frac{1}{m}$ , and fixed $\tau = \frac{1}{500}$ by Method 2. . . . .	39
3.1	MAE and CO of Example 3.3.1 in spatial direction with $M = 2^{12}$ . . .	54
3.2	MAE and CO of Example 3.3.1 in temporal direction with $N = 2^9$ . .	55
3.3	MAE and CO of Example 3.3.2 in temporal direction with $N = 2^9$ . .	56
3.4	MAE and CO of Example 3.3.2 in spatial direction with $M = 2^7$ . . .	57
3.5	MAE and CO of Example 3.3.3 in temporal direction with $N = 2^9$ . .	58
3.6	MAE and CO of Example 3.3.3 in spatial direction with $M = 2^9$ . . .	59
4.1	MAE and CO of Example 4.3.1 with $z(t) = t$ , and $w(t) = 1$ for different values of $\alpha$ . . . . .	75
4.2	MAE and CO of Example 4.3.1 with $z(t) = t$ , and $w(t) = e^t$ , $t + 1$ for different values of $\alpha$ . . . . .	75
4.3	MAE and CO of Example 4.3.1 with $z(t) = t^{1/2}$ , and $w(t) = 1$ for different values of $\alpha$ . . . . .	76
4.4	MAE and CO of Example 4.3.2 with different step-sizes $\Delta t$ and $\Delta x$ for $\alpha = 0.95$ . . . . .	77

---

4.5	MAE and CO of Example 4.3.2 with different step-sizes $\Delta t$ and $\Delta x$ for $\alpha = 0.6$ and $0.45$ . . . . .	77
4.6	MAE and CO of Example 4.3.2 with different $\Delta t$ , fixing $\Delta x = 0.001$ for different values of $\alpha$ . . . . .	78
4.7	MAE and CO of Example 4.3.2 with different $\Delta x$ , fixing $\Delta t = 0.002$ for different values of $\alpha$ . . . . .	79
4.8	MAE and CO of Example 4.3.3 with different step-sizes $\Delta t$ and $\Delta x$ for $\alpha = 0.9, 0.6$ , and $0.45$ . . . . .	79
4.9	MAE and CO of Example 4.3.3 with different $\Delta t$ , fixing $\Delta x = 0.001$ for different values of $\alpha$ . . . . .	80
4.10	MAE and CO of Example 4.3.3 with different $\Delta x$ , fixing $\Delta t = 0.002$ for different values of $\alpha$ . . . . .	81
4.11	CPU time (seconds) in calculating MAE for Example 4.3.2 and Example 4.3.3 at different step sizes. . . . .	81
5.1	MAE and CO for Example 5.4.1 with $\lambda = 0.5$ , $c^2 = 1$ , $\alpha = 0.2$ . . . . .	98
5.2	MAE and CO for Example 5.4.1 with $\lambda = 0.5$ , $c^2 = 1$ , $\alpha = 0.4$ . . . . .	98
5.3	MAE and CO for Example 5.4.2 with $\lambda = 0.5$ , $c^2 = 0.5$ , $\alpha = 0.4$ , $\Delta x = 1/512$ . . . . .	98
5.4	MAE and CO for Example 5.4.2 with $\lambda = 0.5$ , $c^2 = 0.5$ , $\alpha = 0.4$ , $\Delta t = 1/512$ . . . . .	99

# Abbreviations

<b>ABC</b>	Atangana-Baleanu Caputo
<b>ADE</b>	Advection-Diffusion Equation
<b>CO</b>	Convergence Order
<b>FADE</b>	Fractional Advection-Diffusion Equation
<b>FDM</b>	Finite Difference Method
<b>FEA</b>	Finite Element Analysis
<b>FEM</b>	Finite Element Method
<b>FPDE</b>	Fractional Partial Differential Equation
<b>GFADE</b>	Generalized Fractional Advection-Diffusion Equation
<b>GFD</b>	Generalized Fractional Derivative
<b>GTFTE</b>	Generalized Time-Fractional Telegraph Equation
<b>LTE</b>	Local Truncation Error
<b>MAE</b>	Maximum Absolute Error
<b>PDE</b>	Partial Differential Equation
<b>TE</b>	Telegraph Equation
<b>TFTE</b>	Time-Fractional Telegraph Equation



# Symbols

$u$	solute concentration
$a$	average fluid velocity
$v$	dispersion coefficient
$V$	kinematic viscosity
$x$	independent variable in space direction
$t$	independent variable in time direction
$H^k(\omega)$	Sobolev space
$H_0^k(\omega)$	closure in $H^k(\omega)$ of the space $C_c^\infty(\omega)$ of infinitely differentiable compactly supported functions
$O$	Big $O$
$\alpha$	fractional order (non-integer)
$E_\alpha$	Mittag-Leffler function of one parameter
$E_{\alpha,\beta}$	Mittag-Leffler function of two parameters
$M(\alpha)$	normalization function
$h$	space step-size
$\tau$	time step-size
$M, N$	number of subintervals
$z(t)$	scale function
$w(t)$	weight function
$f$	source function or unknown function
$I_{a+}^\alpha$	Riemann-Liouville fractional integral of order $\alpha > 0$
${}^R D_t^\alpha$	Riemann-Liouville fractional derivative of order $\alpha$

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${}^C_a D_t^\alpha$	Caputo fractional derivative of order $\alpha$
$D^m$	derivative of integer order $m$
${}^{CF}{}_0 D_t^\alpha$	Caputo-Fabrizio derivative of Caputo type of order $\alpha$
${}^{ABC}{}_0 D_t^\alpha$	Atangana-Baleanu derivative of Caputo type of order $\alpha$
${}_a I_{t,(z,w)}^\alpha$	generalized fractional integral of order $\alpha$
${}_a D_{t,(z,w)}^m$	generalized derivative of integer order $m$
${}_a D_{t,(z,w)}^\alpha$	generalized fractional derivative of order $\alpha$

## PREFACE

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The classical derivative is local operator, but fractional derivatives are non-local. Hence, the latter are good enough to provide a mechanism to incorporate the hereditary properties of many physical phenomena. Fractional partial differential equations (FPDEs) are partial differential equations that involve fractional derivatives. FPDEs are widely used in fluid mechanics, optical fibers, plasma physics, mathematical biology, viscoelasticity, etc. In this thesis, the fractional advection-diffusion equation (FADE), fractional telegraph equation, and fractional Burgers equations are considered and numerical methods are investigated to solve them.

Fractional generalization of the diffusion equation was introduced to describe the anomalous kinetics of a simple dynamical system with chaotic motion. FADE is used in groundwater hydrology to model the movement of passive tracers carried by fluid flow in a porous medium. FADE, in which time derivative is of fractional order, arises from power-law particle residence time distribution and describes particle motion with memory in time. Another considered linear PDE is telegraph equation which is used in the fields of signal analysis, diffusion process of chemicals, heat transfer theory in thermodynamics, wave propagation, transportation charged particles, etc. Some processes in nature are governed by time-fractional telegraph equations, such as the anomalous diffusion processes in blood flow experiments.

A Burgers equation is a non-linear PDE whose solution gives a travelling wave containing front sharpening. This equation describes non-linear propagation with diffusion effects and is a mathematical model of traffic flow. The generalization of the Burgers equation describes the non-linear phenomena more accurately. Fractional Burgers equation models the unidirectional propagation of non-linear acoustic waves through a pipe filled with gas.

This thesis mainly focuses on the FPDEs generated by time-fractional derivatives such as Atangana-Baleanu derivative and generalized fractional derivative. The researchers and scientists investigated many definitions of fractional derivative, like Caputo, Grünwald-Letnikov, Hadamard, Riemann-Liouville, Riesz, etc. These definitions have shown significant roles in fractional calculus but still have some limitations. They contain a singular kernel, which is a restriction in modelling the behaviour of viscoelastic material, electromagnetic systems, etc. The generalizations of fractional derivatives and integrals are studied by many authors and in different ways. In 2012, Agrawal proposed the definition of generalized fractional operators using scale function and weight function. With the scale function, the assumed domain can be changed, and the weight function is used to extend the kernel in operators. With the help of them, a model can be made more flexible. In 2015, Caputo and Fabrizio proposed a new definition for the fractional derivative in term of exponential kernel to solve the problems of non-local dynamical system. Atangana and Baleanu modified the Caputo-Fabrizio derivative using the Mittag-Leffler function as the kernel, which is non-singular and non-local. The benefit of a non-singular kernel is that it does not lead the addition of artificial singularity into the models. Also, it describes the two different waiting times distribution, which is an ideal waiting time distribution, as noted in the biological phenomena like in the spread of cancer.

As the presence of non-integer order derivatives makes the differential equations too complex to be solved analytically, and hence the numerical methods becomes important to study such problems. It is challenging to provide more accurate numerical results, especially for the newly proposed fractional derivatives. This research work aims to develop some high order numerical approximations for Atangana-Baleanu and generalized fractional derivatives. The finite difference method (FDM) is used to maintain simplicity and provide the proficiency of the scheme. For achieving a high order of convergence, Taylor series expansion is used, followed by FDM. The scope of the thesis is to form the numerical schemes for FPDEs like the FADE,

fractional telegraph equation, and fractional Burgers equation; and to study the stability of proposed methods; to provide the experimental analysis to support theoretical statements.