Chapter 1

Introduction

The notion of Weyl transform was introduced by Hermann Weyl in his book on the theory of groups and quantum mechanics [65]. The Weyl transform plays a crucial role in the theory of pseudo-differential operators and in quantum mechanics. In classical mechanics, the observables are functions defined on phase space, whereas in quantum mechanics the observables are operators on Hilbert space. The Weyl transform provides a way to connect classical mechanics with quantum mechanics through a transformation between functions on phase space and operators on Hilbert space. The Weyl transform is closely related to the group Fourier transform on the Heisenberg group.

We begin this chapter with a brief discussion of the Fourier transform on Euclidean space. Then we define and study the basic properties of the group Fourier transform on the Heisenberg group. We then introduce the Weyl transform. Finally, we discuss the role of curvature in Fourier analysis and give an outline of the thesis.

1.1 Fourier transform

Fourier series is used to analyze and transform functions into their constituent frequencies. Named after the French mathematician Jean-Baptiste Joseph Fourier, who introduced it in the early 19^{th} century, this tool has become fundamental in various fields such as signal processing, image analysis, quantum physics, partial differential equations, and applied mathematics. Fourier's pioneering work focused on the idea that any periodic function could be expressed as a sum of sine and cosine functions, which he described in his book [13]. The Fourier transform generalizes this idea to non-periodic functions. By decomposing a function into its frequency components, it enables us to understand and manipulate the underlying frequencies that compose the signal. This transformation is particularly powerful in analyzing signals in various domains, such as electrical engineering for filtering and signal recovery, in audio processing for sound analysis, and in image processing for tasks like compression and enhancement.

The results in this section are well known and easy to find in the literature, see for example [24, 50, 51].

We begin by introducing some notation. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be in $\mathbb{R}^n$. The inner product $x \cdot y$ of x and y, and the norm $\|x\|$ of x are defined by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j, \quad \text{and} \quad \|x\| = \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}}.$$

For $1 \leq p < \infty$, the space $L^p(\mathbb{R}^n)$ is the set of all complex valued measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Then $\left(L^p(\mathbb{R}^n), \|\cdot\|_p\right)$ is a Banach space. A complex valued measurable function f defined on $\mathbb{R}^n$ is called essentially bounded if there exists $M > 0$ such that $|f(x)| \leq M$ for almost all x in $\mathbb{R}^n$. The number M is called an essential upper bound for f. The space $L^\infty(\mathbb{R}^n)$ is the set of all essentially bounded functions on $\mathbb{R}^n$. For $f \in L^\infty(\mathbb{R}^n)$, $\|f\|_\infty$ is the infimum of the essential upper bounds for f. Then $(L^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space. A function f defined on $\mathbb{R}^n$ vanishes at infinity if

$$\lim_{\|x\|\to\infty} f(x) = 0.$$

The space $\mathcal{C}_0(\mathbb{R}^n)$ is the set of continuous functions on $\mathbb{R}^n$ vanishing at infinity. It is well known that $\mathcal{C}_0(\mathbb{R}^n)$ is a closed subspace of $L^\infty(\mathbb{R}^n)$.

Definition 1.1.1. Let $f \in L^1(\mathbb{R}^n)$. The *Fourier transform* of f, denoted by $\widehat{f}$, is the function on $\mathbb{R}^n$ given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n.$$

It is easy to see that $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$, and the map $f \to \widehat{f}$ is a bounded linear transformation from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

The following result, called the *Riemann-Lebesgue lemma*, gives a necessary condition for a function to be a Fourier transform of an L^1 function.

Theorem 1.1.2. *If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f} \in \mathcal{C}_0(\mathbb{R}^n)$.*

The space $L^1(\mathbb{R}^n)$ is endowed with a multiplication, called convolution, which turns it into a commutative Banach algebra.

Definition 1.1.3. Let $f, g \in L^1(\mathbb{R}^n)$. The *convolution* of f and g, denoted by $f * g$, is the function on $\mathbb{R}^n$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy, \quad x \in \mathbb{R}^n.$$

An essential feature of convolution is the fact that the Fourier transform of the convolution of two functions is the pointwise product of their Fourier transforms, i.e., if $f, g \in L^1(\mathbb{R}^n)$, then

$$\widehat{f * g} = \widehat{f}\widehat{g}.$$

More generally, $f * g$ can be defined if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, for $1 \leq p, q \leq \infty$. In fact, we have the following inequality which is known as Young's inequality. Let $p, q, r \in [1, \infty]$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$. Moreover,

$$\|f * g\|_r \leq \|f\|_p \|g\|_q .$$

The integral defining the Fourier transform is not defined in the Lebesgue sense if $f \in L^2(\mathbb{R}^n)$. Nevertheless the Fourier transform has a natural definition on $L^2(\mathbb{R}^n)$, and a particularly elegant theory. We first recall the definition of the Schwartz space. For a detailed discussion, we refer to [41, 50].

We use the standard multi-index notation. A multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers. Let $\mathbb{N}_0$ denote the set of non-negative integers, and let $\mathbb{N}_0^n$ denote the set of all multi-indices. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the symbols

$|\alpha|$, x^α, and D^α are defined as follows:

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

A smooth function $f : \mathbb{R}^n \to \mathbb{C}$ belongs to $\mathcal{S}(\mathbb{R}^n)$, the *Schwartz space* on $\mathbb{R}^n$, if for each pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$,

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha (D^\beta f)(x) \right| < \infty.$$

It is well known that the Schwartz space is a Fréchet space with respect to the family of semi-norms $\|\cdot\|_{\alpha,\beta}$. The topological dual of $\mathcal{S}(\mathbb{R}^n)$, denoted $\mathcal{S}'(\mathbb{R}^n)$, is called the space of *tempered distributions* on $\mathbb{R}^n$.

It follows from the definition that $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. In fact, we have the following result.

Theorem 1.1.4. *If $1 \leq p < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

The space $\mathcal{S}(\mathbb{R}^n)$ is not dense in $L^\infty(\mathbb{R}^n)$. However, $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{C}_0(\mathbb{R}^n)$. It is easy to check that the Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to itself. Moreover, we have the following fundamental result.

Theorem 1.1.5. *The Fourier transform is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, whose inverse is given by the formula*

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi, \quad x \in \mathbb{R}^n.$$

Moreover, $\|f\|_2 = \left\| \widehat{f} \right\|_2$.

The above two theorems assert that the Fourier transform is a bounded linear operator defined on the dense subset $\mathcal{S}(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$. In fact, it is an isometry. Therefore there exists a unique bounded extension $\mathcal{F}$ of this operator to all of $L^2(\mathbb{R}^n)$. The operator $\mathcal{F}$ is called the Fourier transform on $L^2(\mathbb{R}^n)$. Moreover if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\mathcal{F}(f) = \widehat{f}$.

Theorem 1.1.6 (Plancherel)**.** *If $f \in L^2(\mathbb{R}^n)$, then*

$$\|\mathcal{F}(f)\|_2 = \|f\|_2 .$$

In fact, the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$.

We also have the following formula, which is known as the Fourier inversion formula.

Theorem 1.1.7. *If both f and $\widehat{f}$ are in $L^1(\mathbb{R}^n)$, then*

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi$$

for almost every $x \in \mathbb{R}^n$.

Since $L^p(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, $1 \leq p \leq 2$, we can define the Fourier transform of a function in $L^p(\mathbb{R}^n)$. We use the notation $\widehat{f}$ for the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$.

Theorem 1.1.8 (Hausdorff-Young)**.** *The Fourier transform maps $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, if $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R}^n)$, then*

$$\left\|\widehat{f}\right\|_{p'} \leq \|f\|_p .$$

More generally, we can define the Fourier transform of a tempered distribution. For a detailed discussion, we refer to [41, 44].

Definition 1.1.9. Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then the *Fourier transform* of T, denoted by $\widehat{T}$, is the tempered distribution defined by $\widehat{T}(\varphi) = T(\widehat{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

The Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ is the unique weakly continuous extension of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$. In particular, the *Fourier transform* of a finite Borel measure λ on $\mathbb{R}^n$ is given by

$$\widehat{\lambda}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\, d\lambda(x), \quad \xi \in \mathbb{R}^n.$$

The Fourier transform has been generalized to locally compact abelian groups (see e.g., [10, 22, 42]), and even certain classes of locally compact non-abelian groups (see [53]). The group Fourier transform on a non-abelian group is an operator valued function on the unitary dual. Therefore, basic results about the Fourier transform have to take into account operator theoretic properties of the pointwise values of the Fourier transform.

Let G be a locally compact group, and let μ denote the left Haar measure on G. Let us first recall a few elementary definitions from representation theory. For a detailed discussion, we refer to [14, 19].

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{U}(\mathcal{H})$ denote the group of unitary operators acting on $\mathcal{H}$. A homomorphism π of G into $\mathcal{U}(\mathcal{H})$ is said to be a *unitary representation* of G on $\mathcal{H}$, if for every $x \in \mathcal{H}$, the map $g \to \pi(g)x$ is continuous. We denote this unitary representation by $(\pi, \mathcal{H})$.

A subspace M of $\mathcal{H}$ is said to be invariant under π if $\pi(g)x \in M$ for all $g \in G$, whenever $x \in M$. A representation $(\pi, \mathcal{H})$ is said to be irreducible if there is no nontrivial proper closed subspace M of $\mathcal{H}$ that is invariant under π.

Two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are equivalent if there exists a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_1 = U^{-1} \circ \pi_2 \circ U$. The unitary dual $\widehat{G}$ is the set of all equivalence classes of irreducible unitary representations of G.

Given $f \in L^1(G)$ and $(\pi, \mathcal{H}) \in \widehat{G}$, define the operator $\widehat{f}(\pi)$ on $\mathcal{H}$ by

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)\, d\mu(x).$$

The map $f \to \widehat{f}$ is called the *group Fourier transform* on G. For more details we refer to [10].

For a large class of groups G, there exists a measure on $\widehat{G}$ which is called the *Plancherel measure* such that there are analogs of the Fourier inversion formula and the Plancherel theorem. This class of groups include locally compact abelian groups, compact groups, and some locally compact non-abelian groups. One such group is the Heisenberg group.

1.2 Heisenberg group

In this section, we introduce the Heisenberg group and study the group Fourier transform on the Heisenberg group. The Heisenberg group plays an important role in several branches of mathematics such as representation theory, harmonic analysis, several complex variables, partial differential equations, and quantum mechanics. For a detailed discussion on the role of the Heisenberg group in Harmonic analysis, we refer to [25].

Definition 1.2.1. The *Heisenberg group* $\mathbf{H_n}$ is the set $\{(x,y,s)|x,y \in \mathbb{R}^n, s \in \mathbb{R}\}$ with multiplication defined by

$$(x,y,s)(x',y',s') = \left(x+x', y+y', s+s' + \frac{1}{2}(x \cdot y' - y \cdot x')\right).$$

Observe that the Heisenberg group is a locally compact non-abelian group. The representation theory of the Heisenberg group is fairly simple and well understood. The results in this section are well known and easy to find in the literature, see for example [9, 57, 67].

Let $\mathcal{H} = L^2(\mathbb{R}^n)$, and let $\mathcal{U}(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. For $h \in \mathbb{R}$, define $\rho_h : \mathbf{H_n} \to \mathcal{U}(\mathcal{H})$ by

$$(\rho_h(x,y,s)\varphi)(t) = e^{2\pi ihs + \pi ihx\cdot y + 2\pi iy\cdot t}\varphi(t+hx), \quad \varphi \in \mathcal{H},\, t \in \mathbb{R}^n.$$

It is well known that ρ_h is a unitary representation of $\mathbf{H_n}$ on $\mathcal{H}$. Moreover, if $h \neq h'$, then ρ_h and $\rho_{h'}$ are inequivalent. Furthermore, if $h \neq 0$ then ρ_h is irreducible. This representation ρ_h is called the *Schrödinger representation* of $\mathbf{H_n}$ with parameter h. Therefore, we have a family $\{\rho_h \,|\, h \in \mathbb{R} \setminus \{0\}\}$ of irreducible unitary representations of $\mathbf{H_n}$. The following result of Stone and von Neumann states that any irreducible unitary representation of $\mathbf{H_n}$ which is non-trivial on the center is equivalent to some ρ_h (see [9, Theorem 1.50]).

Theorem 1.2.2. *If γ is an irreducible unitary representation of $\mathbf{H_n}$ on a Hilbert space $\mathcal{H}_\gamma$ such that $\gamma(0,0,s) = e^{2\pi ihs}I$ for some $h \neq 0$, then γ is unitarily equivalent to ρ_h.*

The Stone-von Neumann theorem gives a complete classification of all the irreducible unitary representations of the Heisenberg group.

Corollary 1.2.3. *Every irreducible unitary representation of $\mathbf{H_n}$ is unitarily equivalent to one and only one of the following representations:*

(a) $\rho_h(x, y, s)$, $h \in \mathbb{R} \setminus \{0\}$ *acting on* $\mathcal{H}$,

(b) $\sigma_{ab}(x, y, s) = e^{2\pi i(a\cdot x + b\cdot y)}$, $a, b \in \mathbb{R}^n$ *acting on* $\mathbb{C}$.

Since the Heisenberg group $\mathbf{H_n}$ equals $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ as a set, we have the Lebesgue measure $dx\,dy\,ds$ on $\mathbf{H_n}$. Observe that this measure is both left and right invariant. Hence it is the Haar measure on $\mathbf{H_n}$. With this Haar measure, we form the usual function spaces $L^p(\mathbf{H_n})$, $1 \leq p \leq \infty$.

Let $f \in L^1(\mathbf{H_n})$. If $h \neq 0$, define

$$\widehat{f}(\rho_h) = \int_{\mathbf{H_n}} f(x, y, s)\rho_h(x, y, s)\, dx\, dy\, ds, \tag{1.1}$$

where the integral is the weak integral as defined in [44, Definition 3.26].

Observe that $\widehat{f}(\rho_h)$ is a bounded operator on $\mathcal{H}$, and the operator norm satisfies $\left\|\widehat{f}(\rho_h)\right\|_{\text{op}} \leq \|f\|_1$. It is well known that the Plancherel measure on $\widehat{\mathbf{H_n}}$ is $|h|^n\, dh$ on the family $\{\rho_h\}$, and 0 on the family $\{\sigma_{ab}\}$, i.e., the one-dimensional representations σ_{ab} form a set of Plancherel measure zero. There are analogs of the Plancherel theorem and the Fourier inversion formula for the group Fourier transform on $\mathbf{H_n}$. For a detailed discussion of the group Fourier transform on the Heisenberg group, we refer to [15, 16, 17].

For $f \in L^1(\mathbf{H_n})$ and $h \neq 0$, define

$$f^h(x, y) = \int_{\mathbb{R}} e^{2\pi i h s} f(x, y, s)\, ds.$$

Observe that f^h is the inverse Fourier transform of f in the s variable. Then $f^h \in L^1(\mathbb{R}^{2n})$.

Since $\rho_h(x, y, s) = e^{2\pi ihs}\rho_h(x, y, 0)$, it follows that

$$\widehat{f}(\rho_h) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^h(x, y)\rho_h(x, y, 0)\,dx\,dy.$$

Thus we are led to consider operators of the form

$$W_h(g) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x, y)\rho_h(x, y, 0)\,dx\,dy$$

for functions $g \in L^1(\mathbb{R}^{2n})$. When $h = 1$, this operator is called the *Weyl transform*, and is denoted by $W(g)$.

The kernel of ρ_1 is the set $\{(0, 0, k)|k \in \mathbb{Z}\}$. The quotient group $\mathbf{H_n}/\{(0, 0, k)|k \in \mathbb{Z}\}$ is called the reduced Heisenberg group, and is denoted by $\mathbf{H_n^{red}}$. Elements of $\mathbf{H_n^{red}}$ are of the form (x, y, z) where $x, y \in \mathbb{R}^n$ and $z \in \mathbb{C}$ such that $|z| = 1$, and the multiplication is defined by

$$(x, y, z)(x', y', z') = \left(x + x', y + y', zz'e^{\pi i(x\cdot y' - y\cdot x')}\right).$$

The Schrödinger representation ρ regarded as a representation on $\mathbf{H_n^{red}}$ is now faithful, and given by the formula

$$\left(\rho(x, y, z)\varphi\right)(t) = ze^{\pi ix\cdot y + 2\pi iy\cdot t}\varphi(t + x), \quad \varphi \in \mathcal{H}, t \in \mathbb{R}^n. \tag{1.2}$$

Thus the Weyl transform of a function $g \in L^1(\mathbb{R}^{2n})$ is given by

$$(W(g)\varphi)(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x, y)(\rho(x, y, 1)\varphi)(t)\,dx\,dy, \quad \varphi \in \mathcal{H}, t \in \mathbb{R}^n.$$

1.3 Weyl transform

In this section, we recall some important properties of the Weyl transform. As discussed in the previous section, the Weyl transform is the essence of the group Fourier transform on the Heisenberg group. Thus there are analogs of the Riemann-Lebesgue lemma, the Plancherel theorem, and the Hausdorff-Young theorem for the Weyl transform. We give a brief discussion here. For more details, we refer to [9, 57].

Let $\mathcal{H} = L^2(\mathbb{R}^n)$, and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. For $X \in \mathcal{B}(\mathcal{H})$, let $\|X\|_{\mathrm{op}}$ denote the operator norm of X.

Definition 1.3.1. Let $f \in L^1(\mathbb{R}^{2n})$. The *Weyl transform* of f is the operator $W(f) \in \mathcal{B}(\mathcal{H})$ defined by

$$(W(f)\varphi)(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x,y)e^{\pi i(x\cdot y+2y\cdot t)}\varphi(t+x)\,dx\,dy, \quad \varphi \in \mathcal{H}, t \in \mathbb{R}^n.$$

Thus $W(f)$ is an integral operator with kernel K_f given by

$$K_f(t,u) = \int_{\mathbb{R}^n} f(u-t,y)e^{\pi i(u+t)\cdot y}\,dy, \quad t,u \in \mathbb{R}^n. \tag{1.3}$$

Theorem 1.3.2. *The map $W : L^1(\mathbb{R}^{2n}) \to \mathcal{B}(\mathcal{H})$ is a bounded linear map. Moreover if $f \in L^1(\mathbb{R}^{2n})$, then $\|W(f)\|_{\mathrm{op}} \le \|f\|_1$.*

The following theorem is an analog of the Riemann-Lebesgue lemma for the Weyl transform.

Theorem 1.3.3. *If $f \in L^1(\mathbb{R}^{2n})$, then $W(f)$ is a compact operator.*

We now define a multiplication on the space $L^1(\mathbb{R}^{2n})$ which turns it into a non-commutative Banach algebra.

Definition 1.3.4. Let $f, g \in L^1(\mathbb{R}^{2n})$. The *twisted convolution* of f and g, denoted by $f \natural g$, is the function on $\mathbb{R}^{2n}$ defined by

$$(f \natural g)(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-x', y-y')g(x',y')e^{\pi i(x\cdot y' - y\cdot x')}\, dx'\, dy', \quad (x,y) \in \mathbb{R}^{2n}.$$

Observe that if $f, g \in L^1(\mathbb{R}^{2n})$, then $f \natural g \in L^1(\mathbb{R}^{2n})$. However, unlike ordinary convolution, the operation of twisted convolution is not commutative. Like ordinary convolution, twisted convolution extends from $L^1(\mathbb{R}^{2n})$ to other $L^p(\mathbb{R}^{2n})$ spaces and we have the following analog of Young's inequality. Let $p, q, r \in [1, \infty]$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

If $f \in L^p(\mathbb{R}^{2n})$ and $g \in L^q(\mathbb{R}^{2n})$, then $f \natural g \in L^r(\mathbb{R}^{2n})$. Moreover,

$$\|f \natural g\|_r \leq \|f\|_p \|g\|_q .$$

An essential feature of twisted convolution is the fact that if $f, g \in L^1(\mathbb{R}^{2n})$, then

$$W(f \natural g) = W(f)W(g),$$

i.e., the Weyl transform is an algebra homomorphism from $L^1(\mathbb{R}^{2n})$ to $\mathcal{B}(\mathcal{H})$.

For $1 \leq p < \infty$, let $S^p(\mathcal{H})$ denote the p-Schatten class of $\mathcal{H}$. Recall that an operator T is in $S^p(\mathcal{H})$ if

$$\|T\|_{S^p}^p = \operatorname{tr}((TT^*)^{p/2}) < \infty,$$

where tr is the trace of an operator. Let $S^\infty(\mathcal{H}) = \mathcal{B}(\mathcal{H})$, and $\|T\|_{S^\infty}$ be the operator norm of T. The family $\{S^p(\mathcal{H}) \,|\, p \in [1, \infty]\}$ forms a complex interpolation scale (see [41]). This class of operators is discussed in detail in Section 2.2. It is a well-known result that if the kernel of an integral operator is square integrable, then the integral operator is in $S^2(\mathcal{H})$ (see [48, Theorem 3.8.5]).

Observe that if $f \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$, the kernel K_f of $W(f)$ belongs to $L^2(\mathbb{R}^{2n})$, and $\|W(f)\|_{S^2} = \|K_f\|_2$. Therefore $W(f) \in S^2(\mathcal{H})$, and

$$\|W(f)\|_{S^2} = \|f\|_2 \,.$$

Hence, we have the following analogs of the Plancherel theorem and the Hausdorff-Young theorem for the Weyl transform.

Theorem 1.3.5. *The Weyl transform extends to an isometric isomorphism from $L^2(\mathbb{R}^{2n})$ to $S^2(\mathcal{H})$.*

Theorem 1.3.6. *The Weyl transform extends to a bounded linear map from $L^p(\mathbb{R}^{2n})$ to $S^{p'}(\mathcal{H})$, where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, if $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R}^{2n})$, then*

$$\|W(f)\|_{S^{p'}} \leq \|f\|_p \,.$$

We now discuss the analog of the inversion formula for the Weyl transform.

Definition 1.3.7. Let $X \in S^1(\mathcal{H})$. The *Fourier-Wigner transform* of X is the function $\alpha(X) : \mathbb{R}^{2n} \to \mathbb{C}$ defined by

$$\alpha(X)(x, y) = \mathrm{tr}(X\rho(x, y, 1)^{-1}).$$

Observe that if $\varphi, \psi \in \mathcal{H}$, and $X = \varphi \otimes \overline{\psi}$, then $\alpha(X)(x, y)$ is just the matrix coefficient $\langle \rho(x, y, 1)^{-1}\varphi, \psi\rangle$. The following results are well-known and easy to find in the literature, see for example [9].

Theorem 1.3.8. *If $X \in S^1(\mathcal{H})$, then $\alpha(X) \in \mathcal{C}_0(\mathbb{R}^{2n})$.*

Theorem 1.3.9. *The map* $\alpha : S^1(\mathcal{H}) \to \mathcal{C}_0(\mathbb{R}^{2n})$ *is a bounded linear map, and it extends to an isometric isomorphism from* $S^2(\mathcal{H})$ *onto* $L^2(\mathbb{R}^{2n})$*.*

Theorem 1.3.10. *If* $X \in S^1(\mathcal{H})$ *and* $\alpha(X) \in L^1(\mathbb{R}^{2n})$*, then* $W(\alpha(X)) = X$*, and if* $f \in L^1(\mathbb{R}^{2n})$ *and* $W(f) \in S^1(\mathcal{H})$*, then* $\alpha(W(f)) = f$*.*

The Weyl transform has been generalized to the space of tempered distributions (see e.g., [26, 30]). This will be discussed in Section 4.2. In particular, if λ is a finite Borel measure on $\mathbb{R}^{2n}$, the *Weyl Transform* of λ is the operator $W(\lambda) \in \mathcal{B}(\mathcal{H})$ defined by

$$(W(\lambda)\varphi)(t) = \int_{\mathbb{R}^{2n}} (\rho(x, y, 1)\varphi)(t)\, d\lambda(x, y), \quad \varphi \in \mathcal{H}, t \in \mathbb{R}^n. \tag{1.4}$$

For a long time, the notion of Weyl transform was studied only in the case of $\mathbb{R}^{2n}$. In [63], Weil considered the Heisenberg group over a locally compact abelian group. Inspired by this, Hennings [20] extended the notion of the Weyl transform to locally compact abelian groups. Analogs of certain classical results are proved in this new setting in [31, 39]. The notion of Weyl transform has been generalized to certain classes of locally compact non-abelian groups (see [5, 4, 36]).

There has been a longstanding interest in finding quantum or non-commutative analogs of classical results in harmonic analysis. For example, the quantum analog of the Hausdorff-Young theorem (in greater generality) was proven by Kunze [28]. Subsequently, there have been more developments in this direction (see [46, 64] and the references within). In this thesis, we will prove an analog for the Weyl transform of certain results about the $\mathcal{C}_0$ membership and the L^p membership of the Fourier transform of a measure with appropriate curvature assumptions.

1.4 Curvature in Fourier Analysis

Recall that the Fourier transform of a finite Borel measure λ on $\mathbb{R}^n$ is given by

$$\widehat{\lambda}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\, d\lambda(x), \quad \xi \in \mathbb{R}^n.$$

Observe that if λ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}^n$, i.e., $\lambda = fm$ for some $f \in L^1(\mathbb{R}^n)$, then the formula above reduces to the usual definition of the Fourier transform of f, and

$$\lim_{\|\xi\|\to\infty} \widehat{\lambda}(\xi) = 0.$$

In general, the Fourier transform of a measure need not vanish at infinity. For example, the Fourier transform of δ_0, the Dirac measure, is identically equal to 1. The behavior at infinity of the Fourier transform of a finite Borel measure has a long history (see [49] for more).

The first important observation in this connection concerns the case when the measure is supported on the unit sphere Σ in $\mathbb{R}^n$. Let σ be the measure on Σ induced by Lebesgue measure on $\mathbb{R}^n$. When $n > 1$, $\widehat{\sigma}$ has an unexpected decay at infinity. More precisely, Stieltjes proved that

$$\widehat{\sigma}(\xi) = 2\pi \|\xi\|^{(2-n)/2} J_{(n-2)/2}(2\pi \|\xi\|),$$

where J_m is the Bessel function of order m (see [49]). Since $J_m(r) = O(r^{-1/2})$, it follows that

$$|\widehat{\sigma}(\xi)| \le A \|\xi\|^{(1-n)/2},$$

where A is a constant independent of ξ.

During the 20^{th} century, it became increasingly evident that decay estimates of this kind are in fact deducible from curvature properties of the support of the measure (e.g., the spectral synthesis example of Schwartz [45] and work of Herz [21]), and culminated in the work of Stein, which we now describe.

Definition 1.4.1. Suppose M is a smooth submanifold of $\mathbb{R}^n$. By a *smooth measure* on M, we mean a measure of the form $\mu = \psi\sigma$, where σ is the measure on M induced by the Lebesgue measure on $\mathbb{R}^n$ and ψ is a smooth function on $\mathbb{R}^n$ whose support intersects M in a compact set.

Let S be a hypersurface in $\mathbb{R}^n$. Let $s_0 \in S$. By a rotation and translation of the ambient space $\mathbb{R}^n$, the point s_0 may be moved to the origin, and the tangent space at s_0 becomes the hyperplane $x_n = 0$. Near the origin, the surface S is a graph

$$x_n = \varphi(x_1, ..., x_{n-1}),$$

where φ is a smooth function on $\mathbb{R}^{n-1}$ such that $\varphi(0) = 0$, and $\nabla\varphi(0) = 0$. Recall that the Hessian of φ at 0 is the $(n-1) \times (n-1)$ matrix

$$\left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k}\right)(0).$$

The eigenvalues of the Hessian of φ at 0 are called the *principal curvatures* of S at s_0, and the product of the principal curvatures of S at s_0 is called the *Gaussian curvature* of S at s_0.

The following theorem establishes the connection between the curvature properties of a hypersurface and the decay of the Fourier transform of a smooth measure supported on the hypersurface. For a proof, see [49, p348, Theorem 1] and [24, Theorem 7.7.14].

Theorem 1.4.2. *Suppose S is a smooth hypersurface in $\mathbb{R}^n$, $n \geq 2$, whose Gaussian curvature is nonzero everywhere. Let μ be a smooth measure on S. Then*

$$|\widehat{\mu}(\xi)| \leq A \, \|\xi\|^{(1-n)/2},$$

where A is a constant independent of ξ.

The following result is an immediate corollary to Theorem 1.4.2.

Theorem 1.4.3. *Suppose S is a smooth hypersurface in $\mathbb{R}^n$, $n \geq 2$, whose Gaussian curvature is nonzero everywhere. Let μ be a smooth measure on S. Then $\widehat{\mu} \in \mathcal{C}_0(\mathbb{R}^n)$. Moreover, $\widehat{\mu} \in L^p(\mathbb{R}^n)$ if $p > 2n/(n-1)$.*

Naturally, the question of what happens if we consider a submanifold of arbitrary codimension arises. Let M be a smooth m-dimensional submanifold of $\mathbb{R}^n$, $1 \leq m \leq n-1$. The assumption regarding curvature has to be replaced by the more general assumption that, at each point, M has at most a finite order of contact with any affine hyperplane. Such submanifolds are called *submanifold of finite type*.

The precise meaning of submanifold of finite type is the following. Let $x_0 \in M$. In a sufficiently small neighborhood of x_0, M is the image of a smooth map $\varphi : U \to \mathbb{R}^n$, where U is a neighborhood of the origin in $\mathbb{R}^m$. If for each unit vector $\eta \in \mathbb{R}^n$, there exists a multi-index $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \geq 1$ such that

$$D^{\alpha}[\varphi(x) \cdot \eta] \neq 0 \tag{1.5}$$

at $x = x_0$, then M is of finite type at x_0. The smallest k for which there exists α with $|\alpha| = k$ such that α satisfies (1.5) is called the type of x_0. Also, if $K \subseteq U$ is a compact set, the type of K is defined to be the maximum of the types of the $x_0 \in K$.

The following theorem establishes the decay of the Fourier transform of a measure supported on a submanifold of finite type. For a proof, see [49, p351, Theorem 2].

Theorem 1.4.4. *Suppose M is a finite type smooth submanifold of $\mathbb{R}^n$, $n \geq 2$. Let $\mu = \psi\sigma$ be a smooth measure on M. Then*

$$|\widehat{\mu}(\xi)| \leq A \|\xi\|^{-1/k},$$

where k is the type of M inside the support of ψ.

The following two results are immediate corollaries to Theorem 1.4.4.

Theorem 1.4.5. *Suppose M is a finite type smooth submanifold of $\mathbb{R}^n$, $n \geq 2$. Let μ be a smooth measure on M. Then $\widehat{\mu} \in \mathcal{C}_0(\mathbb{R}^n)$.*

Theorem 1.4.6. *Suppose M is a finite type smooth submanifold of $\mathbb{R}^n$, $n \geq 2$. Let $\mu = \psi\sigma$ be a smooth measure on M. Then $\widehat{\mu} \in L^p(\mathbb{R}^n)$ if $p > nk$, where k is the type of M inside the support of ψ.*

These results have important consequences, such as the Fourier restriction theorem (see [49, p352]), and the existence of L^p functions whose translates are linearly dependent (see [8]).

Since the Weyl transform is closely related to the group Fourier transform on the Heisenberg group, we expect to find analogs of the above mentioned results for the Weyl transform. The analog of the proposition that the Fourier transform of a finite Borel measure belongs to $\mathcal{C}_0$ is the proposition that the Weyl transform of the measure is a compact operator. Also, the Schatten classes are quantum analogs of the L^p spaces. We will prove in this thesis that under appropriate curvature assumptions, the Weyl transform of a smooth measure is compact and belongs to a Schatten class.

1.5 Outline of the thesis

The rest of the thesis is organized as follows.

In ***Chapter 2***, we prove an analog of Theorem 1.4.3 for the Weyl transform, i.e., we prove that the Weyl transform of a smooth measure supported on a smooth hypersurface of positive Gaussian curvature is compact, and belongs to a Schatten class. The results in this chapter are from [32].

In ***Chapter 3***, we prove an analog of Theorem 1.4.5 for the Weyl transform with the additional assumption of real analyticity of the submanifold, i.e., we prove that the Weyl transform of a smooth measure supported on a real-analytic submanifold of finite type is compact. The results in this chapter are from [34].

In ***Chapter 4***, we prove that the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is p-th power traceable if and only if the Fourier transform of the distribution is p-th power integrable . Moreover, we prove that the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is compact if and only if the Fourier transform of the distribution vanishes at infinity. The results in this chapter are from [33].

Finally, in ***Chapter 5***, we provide applications of the results obtained in the thesis. We describe the conditions under which the quantum translates of a non-zero Schatten class operator are linearly independent. Moreover, we prove an analog of the Fourier restriction theorem for the Fourier-Wigner transform.