Chapter 1

Introduction

For since the fabric of the universe is most perfect and the work of the most wise creator, nothing at all takes place in the universe in which some rule of maximum and minimum does not appear

Leonhard Euler

(1707-1783)

Optimization is about making something as good as possible. Generally, it involves improving a process to increase good outcomes and reduce bad ones. In mathematics, optimization means finding the best option based on a specific criterion from a set of possible choices. The goal is to determine the best combination of input variables that either maximizes or minimizes the result of a function with multiple variables. Nowadays, optimization has a strong theoretical base and is supported by many advanced algorithms and software. It has become a revolutionary tool for modeling and decision-making in various fields such as computer science, engineering, operations research, and economics.

Mathematically, optimization models comprise three significant components: decision variables, objective function, and constraints.

- *Decision variables* designate a value that may vary within the scope of a given optimization problem.
- In a mathematical optimization problem, the *objective function* expresses the problem's main criteria, whose value is either minimized or maximized over the set of feasible alternatives.
- *Constraints* are the logical conditions or allowable values or scopes for the variables in an optimization problem that the solution of a given problem must satisfy.

There are many theories and optimization tools to obtain optimal solutions. However, it is not always possible to properly represent real-world situations with classical mathematics due to the presence of uncertain events or environments. Most often, the recorded or collected data are inherently imprecise or inexact. The data may be affected by measurement errors or by random events. Sometimes, the data may be roughly estimated. To handle such situations, it is more appropriate to assume that the data belongs to some interval or some uncertain set. Due to this, the set-valued or interval-valued optimization problems come into the picture. In this thesis, the focus is to develop some optimality concepts and methods to handle interval-valued and set-valued optimization problems.

The interval optimization or interval-valued optimization problems are one type of set optimization or set-valued optimization problems. However, in the literature, the theories and methodologies have been proposed separately for interval optimization problems. Therefore, we also treat interval optimization problems separately from the considered set optimization problem in this thesis.

Next, a brief introduction to interval analysis and interval optimization problems has been given.

1.1 Interval analysis

In mathematical computations, the collected and recorded data frequently contain measurement errors or are otherwise uncertain due to rounding errors and approximations. Interval analysis aims to provide upper and lower bounds on how these errors and uncertainties impact a computed quantity. The goal is to make these interval bounds as narrow as possible. A key aspect of interval analysis is to develop some techniques or concepts that yield sharp (or nearly sharp) bounds for the solutions of numerical computing problems.

Some researchers proposed the idea of bounding the computational errors with intervals, e.g., [28, 183, 193, 194]. However, R. E. Moore [156] laid the foundation in the field of interval mathematics and gave an analysis of interval arithmetic. In addition to treating rounding errors, Moore extended the use of interval analysis to bound the effect of errors from all sources, including approximation errors and errors in data. After that, interval analysis has become a new and growing branch of applied mathematics.

Many real-world problems involve parameters with inherent uncertainties and imprecisions. Generally, the parameters affected by imprecision and uncertainty can not be measured with exact values. Therefore, one can not use real numbers to determine such parameters. This means that such problems can not be solved by real-valued functions. One way to overcome these kinds of deficiencies is to use interval values to incorporate the uncertainties of parameters. Thus, we need a function whose range is interval to handle such problems. In the next section, we provide a brief introduction to interval-valued functions.

1.2 Interval-valued functions

A function whose involving parameters are intervals is known as an interval-valued function (IVF). Mathematically, an IVF is a function of one or more variables whose domain is a nonempty subset of $\mathbb{R}^n$ and whose range is a subset of $I(\mathbb{R})$. Let X be a nonempty subset of $\mathbb{R}^n$. Then, an IVF $\mathbf{F} : X \to I(\mathbb{R})$ is presented by

$$\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)] \text{ for all } x \in X,$$

where $\underline{F}$ and $\overline{F}$ are lower and upper real-valued functions of $\mathbf{F}$, respectively, on X. The next section describes optimization problems associated with IVFs.

1.3 Interval optimization problems

The optimization problems associated with interval-valued functions are called interval optimization problems (IOPs). The study of IOPs has become a significant topic of research as an attempt to handle the uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. Most of the real-world problems cannot be dealt with conventional optimization problems with real-valued parameters. For example, let a company make two products g_1 and g_2. Let company make x_1 and x_2 quantities of g_1 and g_2 under budget constraints $S \subset \mathbb{R}^n$, respectively. On selling the products g_1 and g_2, suppose that the company earns c_1 and c_2 rupees per unit, respectively. Then, the optimization problem is to maximize $c_1x_1 + c_2x_2$ subject to budget constraint $S \subset \mathbb{R}^2$. However, the price of the product may vary from time to time in the market. Due to uncertainty, we cannot find the exact solution to such problems and cannot handle such problems with real-valued parameters. However, by assuming c_1 and c_2 to be a range of values or closed and bounded intervals, we can get a better solution. In this case, intervals are chosen such that one can make sure that the prices of the products will fall within the corresponding interval even if the prices of the products vary from time to time. Then, the optimization problem becomes an IOP as it involves the intervals.

Mathematically, an interval optimization problem is described by

$$\min_{x \in X} \mathbf{F}(x),$$

where $X \subseteq \mathbb{R}^n$ and $\mathbf{F} : X \to I(\mathbb{R})$ is an interval-valued function.

Next, we give a brief introduction to set-valued optimization.

1.4 Set-valued optimization

Set-valued optimization is a generalization of classical optimization that deals with objective functions mapping decision variables to sets of possible outcomes rather than single scalar values. This class is more general as such problems may include scalars, fuzzy numbers, intervals, vectors, etc. The theory of set optimization has gained attention within the optimization community, with numerous researchers exploring set optimization problems due to their broad applicability across various fields. Set optimization has found extensive use in applied mathematics, engineering, economics, finance, and medical sciences. Notably, several problems in uncertain optimization [8, 109], economics [142, 159], game theory [94, 184], optimal control [17, 95], and mathematical finance [93, 153] can be formulated as set optimization problems.

Set-valued optimization has greatly benefited from the advancements in nonsmooth and variational analysis, offering not only elegant proofs for established results in scalar, vector, and nonsmooth optimization but also introducing a range of powerful new techniques to these critical areas of applied mathematics. More importantly, set-valued optimization provides a unified framework that is particularly well-suited for studying a variety of complex optimization problems that arise in different research domains. Examples include duality principles in vector optimization, gap functions for vector variational inequalities, inverse problems in partial differential equations and variational inequalities, fuzzy optimization, image processing, viability theory, and mathematical economics. Given the natural occurrence of set-valued maps in many areas of pure and applied mathematics, set-valued optimization is poised to remain a significant and dynamic field of research in the foreseeable future.

Next, a brief introduction of set-valued functions, followed by set optimization problems, is given.

1.5 Set-valued functions

Set-valued functions, also known as multifunctions, are a generalization of classical (single-valued) functions, where each input is mapped to a set of possible outputs rather than a single output. These functions play a significant role in various branches of mathematics, including optimization, game theory, economics, and differential equations, among others.

Mathematically, a map $F : \mathcal{X} \rightrightarrows \mathcal{Y}$, which maps each element $x \in \mathcal{X}$ to a subset $F(x) \subseteq \mathcal{Y}$, is called a set-valued map, where $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces. Sometimes, the set-valued map F is defined as $F : \mathcal{X} \to \mathscr{P}(\mathcal{Y})$, where $\mathscr{P}(\mathcal{Y})$ is the power set of $\mathcal{Y}$. Since an empty set $\emptyset \in \mathscr{P}(\mathcal{Y})$, therefore $F(x)$ may be an empty set for some $x \in \mathcal{X}$. Considering this, the domain of a set-valued map F, denoted by $\mathrm{Dom}(F)$, is defined as $\mathrm{Dom}(F) = \{x \in \mathcal{X} : F(x) \neq \emptyset\}$.

1.6 Set optimization problem

An optimization problem that involves a set-valued function is called a set optimization problem (SOP). These problems arise when the outcome of a decision process is naturally represented as a set of possible solutions rather than a single value. Mathematically, a set optimization problem is formulated as

$$\min F(x) \text{ subject to } x \in \mathbb{R}^n,$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued function with $\mathrm{Dom}(F) = \mathbb{R}^n$.

Unlike classical optimization, where each decision variable corresponds to a single objective value, set optimization involves mappings that assign a set of possible outcomes to each decision point. This leads to a more complex framework, and hence the definition of solutions of set optimization problems is different from classical optimization problems. There are two common ways to define the solution concepts in set optimization. One is the vector approach, and the other is the set approach. The main disadvantage of using the vector approach is that it reduces the complexity of the set to a single point or vector, which may not adequately capture the entire properties of the set. This is particularly limiting when the decision-maker's preferences involve comparing entire sets rather than specific points within those sets. The set approach overcomes the drawback of the vector approach by allowing comparisons based on the entirety of the sets, which is more aligned with scenarios, where decision-makers are concerned with the overall properties of the solution sets. This approach can better accommodate preferences related to uncertainty, risk, or other multi-faceted considerations that are inherently set-based. Therefore, the set approach to define the solution concept is more adopted than the vector approach. In this thesis, we shall use the set approach to define the solution of set optimization problems.

Next, a brief literature survey on the interval and set-valued analysis is presented.

1.7 Literature survey

In this section, we start with the development of interval analysis and its calculus. After that, a brief discussion has been given on the development of notions such as optimality conditions and solutions of smooth and nonsmooth interval optimization problems. Further, we extend the literature survey to set-valued analysis in which we discuss the development of order relations, solutions concepts and methods for set optimization problems.

1.7.1 Literature on interval analysis

One of the key objectives of interval analysis is to establish upper and lower bounds on the impact of errors and uncertainties in computed quantities. The work in this direction was started by Warmus and Sunaga. They proposed the concept of bounding rounding errors using intervals as referred in [183, 193, 194]. However, the formalization of interval analysis as a systematic approach began with R.E. Moore's seminal book *Interval Analysis* [156] in 1966. Moore's work elevated this basic concept into a practical tool for rigorous error and uncertain analysis. In [156], Moore introduced the arithmetic for intervals. The interval arithmetic introduced in [156] was designed for intervals with finite endpoints. Subsequent advancements were made by Hanson [96] and Kahan [127]. They proposed extensions to interval arithmetic that allowed for intervals with infinite endpoints. Moreover, Hansen [98] developed a practical interval Newton algorithm in which the division by an interval containing zero is permitted. An error analysis and tools for analysis of algorithms containing intervals using the interval arithmetic have been given by Ris [169]. After that, many researchers have used the interval arithmetic given by Moore, for instance, see [97, 99, 100, 130, 160] and references therein. There are a few limitations in Moore's interval arithmetic, especially in finding the additive inverse. Mathematically, for a compact interval $\mathbf{A}$, $\mathbf{A} \oplus (-\mathbf{A}) \neq \mathbf{A}$ (unless $\mathbf{A} = \{a\}$ is a singleton). This deficiency has been partially overcome by Hukuhara [107]. In [107], Hukuhara gave a concept of difference of two intervals, named as Hukuhara difference or H-difference, and it is denoted by $\ominus_H$. This difference provides the additive inverse of compact intervals. However, the difference $\mathbf{A} \ominus_H \mathbf{B}$ of two intervals $\mathbf{A}$ and $\mathbf{B}$ exists only when the width of $\mathbf{A}$ is greater than or equal to the width of $\mathbf{B}$. This difference is modified by Markov [152] and is known as nonstandard subtraction. The difference in [152] is also based on the width of the intervals. Stefanini and Bede [179] introduced a more generalized concept of difference, which is known as generalized Hukuhara difference ($\ominus_{gH}$-difference). This gH-difference provides the additive inverse and is applicable to all pairs of compact intervals. This gH-difference has been adopted by many researchers comprehensively. Many important notions in calculus for interval-valued functions, such as continuity partial derivative, gradient, directional derivative, and differentiability, have been defined using this gH-difference. Further, a brief survey on the development of the calculus of interval-valued functions has been given.

1.7.2 Literature on calculus of interval-valued functions

Calculus plays a significant role in observing the characteristics of IVFs. The foundations of IVF calculus began in 1967 when Hukuhara [107] introduced the concept of H-differentiability for IVFs using H-difference. However, H-differentiability was later found to be restrictive, as pointed out in [40]. To overcome the limitations of H-differentiability, Bede and Gal [20] proposed a strongly generalized derivative (G-derivative) for IVFs and derived a Newton-Leibniz-type formula using this differentiability. Another form of the differentiability of IVFs was given by Markov by defining a new difference approach in [152]. Using this differentiability, Markov proved the mean value theorem for IVFs. A major advancement occurred in 2009 when Stefanini and Bede [179] introduced generalized Hukuhara differentiability (gH-differentiability) for IVFs using the generalized Hukuhara difference. This concept laid the groundwork for further developments in the direction of calculus for IVFs. Using the gH-difference, many important characteristics of IVFs, such as gH-continuity, gH-derivative, partial gH-derivatives, gH-gradients have been proposed by researchers, for instance, see [72, 73, 179]. Recognizing the restrictive nature of H-differentiability, Chalco et al. [39] proposed the π-derivative, a generalization of both the Hukuhara and G-derivatives. Chalco-Cano et al. [40] further expanded the calculus of IVFs by refining the generalized Hukuhara difference (gH-difference), and proved that the G-derivative is equivalent to the gH-derivative. Further, Lupulescu [149] introduced the concept of delta generalized Hukuhara differentiability on time scales, utilizing the

gH-difference approach. Subsequently, Stefanini and Bede [180] defined the level-wise gH-differentiability. Building on these developments, Ghosh [72] analyzed the gH-differentiability of multi-variable IVFs and applied it to propose a Newton method for capturing the efficient point of IOPs. Recently, Guo et al. [88] improved this area by defining a gH-symmetric derivative for IVFs. Ghosh et al. [74] generalized key derivative notions such as directional, Gâteaux, and Fréchet derivatives for IVFs. Recently, some fundamental concepts such as gH-subdifferentiability, gH-Clarke derivative, gH-Hadamard derivative, and other various notions have been proposed for IVFs also, for instance, see [10, 12, 43, 44, 78, 87, 135–137, 178].

1.7.3 Literature on interval optimization problems

In this section, we provide a concise overview of the literature on interval optimization problems, their solution concepts, and the optimality conditions for IOPs. We begin by exploring the ordering relations defined on intervals in literature.

Since intervals are not linearly ordered, many partial ordering relations are defined to introduce solution concepts for IOPs [25, 114, 123, 176, 196, 197]. Most of these partial order relations are discussed in [114] by Ishibuchi and Tanaka. By using these partial ordered relations, many theories and methods have been developed to solve IOPs [42, 47, 108, 112, 120, 157, 177, 195, 196, 198]. However, these references are insufficient to cover the various ordering of intervals and related solution concepts. Therefore, one may find a systematic review in [76, 123, 146] and their references.

Over the past two decades, IOPs have gained significant attention as a key area of research. The groundwork for this field was laid by Tanaka and Asai [185] in 1984, who discussed linear programming problems with interval coefficients in the objective function. A decade later, Tong [177] extended this approach to include problems, where both the objective function and constraints involved interval numbers. After that, Chanas and Kuchta [41, 42] converted linear optimization problems with uncertainty into deterministic ones by introducing an order relation of interval numbers. Liu and Da [147] developed a method for interval number optimization that employed fuzzy constraints to address linear problems. Sengupta et al. [176] focused on linear interval number programming problems, where both the objective function and inequality constraints were defined by interval numbers. They introduced the "acceptability index" as a novel way to solve uncertain linear programming problems. Subsequently, Quan et al. [168] treated interval numbers as random variables with uniform distributions and used a possibility degree to address multi-criteria decision-making problems. Moving beyond linear problems, Ma [150] tackled nonlinear interval number programming and proposed a deterministic method to obtain intervals for nonlinear objective functions, formulating a three-objective optimization problem. Wu [195] further contributed to the field by providing Karush-Kuhn-Tucker (KKT) conditions for IOPs using Hukuhara differentiability. After that, Chalco-Cano et al. [39] illustrated KKT conditions based on generalized Hukuhara differentiability for IOPs. Bhurjee and Panda [25], and Bhurjee and Pradhan [24] expanded this research in this direction by defining interval-valued functions in parametric form and exploring their properties and proposed methods to study the existence of efficient solutions in optimization problems involving IVFs. Ghosh advanced the computational aspect by proposing a Newton method [72] and its updated version [73] for finding the efficient point of IOPs. Apart from smooth IOPs, nonsmooth IOPs have also been getting attention in the past few years. Some authors converted nonsmooth IOPs into real-valued multiobjective problems to solve them [2, 13, 121, 188]. Optimality conditions using gH-Clarke derivative are introduced by Chauhan et al. [44], and a notion of weak sharp minima using gH-subifferentiability is presented by Kumar et al. [138] to solve nonsmooth IOPs. Recently, Upadhayay et al. proposed Newton and quasi-Newton methods for IOPs using their certain equivalence in [186, 187]. For more details about theories and methods to find efficient solutions for IOPs, see [10, 23, 26, 73, 77] and their references.

1.7.4 Literature on set-valued optimization

In this section, we present a brief literature survey on set-valued optimization. Initially, an overview of set-valued analysis in optimization is given. After that, we discuss the work on set order relations to define the solution concepts in set optimization. Further, detailed information about the existing methods to solve the optimization problems has been given.

The early glimpses of set-valued analysis in optimization have been found in [29, 49, 50, 70, 167]. In [29, 70], theory of maximization of multivalued functions has been proposed. In [49, 50], Corley defined the maximization with respect to a cone of a set-valued function into possibly infinite dimensions. Some necessary and sufficient optimality conditions and a Lagrangian theory have been developed in [49, 50] for set-valued functions in terms of tangent cones. However, a systematic and comprehensive study of set-valued analysis in optimization has been given in book *Set-valued Analysis* [15] by Aubin and Frankowska. After that, set-valued optimization has become a new and useful topic of research in optimization. For an overview and more details, one may refer to [9, 45, 58, 92, 115, 117, 131].

There are two approaches (vector and set) to define the solutions of set optimization problems. As already discussed in Section 1.6, we use the set approach in this thesis to define the solution of the set optimization problem. The credit for the set approach goes to Kuroiwa [139]. In [139], six order relations have been proposed. These relations have been further studied and applied by many researchers, for instance, see [4, 102, 140, 141, 159, 203]. Further, Jahn and Ha [119] have proposed a new set order relation called minmax set order relations to deal with the solutions of such set optimization problems, where the above mentioned six kinds of set order relations fail. Next, Chen et al. [46] introduced a weighted set order relation, which combines Kuroiwa's [139] upper and lower set order relations into a single framework. In 2018, Karaman et al. [128] introduced set order relations for families of sets using the Minkowski difference.

It is very difficult to handle set optimization directly due to its complex framework. However, there are several techniques to handle set optimization problems, for instance, see [7, 14, 60, 86, 102, 117, 125, 128, 200]. Scalarization and vectorization are two of them. In the scalarization technique, a set optimization problem is converted to a scalar optimization problem using some scalarization functions such as Gerstewitz, Drummond and Svaiter, and Hiriart-Urruty. An overview of these scalarizing functions has been given in [31]. In the vectorization technique, the set optimization problem is transformed into a family of multiobjective optimization problems and then is solved.

1.7.5 Literature on methods for set optimization problems

In this thesis, we focus on methods to solve a particular class of set optimization problems. In this class, the objective function of the problem has finitely many vector-valued functions. We present a brief literature review below on the work based on this direction.

Various types of algorithms have been proposed in the literature, including derivative-free algorithms, sorting-type algorithms, and scalarization-based algorithms. Derivative-free methods, as outlined in [116, 118, 132], are based on the strategy introduced in [48]. In [116], set-valued mappings are considered, where the epigraphical and hypographical multifunctions are convex. This convexity assumption is relaxed in [132] by utilizing an upper set less relation. In [118], a new derivative-free strategy is proposed, where multiple directions are chosen simultaneously at each iteration rather than just one. This approach is referred to as the rooted tree method because it generates a tree structure, with the initial point as the root and the leaves representing potential solutions.

Sorting-type algorithms have also been developed for set optimization problems with finite feasible sets. For example, [84, 85, 133, 134] propose forward and backward reduction procedures that work more efficiently than methods comparing every pair of sets. In [84, 85], an enumeration of the images of the set-valued mappings is achieved through a strongly monotone functional, followed by a forward iteration procedure.

Additionally, scalarization-based approaches have been introduced, especially for set-valued mappings with specific structures, often encountered in the robust counterpart of vector optimization problems under uncertainty [57,59,109,110,124,174]. Techniques like linear scalarization and ϵ-constraint methods are used in [57,109], while the weighted Chebyshev scalarization and its variants can be found in [124,174].

Recently, a steepest descent method [32] has been introduced for unconstrained set-valued optimization problems in which the objective set-valued mapping has a finite number of continuously differentiable selections.

In the next section, some required basic definitions, lemmas, propositions, and theorems have been given.

1.8 Preliminaries

Throughout the thesis, we denote the elements of $I(\mathbb{R})$ by bold capital letters, for instance, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$.

Next, we give the definition of fundamental operations on intervals from [156].

1.8.1 Fundamental operations on intervals

Arithmetic operations of two intervals $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ are defined by $\mathbf{A} \oplus \mathbf{B} = [\underline{a} + \underline{b},\ \overline{a} + \overline{b}],\ \mathbf{A} \ominus \mathbf{B} = [\underline{a} - \overline{b},\ \overline{a} - \underline{b}],\ \mathbf{A} \otimes \mathbf{B} = [\min\{\underline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \overline{a}\overline{b}\}]$

and

$$\lambda \odot \mathbf{A} = \mathbf{A} \odot \lambda = \begin{cases} [\lambda \underline{a},\ \lambda \overline{a}], & \text{if } \lambda \geq 0 \\ [\lambda \overline{a},\ \lambda \underline{a}], & \text{if } \lambda < 0, \end{cases}$$

where λ is a real constant.

The norm [156] of an interval $\mathbf{A} = [\underline{a}, \overline{a}]$ in $I(\mathbb{R})$ is defined by $\|\mathbf{A}\|_{I(\mathbb{R})} = \max\{|\underline{a}|, |\overline{a}|\}$.

The norm of an interval vector $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in I(\mathbb{R})^n$ is given by (see [156])

$$\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n} = \sum_{i=1}^{n} \|\mathbf{A}_i\|_{I(\mathbb{R})}.$$

It is to note that a real number p can be represented by the interval $[p, p]$.

Definition 1.1 (*gH*-difference of intervals [179]). *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two elements in $I(\mathbb{R})$. The gH-difference between $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A} \ominus_{gH} \boldsymbol{B}$, is defined by the interval $\boldsymbol{C}$ such that*

$$\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{C} \quad \textit{or} \quad \boldsymbol{B} = \boldsymbol{A} \ominus \boldsymbol{C}.$$

It is to be noted that for $\boldsymbol{A} = [\underline{a},\ \overline{a}]$ and $\boldsymbol{B} = \left[\underline{b},\ \overline{b}\right]$,

$$\boldsymbol{A} \ominus_{gH} \boldsymbol{B} = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\},\ \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right],$$

and $\boldsymbol{A} \ominus_{gH} \boldsymbol{A} = \boldsymbol{0}$.

Definition 1.2 (Algebraic operations on $I(\mathbb{R})^n$). *Let $\widehat{\boldsymbol{A}} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n)$ and $\widehat{\boldsymbol{B}} = (\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_n)$ be two elements in $I(\mathbb{R})^n$. An algebraic operation $\star$ between $\widehat{\boldsymbol{A}}$ and $\widehat{\boldsymbol{B}}$, denoted by $\widehat{\boldsymbol{A}} \star \widehat{\boldsymbol{B}}$, is defined by*

$$\widehat{\boldsymbol{A}} \star \widehat{\boldsymbol{B}} = (\boldsymbol{A}_1 \star \boldsymbol{B}_1, \boldsymbol{A}_2 \star \boldsymbol{B}_2, \ldots, \boldsymbol{A}_n \star \boldsymbol{B}_n),$$

where $\star \in \{\oplus,\ \ominus,\ \ominus_{gH}\}$.

Definition 1.3 (Product between elements of $\mathbb{R}^n$ and $I(\mathbb{R})^n$). *For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and a vector of intervals $\widehat{\boldsymbol{A}} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n) \in I(\mathbb{R})^n$ with $\boldsymbol{A}_i = [\underline{a}_i, \overline{a}_i]$ for each $i = 1, 2, \ldots, n$, the product between x and $\widehat{\boldsymbol{A}}$, denoted by $x^\top \odot \widehat{\boldsymbol{A}}$, is defined by*

$$x^\top \odot \widehat{\boldsymbol{A}} = \left[\min\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}, \max\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}\right].$$

Remark 1.1 *It is to notice that if all the components of $\widehat{\boldsymbol{A}}$ are degenerate intervals, i.e., $\widehat{\boldsymbol{A}} \in \mathbb{R}^n$, then the product $x^\top \odot \widehat{\boldsymbol{A}}$ reduces to the standard inner product of $x \in \mathbb{R}^n$ with $\widehat{\boldsymbol{A}}$.*

Definition 1.4 (Dominance of intervals [197]). *Let $\boldsymbol{A} = [\underline{a}, \overline{a}]$ and $\boldsymbol{B} = [\underline{b}, \overline{b}]$ be two elements in $I(\mathbb{R})$.*

(i) *$\boldsymbol{B}$ is said to be dominated by $\boldsymbol{A}$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, and then we write $\boldsymbol{A} \preceq \boldsymbol{B}$;*

(ii) *$\boldsymbol{B}$ is said to be strictly dominated by $\boldsymbol{A}$ if $\boldsymbol{A} \preceq \boldsymbol{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$, and then we write $\boldsymbol{A} \prec \boldsymbol{B}$. Equivalently, $\boldsymbol{A} \prec \boldsymbol{B}$ if and only if any of the following holds: '$\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$' or '$\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$' or '$\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$';*

(iii) *if neither $\boldsymbol{A} \preceq \boldsymbol{B}$ nor $\boldsymbol{B} \preceq \boldsymbol{A}$, we say that none of $\boldsymbol{A}$ and $\boldsymbol{B}$ dominates the other, or $\boldsymbol{A}$ and $\boldsymbol{B}$ are not comparable. Equivalently, $\boldsymbol{A}$ and $\boldsymbol{B}$ are not comparable if either '$\underline{a} < \underline{b}$ and $\overline{a} > \overline{b}$' or '$\underline{a} > \underline{b}$ and $\overline{a} < \overline{b}$';*

(iv) *if either $\boldsymbol{B} \preceq \boldsymbol{A}$ or $\boldsymbol{A}$ and $\boldsymbol{B}$ are not comparable, then we say $\boldsymbol{B}$ is not dominated by $\boldsymbol{A}$ and write $\boldsymbol{A} \not\prec \boldsymbol{B}$.*

1.8.2 Calculus of IVFs

Throughout this subsection, $\mathbf{F}$ is an IVF defined on a nonempty subset X of $\mathbb{R}^n$ unless mentioned otherwise.

Definition 1.5 (gH-continuity [73]). *Let $\boldsymbol{F}$ be an IVF and let $\bar{x}$ be a point of X and $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$. The function $\boldsymbol{F}$ is said to be gH-continuous at $\bar{x}$ if*

$$\lim_{\|h\| \to 0} \big(\boldsymbol{F}(\bar{x} + h) \ominus_{gH} \boldsymbol{F}(\bar{x})\big) = \boldsymbol{0}.$$

Lemma 1.1 (See [75]). *$\underline{F}$ and $\overline{F}$ are continuous on X if and only if $\boldsymbol{F}$ is gH-continuous on X, where $\underline{F}$ and $\overline{F}$ are lower and upper real-valued functions of F, respectively.*

Definition 1.6 (gH-derivative [179]). *The gH-derivative of an IVF $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ at $\bar{x} \in \mathbb{R}$ is defined by*

$$\boldsymbol{F}'(\bar{x}) = \lim_{h \to 0} \frac{\boldsymbol{F}(\bar{x}+h) \ominus_{gH} \boldsymbol{F}(\bar{x})}{h}, \text{ provided the limit exists.}$$

Remark 1.2 (See [179]). *Let $\boldsymbol{F} = [\underline{F}, \overline{F}]$ be an IVF on X, where $\underline{F}$ and $\overline{F}$ are real-valued functions defined on X. Then, the gH-derivative of $\boldsymbol{F}$ at $\bar{x} \in X$ exists if the derivatives of $\underline{F}$ and $\overline{F}$ at $\bar{x}$ exist and*

$$\boldsymbol{F}'(\bar{x}) = \left[\min\left\{\underline{F}'(\bar{x}), \overline{F}'(\bar{x})\right\}, \max\left\{\underline{F}'(\bar{x}), \overline{F}'(\bar{x})\right\}\right].$$

Definition 1.7 (gH-partial derivative [73]). *Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^\top$ be a point of X. For a given $i \in \{1, 2, \ldots, n\}$, we define a function $\boldsymbol{G}_i$ by $\boldsymbol{G}_i(x_i) = \boldsymbol{F}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n)$. If gH-derivative of $\boldsymbol{G}_i$ exists at $\bar{x}_i$, then we say that $\boldsymbol{F}$ has the ith gH-partial derivative at $\bar{x}$. We denote the ith gH-partial derivative of $\boldsymbol{F}$ at $\bar{x}$ by $D_i\boldsymbol{F}(\bar{x})$, i.e., $D_i\boldsymbol{F}(\bar{x}) = \boldsymbol{G}_i'(\bar{x}_i)^\top$.*

Definition 1.8 (gH-gradient [73]). *The gH-gradient of $\boldsymbol{F}$ at a point $\bar{x} \in X$, denoted by $\nabla\boldsymbol{F}(\bar{x}) \in I(\mathbb{R})^n$, is defined by*

$$\nabla\boldsymbol{F}(\bar{x}) = (D_1\boldsymbol{F}_1(\bar{x}), D_2\boldsymbol{F}_2(\bar{x}), \ldots, D_n\boldsymbol{F}_n(\bar{x}))^\top.$$

Lemma 1.2 *Let $\boldsymbol{A}, \boldsymbol{B},$ and $\boldsymbol{C}$ are in $I(\mathbb{R})$. Then, for any real number r,*

(i) $[r, r] \preceq \boldsymbol{A}$ *and* $\boldsymbol{A} \preceq \boldsymbol{B} \ominus_{gH} \boldsymbol{C} \implies \boldsymbol{C} \oplus [r, r] \preceq \boldsymbol{B}$ *and*

(ii) $\big((1-\lambda) \odot \boldsymbol{A} \oplus \lambda \odot \boldsymbol{B}\big) \ominus_{gH} \boldsymbol{A} = \lambda \odot \big(\boldsymbol{B} \ominus_{gH} \boldsymbol{A}\big)$ *for any* $\lambda \in [0, 1]$.

Proof: See Appendix A.1. □

Definition 1.9 (Convex IVF [195]). *Let X be a nonempty convex subset of $\mathbb{R}^n$. An*

IVF $\boldsymbol{F} : X \to I(\mathbb{R})$ is said to be convex on X if for any x_1 and x_2 in X,

$$\boldsymbol{F}(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 \odot \boldsymbol{F}(x_1) \oplus \lambda_2 \odot \boldsymbol{F}(x_2) \text{ for all } \lambda_1, \lambda_2 \in [0,1] \text{ with } \lambda_1 + \lambda_2 = 1.$$

Lemma 1.3 (See [195]). *Let X be a nonempty convex subset of $\mathbb{R}^n$, and $\boldsymbol{F} = [\underline{F}, \overline{F}]$ be an IVF on X, where $\underline{F}$ and $\overline{F}$ are real-valued functions defined on X. Then, $\boldsymbol{F}$ is convex on X if and only if $\underline{F}$ and $\overline{F}$ are convex on X.*

Theorem 1.1 (See [75]). *If an IVF $\boldsymbol{F}$ on $\mathbb{R}^n$ is convex, then $\boldsymbol{F}$ is gH-continuous on $\mathbb{R}^n$.*

Definition 1.10 (gH-directional derivative [74]). *Let $\boldsymbol{F}$ be an IVF on X. Let $\bar{x} \in X$ and $d \in \mathbb{R}^n$. If the limit*

$$\lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{x} + \lambda d) \ominus_{gH} \boldsymbol{F}(\bar{x}) \right)$$

exists, then the limit is said to be gH-directional derivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction d, and it is denoted by $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(d)$.

Definition 1.11 (gH-differentiability [73]). *An IVF $\boldsymbol{F}$ is said to be gH-differentiable at $\bar{x} \in X$ if there exist two IVFs $\boldsymbol{E}(\boldsymbol{F}(\bar{x}); h)$ and $\boldsymbol{L}_{\bar{x}} : \mathbb{R}^n \to I(\mathbb{R})$ such that*

$$\boldsymbol{F}(\bar{x} + h) \ominus_{gH} \boldsymbol{F}(\bar{x}) = \boldsymbol{L}_{\bar{x}}(h) \oplus \|h\| \odot \boldsymbol{E}(\boldsymbol{F}(\bar{x}); h)$$

for $\|h\| < \delta$ for some $\delta > 0$, where $\lim\limits_{\|h\| \to 0} \boldsymbol{E}(\boldsymbol{F}(\bar{x}); h) = \boldsymbol{0}$ and $\boldsymbol{L}_{\bar{x}}$ is such a function that satisfies

(i) *$\boldsymbol{L}_{\bar{x}}(x + y) = \boldsymbol{L}_{\bar{x}}(x) \oplus \boldsymbol{L}_{\bar{x}}(y)$ for all $x, y \in X$, and*

(ii) *$\boldsymbol{L}_{\bar{x}}(cx) = c \odot \boldsymbol{L}_{\bar{x}}(x)$ for all $c \in \mathbb{R}$ and $x \in X$.*

Theorem 1.2 (See [73]). *Let $\boldsymbol{F} : X \to I(\mathbb{R})$ be gH-differentiable at $\bar{x}$. Then, $\boldsymbol{L}_{\bar{x}}$ exists for every $h \in \mathbb{R}^n$ and*

$$\boldsymbol{L}_{\bar{x}} = \sum_{i=1}^{n} h_i \odot D_i \boldsymbol{F}(\bar{x}),$$

where $\sum_{i=1}^{n} h_i \odot D_i \boldsymbol{F}(\bar{x}) = h_1 \odot D_1 \boldsymbol{F}(\bar{x}) \oplus h_2 \odot D_2 \boldsymbol{F}(\bar{x}) \oplus \cdots \oplus h_n \odot D_n \boldsymbol{F}(\bar{x})$.

Remark 1.3 (See [73]). *Let $\boldsymbol{F} : X \to I(\mathbb{R})$ be gH-differentiable at $\bar{x} \in X$. Then, there exists a nonzero λ and $\delta > 0$ such that*

$$\lim_{\lambda \to 0} \tfrac{1}{\lambda} \odot \big(\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})\big) = \boldsymbol{L}_{\bar{x}}(h)$$

for all $h \in \mathbb{R}^n$ with $|\lambda| \|h\| < \delta$, where $\boldsymbol{L}_{\bar{x}}$ is an IVF, defined in Definition 1.11 of gH-differentiability.

Lemma 1.4 *Let $\boldsymbol{F} : X \to I(\mathbb{R})$ be a gH-differentiable at $\bar{x} \in X$. Then, $\boldsymbol{F}$ has gH-directional derivative at $\bar{x}$ for every direction $h \in \mathbb{R}^n$ and*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n,$$

where $\boldsymbol{L}_{\bar{x}}$ is as defined in Definition 1.11.

Proof: Since $\mathbf{F}$ is gH-differentiable at $\bar{x}$, by Remark 1.3, we have

$$\begin{aligned} & \lim_{\lambda \to 0} \tfrac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\big) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n \\ \implies & \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\big) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n. \end{aligned}$$

Hence, by Definition 1.10, we conclude that $\mathbf{F}$ has gH-directional derivative at $\bar{x}$ and

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(h) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n.$$

$\square$

Definition 1.12 (Proper IVF). *An extended IVF* $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ *is called a proper IVF if there exists* $\bar{x} \in X$ *such that* $\boldsymbol{F}(\bar{x}) \prec [+\infty, +\infty]$ *and* $[-\infty, -\infty] \prec \boldsymbol{F}(x)$ *for all* $x \in X$.

Definition 1.13 (Effective domain of IVF). *The effective domain of an extended IVF* $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ *is the collection of all such points at which* $\boldsymbol{F}$ *is finite. It is denoted by* $\mathrm{dom}(\boldsymbol{F})$, *i.e.,*

$$\mathrm{dom}(\boldsymbol{F}) = \left\{x \in X : \|\boldsymbol{F}(x)\|_{I(\mathbb{R})} < +\infty\right\}.$$

Definition 1.14 (Linear IVF). *An IVF* $\boldsymbol{F} : X \to I(\mathbb{R})$ *is said to be linear if the following two conditions hold:*

(i) $\boldsymbol{F}(cx) = c \odot \boldsymbol{F}(x)$ *for all* $x \in X$ *and for all* $c \in \mathbb{R}$, *and*

(ii) *for all* $x,\ y \in X$,

$$\boldsymbol{F}(x) \oplus \boldsymbol{F}(y) = \boldsymbol{F}(x + y).$$

Definition 1.15 (Sublinear IVF [44]). *Let* $\boldsymbol{F}$ *be an IVF on a linear subspace* S *of a real normed linear space. Then,* $\boldsymbol{F}$ *is sublinear on* S *if*

(i) $\boldsymbol{F}(\beta x) = \beta \odot \boldsymbol{F}(x)$ *for all* $x \in S$ *and* $\beta \geq 0$ *and*

(ii) $\boldsymbol{F}(y + x) \not\succ \boldsymbol{F}(y) \oplus \boldsymbol{F}(x)$ *for all* $y, x \in S$.

Theorem 1.3 *Let* X *be a nonempty convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F} : X \to I(\mathbb{R})$ *be a convex IVF with* $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$, *where* $\underline{F}$ *and* $\overline{F}$ *are real-valued functions defined on* X. *Then, at any* $\bar{x} \in X$, *gH-directional derivative* $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(d)$ *exists and*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(d) = \Big[\min\Big\{\underline{F}_{\mathscr{D}}(\bar{x})(d), \overline{F}_{\mathscr{D}}(\bar{x})(d)\Big\}, \max\Big\{\underline{F}_{\mathscr{D}}(\bar{x})(d), \overline{F}_{\mathscr{D}}(\bar{x})(d)\Big\}\Big].$$

Proof: Similar to the proof of Theorem 3.1 in [74]. □

Definition 1.16 (gH-Lipschitz continuous IVF [74]). *An IVF* $\boldsymbol{F}$ *is said to be* gH*-Lipschitz continuous on* X *if there exists* $M > 0$ *such that*

$$\|\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(y)\|_{I(\mathbb{R})} \leq M\|x - y\| \text{ for all } x, y \in X.$$

The constant M *is called a Lipschitz constant.*

Lemma 1.5 (See [75]). *Let* $\boldsymbol{F}$ *be an IVF on* $\mathbb{R}^n$ *such that*

$$\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(y) \preceq r\|x - y\| \text{ for all } x, y \in \mathbb{R}^n,$$

for some $r \in \mathbb{R}_+$. *Then,*

$$\|\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(y)\|_{I(\mathbb{R})} \leq r\|x - y\| \text{ for all } x, y \in \mathbb{R}^n.$$

Lemma 1.6 *For* $\boldsymbol{T}$, $\boldsymbol{U}$, *and* $\boldsymbol{V}$ *in* $I(\mathbb{R})$,

$$\left(\tfrac{1}{2} \odot \boldsymbol{T} \oplus \tfrac{1}{2} \odot \boldsymbol{U}\right) \ominus_{gH} \boldsymbol{V} \not\preceq \tfrac{1}{2} \odot \left(\boldsymbol{T} \ominus_{gH} \boldsymbol{V}\right) \oplus \tfrac{1}{2} \odot \left(\boldsymbol{U} \ominus_{gH} \boldsymbol{V}\right).$$

Proof: See Appendix A.2. □

Lemma 1.7 *For any* $x \in \mathbb{R}^n$, $\epsilon > 0$, $\widehat{\boldsymbol{T}} \in I(\mathbb{R})^n$, *and* $\boldsymbol{U}, \boldsymbol{V} \in I(\mathbb{R})$,

$$x^\top \odot \widehat{\boldsymbol{T}} \preceq \boldsymbol{U} \ominus_{gH} \boldsymbol{V} \oplus \epsilon \implies \boldsymbol{V} \ominus_{gH} \boldsymbol{U} \preceq (-x)^\top \odot \widehat{\boldsymbol{T}} \oplus \epsilon.$$

Proof: See Appendix A.3. □

Lemma 1.8 *For* $x \in \mathbb{R}^n$ $\epsilon > 0$, *and* $\widehat{\boldsymbol{T}} = (\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_n) \in I(\mathbb{R})^n$,

$$x^\top \odot \widehat{\boldsymbol{T}} \oplus \epsilon \preceq \|x\| \odot \left[\|\widehat{\boldsymbol{T}}\|_{I(\mathbb{R})^n} + \epsilon, \|\widehat{\boldsymbol{T}}\|_{I(\mathbb{R})^n} + \epsilon\right].$$

Proof: See Appendix A.4. □

Definition 1.17 (gH-locally Lipschitz continuous IVF). *An IVF $\boldsymbol{F}$ is said to be gH-locally Lipschitz continuous on X if there exists $M' > 0$ and $\delta > 0$ such that*

$$\|\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(y)\|_{I(\mathbb{R})} \leq M'\|x - y\| \text{ and } \|x - y\| \leq \delta \text{ for all } x, y \in X.$$

Definition 1.18 (Weak efficient solution [11]). *A point $\bar{x} \in S$ is said to be a weak efficient solution of IOP (2.22) if $\boldsymbol{F}(\bar{x}) \preceq \boldsymbol{F}(x)$ for all $x \in S$.*

Definition 1.19 (Supremum of a subset of $\overline{I(\mathbb{R})}$ [135]). *Let $\boldsymbol{S}$ be a nonempty subset of $\overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is said to be an upper bound of $\boldsymbol{S}$ if $\boldsymbol{B} \preceq \bar{\mathbf{A}}$ for all $\boldsymbol{B}$ in $\boldsymbol{S}$. An upper bound $\bar{\mathbf{A}}$ of $\boldsymbol{S}$ is called a supremum of $\boldsymbol{S}$, denoted by $\sup \boldsymbol{S}$, if for all upper bounds $\boldsymbol{C}$ of $\boldsymbol{S}$ in $I(\mathbb{R})$, $\bar{\mathbf{A}} \preceq \boldsymbol{C}$. Moreover, if the supremum of the set $\boldsymbol{S}$ belongs to the set itself, then it is called the maximum of $\boldsymbol{S}$, denoted by $\max \boldsymbol{S}$.*

Remark 1.4 (See [135]). *Let Λ be an index set, and $\lambda \in \Lambda$. For any subset $\boldsymbol{S} = [a_\lambda, b_\lambda]$ of $\overline{I(\mathbb{R})}$, we have $\sup \boldsymbol{S} = \left[\sup\limits_{\lambda\in\Lambda} a_\lambda,\ \sup\limits_{\lambda\in\Lambda} b_\lambda\right]$.*

Lemma 1.9 (See [135]). *Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two proper extended IVFs defined on S, which is a nonempty subset of X. Then,*

(i) $\inf\limits_{x\in S} \mathbf{F}_1(x) \oplus \inf\limits_{x\in S} \mathbf{F}_2(x) \preceq \inf\limits_{x\in S} \big(\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\big)$ *and*

(ii) $\sup\limits_{x\in S} \big(\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\big) \preceq \sup\limits_{x\in S} \mathbf{F}_1(x) \oplus \sup\limits_{x\in S} \mathbf{F}_2(x)$.

Definition 1.20 (Lower limit and gH-lower semicontinuity of an extended IVF [135]). *The lower limit of an extended IVF $\boldsymbol{F}$ at $\bar{x} \in X$, denoted by $\liminf\limits_{x\to\bar{x}} \boldsymbol{F}(x)$, is defined by*

$$\liminf_{x\to\bar{x}} \boldsymbol{F}(x) \quad = \quad \lim_{\delta\downarrow 0}(\inf\{\boldsymbol{F}(x) : x \in B_\delta(\bar{x})\}) = \sup_{\delta>0}(\inf\{\boldsymbol{F}(x) : x \in B_\delta(\bar{x})\}),$$

where $B_\delta(\bar{x})$ is an open ball with radius δ centered at $\bar{x}$. $\boldsymbol{F}$ is called gH-lower semicontinuous (gH-lsc) at a point $\bar{x}$ if $\boldsymbol{F}(\bar{x}) \preceq \liminf\limits_{x\to\bar{x}} \boldsymbol{F}(x)$. Further, $\boldsymbol{F}$ is called gH-lsc on X if $\boldsymbol{F}$ is gH-lsc at every $\bar{x} \in X$.

Remark 1.5 *By Note 5 in* [135], *we see that* $\boldsymbol{F}$ *is* gH*-lsc at* $\bar{x} \in X$ *if and only if* $\underline{F}$ *and* $\overline{F}$ *both are lsc at* $\bar{x}$.

Lemma 1.10 *Let* $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ *be a proper convex IVF. Then, for all* $x, y \in \mathrm{dom}(\boldsymbol{F})$, *we have*

$$\boldsymbol{F}_{\mathscr{D}}(x)(y-x) \preceq \boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(x).$$

Proof: By Definition 1.10 of gH-directional derivative, we have

$$\mathbf{F}_{\mathscr{D}}(x)(d) = \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \big(\mathbf{F}(x + \lambda d) \ominus_{gH} \mathbf{F}(x)\big)\,. \tag{1.1}$$

By taking $d = y - x$ in (1.1), we get

$$\begin{aligned}
\mathbf{F}_{\mathscr{D}}(x)(d) &= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \big(\mathbf{F}(x + \lambda(y-x)) \ominus_{gH} \mathbf{F}(x)\big) \\
&= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \big(\mathbf{F}((1-\lambda)x + \lambda(y)) \ominus_{gH} \mathbf{F}(x)\big) \\
&\preceq \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \big\{((1-\lambda) \odot \mathbf{F}(x) \oplus \lambda \odot \mathbf{F}(y)) \ominus_{gH} \mathbf{F}(x)\big\} \\
&= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \lambda \big(\mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)\big) \text{ by (ii) of Lemma 1.2} \\
&= \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x).
\end{aligned}$$

□

Definition 1.21 (Convergence of a sequence in $I(\mathbb{R})^n$ [75]). *A sequence* $\widehat{\boldsymbol{G}} : \mathbb{N} \to I(\mathbb{R})^n$ *is said to be convergent if there exists a* $\widehat{\boldsymbol{G}} \in I(\mathbb{R})^n$ *such that*

$$\|\widehat{\boldsymbol{G}}_k \ominus_{gH} \widehat{\boldsymbol{G}}\|_{I(\mathbb{R})^n} \to 0 \text{ as } k \to \infty,$$

where $\widehat{\boldsymbol{G}}(k) = \widehat{\boldsymbol{G}}_k$, $k \in \mathbb{N}$.

Remark 1.6 (See [75]). *It is noteworthy that if a sequence* $\left\{\widehat{\boldsymbol{G}}_k\right\}$ *in* $I(\mathbb{R})^n$, *where* $\widehat{\boldsymbol{G}}_k = (\boldsymbol{G}_{k1}, \boldsymbol{G}_{k2}, \ldots, \boldsymbol{G}_{kn}) \in I(\mathbb{R})^n$ *with* $\boldsymbol{G}_{ki} = [\underline{g}_{ki}, \overline{g}_{ki}]$, *converges to* $\widehat{\boldsymbol{G}} = (\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots, \boldsymbol{G}_n)$

$\in I(\mathbb{R})^n$ *with* $\boldsymbol{G}_i = [\underline{g}_i, \overline{g}_i]$, *then according to Definition 1.2 and norm on* $I(\mathbb{R})^n$, *the corresponding sequence* $\{\boldsymbol{G}_{ki}\}$ *in* $I(\mathbb{R})$ *converges to* $\boldsymbol{G}_i \in I(\mathbb{R})$ *for each* $i = 1, 2, \ldots, n$. *Also, by Definition 1.21, the sequences* $\underline{g}_{ki}$ *and* $\overline{g}_{ki}$ *in* $\mathbb{R}$ *converge to* $\underline{g}_i$ *and* $\overline{g}_i$ *in* $\mathbb{R}$, *respectively, for each* $i = 1, 2, \ldots, n$.

1.8.3 Results from convex analysis

Apart from the results of interval analysis, we use the following results from classical convex analysis throughout the thesis.

Definition 1.22 (Projection [170]). *Let* A *be a nonempty closed set in* $\mathbb{R}^n$. *Then, the projection of a point* $x \in \mathbb{R}^n$ *onto the set* A *is denoted by* $P(x \mid A)$, *and is defined by*

$$P(x \mid A) = \{y \in A : \|x - y\| = \inf\{\|x - u\| : u \in A\}\}.$$

Definition 1.23 (Polar cone [170]). *Let* A *be a nonempty set in* $\mathbb{R}^n$. *Then, the polar cone of the set* A *is*

$$A^o = \{x^* \in \mathbb{R}^n : \langle x^*, x\rangle \leq 0 \ \ \textit{for all } x \in A\}.$$

Definition 1.24 (Tangent cone [170]). *Let* A *be a nonempty closed convex set in* $\mathbb{R}^n$. *Then, the tangent cone to the set* A *at* $x \in A$ *is defined by*

$$T_A(x) = \operatorname{cl}\left(\bigcup_{t>0} \frac{A - x}{t}\right).$$

Definition 1.25 (Normal cone [170]). *The normal cone to a nonempty set* A *in* $\mathbb{R}^n$ *at* x *is polar of the tangent cone at* x *to the* A, *i.e.,* $N_A(x) = T_A(x)^o$. *Therefore,*

$$N_A(x) = \left\{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leq 0, \ \textit{for any } y \in A\right\}.$$

Lemma 1.11 (See [103]). *Consider a convex set $S \subseteq \mathbb{R}^n$. Then, $\bar{x}$ is an element in the closure of S if and only if $\langle x, \bar{x} \rangle \leq \psi_S^*(x)$ for all $x \in \mathbb{R}^n$, where ψ_S^* is the support function of S, i.e., $\psi_S^*(x) = \sup_{s \in S} \langle x, s \rangle$.*

Lemma 1.12 (See [34]). *Let C be a nonempty closed convex subset of $\mathbb{R}^n$.*

(i) *For all $y \in \mathbb{R}^n$, $\operatorname{dist}(y, C) = \sup_{x \in C} \operatorname{dist}(y, x + T_C(x))$, where the distance function is given by $\operatorname{dist}(y, C) = \inf_{\overline{x} \in C} \|y - \overline{x}\|$.*

(ii) *Define $\rho(x) = \operatorname{dist}(x, C)$. Then, for all $x \in C$ and $d \in \mathbb{R}^n$,*

$$\rho_{\mathscr{D}}(x)(d) = \operatorname{dist}(d, T_C(x)) = \psi^*_{\mathbb{B} \cap N_C(x)}(d).$$

Moreover, if $d \in N_C(x)$, then $\rho_{\mathscr{D}}(x)(d) = \operatorname{dist}(d, T_C(x)) = \psi^*_{\mathbb{B} \cap N_C(x)}(d) = \|d\|$.

We recall, below, some basic results and definitions from set optimization.

Definition 1.26 (Partial ordering on $\mathbb{R}^m$ [82]). *The closed, convex, solid, and pointed cone K generates a partial ordering $\preceq$ and $\prec$ on $\mathbb{R}^m$, defined as follows: for any $x, y \in \mathbb{R}^m$,*

$$x \preceq y \iff y - x \in K, \quad \text{and } x \prec y \iff y - x \in \operatorname{int}(K).$$

Definition 1.27 (Minimal and weakly minimal elements of a set [115]). *Let $A \in \mathscr{P}(\mathbb{R}^m)$. An element $a \in A$ is said to be minimal if $(a - K) \cap A = \{a\}$ and weakly minimal if $(a - \operatorname{int}(K)) \cap A = \emptyset$.*

The collections of all minimal and weakly minimal elements of A are denoted by $\operatorname{Min}(A, K)$ and $\operatorname{WMin}(A, K)$, respectively.

Proposition 1.1 (Theorem 6.3c [115]). *Let $A \in \mathscr{P}(\mathbb{R}^m)$ be a nonempty compact set. Then, A satisfies the so-called domination property with respect to K, i.e., $A + K = \operatorname{Min}(A, K) + K$.*

Next, we recall the notion of Drummond-Svaiter and Gerstewitz scalarizing functions along with their properties that will be used further in this thesis. These functions play a significant role in the proposed algorithms in Chapter 4 and Chapter 5. We start with an overview of Drummond-Svaiter scalarizing function.

Let $C \subseteq K^* \setminus \{0\}$ be a compact set such that $K^* = cone(\text{conv}(C))$, i.e., K^* is the conic hull of the convex hull of C. Note that if $K = \mathbb{R}^m_+$, then $K^* = K$. In this case, we may take C as the canonical basis of $\mathbb{R}^m$. If K is a polyhedral cone, then K^* is also a polyhedral cone, and in this case, C can be taken as the finite set of extremal rays of K^*. For generic K, in this thesis, the set C is taken as

$$C = \{w \in K^* : \|w\| = 1\}.$$

The Drummond-Svaiter scalarizing function (see [56]) $\varphi : \mathbb{R}^m \to \mathbb{R}$ is given by

$$\varphi(y) = \sup\{\langle y, w\rangle : w \in C\}.$$

It is to be noted that since C is a compact set, therefore φ is well defined. Some useful properties of function φ are given in the following lemma.

Lemma 1.13 (See [56]). *Let $y, y' \in \mathbb{R}^m$. The following properties hold true for the function φ:*

(i) $\varphi(y + y') \leq \varphi(y) + \varphi(y')$ *and* $\varphi(y) - \varphi(y') \leq \varphi(y - y')$;

(ii) *if* $y \preceq y'$, *then* $\varphi(y) \leq \varphi(y')$, *and if* $y \prec y'$, *then* $\varphi(y) < \varphi(y')$;

(iii) φ *is Lipschitz continuous with constant 1.*

It is known from [56] that

$$-K = \{y \in \mathbb{R}^m : \varphi(y) \leq 0\} \text{ and } -\text{int}(K) = \{y \in \mathbb{R}^m : \varphi(y) < 0\}. \tag{1.2}$$

Next, the definition of the Gerstewitz scalarizing function and its properties are given.

Definition 1.28 (Gerstewitz function [71]). *Let $e \in \operatorname{int}(K)$ and $x \in \mathbb{R}^m$. The Gerstewitz function $\Psi_e : \mathbb{R}^m \to \mathbb{R}$ associated to e and K is defined by*

$$\Psi_e(x) = \min\{t \in \mathbb{R} : te \in x + K\}.$$

We fix e as a given element in $\operatorname{int}(K)$ throughout in Chapter 5.

Proposition 1.2 (See [131]). *The Gerstewitz function Ψ_e has the following characteristics.*

(i) *Ψ_e is sublinear and Lipschitz on $\mathbb{R}^m$.*

(ii) *Ψ_e is positively homogeneous of degree 1.*

(iii) *Ψ_e is monotone with respect to the partial order $\preceq$, i.e., for all $x, y \in \mathbb{R}^m$*

$$x \preceq y \implies \Psi_e(x) \preceq \Psi_e(y) \text{ and } x \prec y \implies \Psi_e(x) \prec \Psi_e(y), \text{ respectively.}$$

(iv) *Ψ_e has the representability property:*

$$-K = \{x \in \mathbb{R}^m : \Psi_e(x) \leq 0\} \text{ and } -\operatorname{int}(K) = \{x \in \mathbb{R}^m : \Psi_e(x) < 0\}.$$

There are many set ordering relations in the literature [119]. In this thesis, we use the lower set less relation that is given below.

Definition 1.29 (Lower set less relation [141]). *Let $A, B \in \mathscr{P}(\mathbb{R}^m)$ be two arbitrary sets. The lower set less relation $\preceq^l$ is defined on $\mathscr{P}(\mathbb{R}^m)$ by*

$$A \preceq^l B \iff B \subseteq A + K.$$

Similarly, the strict lower set less relation $\prec^l$ *is defined on* $\mathscr{P}(\mathbb{R}^m)$ *by* $A \prec^l B \iff B \subseteq A + \operatorname{int}(K)$.

The focus of this thesis is on both unconstrained and constrained set optimization problems. A necessary optimality condition and conjugate gradient methods are proposed for the following unconstrained set optimization problem (SOP):

$$\text{minimize} \;\; F(x) \;\; \text{subject to} \;\; x \in \mathbb{R}^n, \tag{1.3}$$

and a necessary optimality condition along with a projected gradient method is proposed for the following constrained set optimization problem (CSOP):

$$\text{minimize} \;\; F(x) \;\; \text{subject to} \;\; x \in \mathcal{S}, \tag{1.4}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a nonempty set-valued map, and $\emptyset \neq \mathcal{S} \subseteq \mathbb{R}^n$. For these problems, we aim to identify weakly minimal solutions (Definition 1.30) with respect to a given ordering cone K in $\mathbb{R}^m$. We take the following assumption on problems SOP (1.3) and CSOP (1.4) throughout the thesis.

Assumption 1 *We assume that the objective set-valued mapping* F *of SOP* (1.3) *and CSOP* (1.4) *is given by*

$$F(x) = \left\{ f^1(x), f^2(x), \ldots, f^p(x) \right\} \;\textit{for all}\; x \in \mathbb{R}^n,$$

where $f^1, f^2, \ldots, f^p : \mathbb{R}^n \to \mathbb{R}^m$ *are continuously differentiable vector-valued functions. Moreover, it is also assumed that the set* $\mathcal{S}$ *in CSOP* (1.4) *is nonempty, closed, and convex.*

Remark 1.7 *The assumptions that the set* $\mathcal{S}$ *is nonempty, closed, and convex in Assumption 1 are necessary due to the implicit use of the projection in the proposed projected gradient method in Chapter 5.*

In the next definition, the concept of a local weakly minimal solution for SOP (1.3) and CSOP (1.4) is described.

Definition 1.30 (Local weakly minimal solution). *A point $\bar{x} \in \mathcal{S}$ is said to be a local weakly minimal solution of CSOP (1.4) if for any neighborhood U of $\bar{x}$ there does not exist any $x \in U$ such that $F(x) \prec^l F(\bar{x})$. Moreover, if $U = \mathcal{S}$, then $\bar{x}$ is a weakly minimal solution of CSOP (1.4).*

Furthermore, if $\mathcal{S} = \mathbb{R}^n$ in Definition 1.30, then $\bar{x}$ is a local weakly minimal solution of SOP (1.3). Subsequently, if $\mathcal{S} = \mathbb{R}^n$ and $U = \mathbb{R}^n$ in Definition 1.30, then $\bar{x}$ is a local weakly minimal solution of SOP (1.3).

Below, we mention two lemmas that will be useful further in this thesis.

Lemma 1.14 (See [162]). *Let $\hat{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable vector-valued function. If the Jacobian of $\hat{f}$ is Lipschitz continuous with Lipschitz constant L, then for any $d \in \mathbb{R}^n$, the function $\varphi(\mathrm{J}\,\hat{f}(\cdot)d)$ is Lipschitz continuous with Lipschitz constant $L\|d\|$.*

Lemma 1.15 (See [162]). *Let $\hat{a}, \hat{b}$ and $\hat{c} \neq 0$ be three real numbers. Then, we have*

(i) $\hat{a}\hat{b} \leq \frac{\hat{a}^2}{2} + \frac{\hat{b}^2}{2}$,

(ii) $2\hat{a}\hat{b} \leq 2\hat{c}^2\hat{a}^2 + \frac{\hat{b}^2}{2\hat{c}^2}$,

(iii) $(\hat{a} + \hat{b})^2 \leq 2\hat{a}^2 + 2\hat{b}^2$, *and*

(iv) $(\hat{a} + \hat{b})^2 \leq (1 + 2\hat{c}^2)\hat{a}^2 + (1 + \frac{1}{2\hat{c}^2})\hat{b}^2$.

1.9 Objective of the thesis

The primary aim of this thesis is to advance both theoretical frameworks and numerical methods in the fields of interval and set optimization. From a theoretical standpoint, the first goal is to introduce the concept of weak sharp minima for IVFs using the notion of subdifferentiability for IVFs. The second theoretical objective is to establish optimality concepts based on ϵ-subdifferentiability for IVFs, to provide a pathway to obtain approximate solutions for IOPs. On the numerical side, the focus is on developing two first-order methods: one for unconstrained and another for constrained SOPs whose objective functions have finitely many vector-valued functions. The specific objectives of this thesis are as follows.

- To define the concept of the support function of a set in $I(\mathbb{R})^n$.
- To define and analyze the notion of weak sharp minima and its characterizations for IVFs.
- To propose the notion of ϵ-subdifferentiability for IVFs.
- To analyze the characteristics such as nonemptiness, closeness, convexity, boundedness, etc., of ϵ-subdifferential set of IVFs.
- To give the optimality concepts for IOPs, using ϵ-subdifferentiability for IVFs, to find the approximate solutions of IOPs.
- To develop a nonlinear conjugate gradient method without imposing any restriction on the conjugate parameter and its variants for SOPs associated with objective functions having finitely many vector-valued functions.
- To develop a projected gradient method for constrained SOPs associated with objective functions having finitely many vector-valued functions.

Next, a chapter-wise organization of the thesis is given below.

1.10 Organization of the thesis

This thesis is divided into six chapters. Chapter 1 provides an introduction to interval and set optimization, including a brief literature review. It also presents the necessary preliminaries, notations, and basic definitions required for the thesis. Chapters 2 and 3 focus on the theoretical concepts of IVFs and IOPs. Further, in Chapters 4 and 5, numerical methods for unconstrained and constrained SOPs are developed. The thesis is concluded with Chapter 6, in which a conclusion of the entire thesis and some potential directions for future research are given. A detailed description of the organization of the chapters is given below.

In Chapter 2, we propose the concept of the support function of a subset of $I(\mathbb{R})^n$; alongside, a few necessary results on extended support function are also given. Next, we define the notion of gH-subdifferentiability for convex IVF along with the properties of gH-subdifferential set of convex IVF, which are required further in the chapter. Next, the concept of weak sharp minima for convex IVFs is presented. Further, we give primal and dual characterizations of weak sharp minima. Subsequently, applications of the proposed study are given.

In Chapter 3, the concepts of gH_ϵ-directional derivative and gH_ϵ-subdifferentiability for convex IVFs are proposed. Essential characteristics such as closedness, convexity, nonemptiness, and boundedness of gH_ϵ-subdifferential set are also presented. The difference between gH-subdifferentiability and gH_ϵ subdifferentiability is shown with the suitable examples. Further, the definition of ϵ-solution of an IOP is given. With the help of the gH_ϵ-subdifferentiability, two optimality conditions are given to find approximate solutions of IOPs. Next, a theorem that helps us to find the gH_ϵ-subgradient of the objective function of the interval minimax optimization problem is proved.

In Chapter 4, some optimality conditions are given for unconstrained set-valued optimization problems under consideration. Based upon these optimality conditions, a nonlinear conjugate gradient method without imposing any condition on conjugate parameter is proposed further. The well-definedness and global convergence of this method are then demonstrated. Following this, the Fletcher–Reeves and conjugate descent variants are introduced, along with proofs of their global convergence. Lastly, the chapter presents numerical results to evaluate the performance of the proposed methods. In numerical results, a comparison of the proposed methods in this chapter with the existing steepest descent method is shown.

In Chapter 5, some optimality conditions are derived for the constrained set-valued optimization problems under consideration. Based upon these optimality conditions, a projected gradient method and its variant are then proposed. Subsequently, the well-definedness and the global convergence of the proposed methods are also shown. Numerical results of the method are also exhibited in this chapter.

In Chapter 6, a concluding remark on the entire work of the thesis and some potential future directions are given.

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