Chapter 5

Pure Direct Projective Modules

In this Chapter, we introduce the notion of Pure direct projective modules, which is an extension of the class of direct projective modules, and the dual notion of pure direct injective modules. Specifically, a module M is classified as a pure direct projective module if a quotient module formed by a pure submodule of M is isomorphic to a direct summand of M, then the pure submodule is a direct summand of M. Also, in this chapter, we study the direct sums and direct summands of pure direct projective modules. Further, we characterize semi-simple, pure semisimple, pure hereditary, and von Neumann regular rings in terms of pure direct projective modules.

5.1 Pure Direct Projective Modules

We first define pure direct projective modules and give some of their examples.

Definition 5.1.1. *An R-module M is said to be a pure direct projective module or pure D_2 module if for any pure submodule B of M satisfies $M/B \cong A$ and $A \leq^{\oplus} M$, then B is a direct summand of M.*

Since every direct summand of R-module M is a pure submodule, so in general, direct projective modules are pure direct projective modules. Next, we discuss some examples of pure direct projective modules.

Example 5.1.2.

1. *Pure direct projective module is a strict generalization of the direct projective module. Since every direct summand of right R-module M is also a pure submodule of the module M. Hence every direct projective module is a pure direct projective module, but its converse need not be true. For example $\mathbb{Z}$ module $\mathbb{Z} \bigoplus \mathbb{Z}_n$, where $n \in \mathbb{N}$, which is pure direct projective but not a direct projective module. Since both $\mathbb{Z}$ and $\mathbb{Z}_n$ has no pure submodules.*

2. *We can similarily construct another example $\mathbb{Q} \bigoplus \mathbb{Q}/\mathbb{Z}$ over the ring $\mathbb{Z}$. This module is a pure direct projective but not a direct projective.*

3. *Every Quasi Dedekind R-module M is always pure direct projective, because for every $s \in End_R(M)$ we have $Ker(s) = 0$, so from Proposition 5.1.4 M is pure direct projective.*

4. *If $M = \bigoplus_{i \in \mathbb{N}} N_i$ and each N_i is a uniserial module. Then M is a pure direct projective. An R module M is said to be uniserial if submodules of M are totally ordered by inclusion, that is if N and L are two submodules of M then either $N \subseteq L$ or $L \subseteq N$.*

5. *If M is a pure simple R-module then M is a pure direct projective module. An R module M is pure simple if only pure submodules of M are trivial.*

Lemma 5.1.3. *A right R-module M is pure direct projective if, the pure exact sequence*

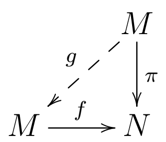
$$0 \longrightarrow Ker(f) \longrightarrow M \longrightarrow M/(Ker(f) \longrightarrow 0$$

splits for $f \in End_R(M)$ *such that* $Img(f) \leq^{\oplus} M$ *.*

Proof:- Since the sequence is a pure exact, $M/Ker(f)$ is flat. By the Fundamental Theorem of module homomorphism, we get $M/Ker(f) \cong Img(f) \leq^{\oplus} M$. Therefore, the sequence splits.

Proposition 5.1.4. *The following conditions are equivalent for an R-module M:*

1. *M is a pure direct projective module;*
2. *If* $Img(f)$ *is a direct summand of M with* $Ker(f)$ *is a pure submodule of M for* $f \in End_R(M)$*, then* $Ker(f)$ *is a direct summand of M;*
3. *If N is any direct summand of M and an epimorphism* $f : M \longrightarrow N$ *such that* $Ker(f)$ *is pure in M, then there exists a* $g \in End_R(M)$ *such that following diagram commutes,*



equivalently, $f \circ g = \pi$*, where* π *is projection map.*

Proof:-

(1) $\Rightarrow$ (3), (1) $\Rightarrow$ (2) Since, $M/Ker(f) \cong Img(f)$ (By Fundamental theorem of Module Homomorphism) and $Ker(f)$ is a pure submodule of M, hence $Ker(f) \leq^{\oplus} M$.

(2) $\Rightarrow$ (1) Consider $M/B \cong N \leq^{\oplus} M$ with B pure submodule of M, let $M/B \cong eM$ for some idempotent $e \in End_R(M)$. Let π be a projection map from M to N and f be an epimorphism from M to N. We set for some s as $s = e \circ f^{-1} \circ \pi$, then $Ker(s) \cong B$ and $Img(s) \cong eM$, Since $Ker(s)$ is a direct summand in M implies B is direct summand of M.

(3) $\Rightarrow$ (1) Let B be any pure submodule of M such that it satisfies $M/B \cong N \leq^{\oplus} M$. Consider $h : M \longrightarrow M/B$, since $Ker(h) \cong B$ and pure in M, therefore from the hypothesis, we will have some $g \in End_R(M)$ such that $h \circ g = \pi$. Hence g is required splitting of h thus, $Ker(h)$ is a direct summand of M.

Next, we discuss the Morita equivalence with respect to pure direct projective modules. Two rings say R and S are Morita equivalent if their category of modules say R-module M and S-module M are equivalent.

Proposition 5.1.5. *For two Morita equivalent rings say R and S, let $\phi : Mod - R \longrightarrow Mod - S$ be their Morita equivalence. For any Module M which is pure direct projective, then $\phi(M)$ is pure direct projective and vice-versa.*

Proof:- Let $N \leq M$ be a pure submodule of M, then $\phi(N)$ is a pure submodule of $\phi(M)$ as purity is a Morita equivalent property. Hence proof is obvious.

Proposition 5.1.6. *Let M be a pure direct projective module. If $M = M_1 \bigoplus M_2$ with $f : M_1 \longrightarrow M_2$ be a homomorphism such that $Img(f)$ is a direct summand of M_2 and $Ker(f)$ is a pure submodule of M_1. Then $Ker(f)$ is a direct summand of M_1.*

Proof:- Let $f : M_1 \longrightarrow M_2$ be a module homomorphism. We consider $\pi : M \longrightarrow M_1$ a canonical projection, then $f \circ \pi : M \longrightarrow M_2$. Also $Img(f \circ \pi) = Img(f)$. Therefore $M/Ker(f \circ \pi) \cong Img(f \circ \pi) = Img(f)$. Since $Img(f) \leq^{\oplus} M_2$ and it is a direct summand of module M. By definition of pure direct projective module, and $Ker(f \circ \pi) = M_2 \oplus Ker(f) \leq^p M_2 \bigoplus M_1$, as $Ker(f)$ is pure in M_1, hence $Ker(f \circ \pi)$ is pure in M. Therefore $Ker(f \circ \pi) \leq^{\oplus} M$.

This implies, $Ker(f) \leq^{\oplus} M$. Since $Ker(f)$ is a submodule of M_1 implies $Ker(f) \leq^{\oplus} M_1$.

Corollary 5.1.7. *If $M = M_1 \bigoplus M_2$ with $f : M_1 \longrightarrow M_2$ be an epimorphism such that $Ker(f)$ is pure in M_1, then $Ker(f)$ is a direct summand of M_1.*

Proof:- Follows from Proposition 5.1.6.

Proposition 5.1.8. *If $M = M_1 \bigoplus M_2$ is a pure direct projective module with M_1 is a pure simple and M_2 is a dual Rickart. Then either $Hom_R(M_1, M_2) = 0$ or for each nonzero homomorphism $f : M_1 \to M_2$ with $Ker(f)$ is pure submodule of M, f is a monomorphism.*

Proof:- We consider $Hom(M_1, M_2) \neq 0$ otherwise the case is trivial. If $f : M_1 \to M_2$ is non-zero homomorphism and M_1 is a pure submodule of M, then this implies $Ker(f) \neq 0$ is not a pure submodule, implies $Ker(f) \neq 0$ is not direct summand of M_1. Therefore $Ker(f) = 0$ and $f \in Hom_R(M_1, M_2)$ is a monomorphism.

In [10], RD (relatively divisible) pure submodule P is a pure submodule of M if for every $r \in R$, $rP = rM \cap P$. Similarily, we can define M is RD direct projective module as, if B is RD pure submodule of M and it satisfies $M/B \cong A \leq^{\oplus} M$, then $B \leq^{\oplus} M$.

Proposition 5.1.9. *Every pure direct projective module is RD-pure direct projective module.*

Proof:- Let P be a pure submodule of a module M. Since every pure submodule is RD-pure and we have M as a pure direct projective module. This implies M is RD-pure direct projective.

Corollary 5.1.10. *A flat R-module M is pure direct projective if M is RD-pure direct projective.*

Proof:- This follows from the fact that when R-module M is a flat module then the RD-pure submodule is pure.

Now we investigate when direct projective and pure direct projective modules are equivalent.

Proposition 5.1.11. *Let M be an R-module and for each $f \in End_R(M)$, $Img(f)$ is a direct summand and $M/Ker(f)$ be pure injective submodule of M. Then M is a pure direct projective module if and only if M is a direct projective module.*

Proof:- Since $M/Ker(f)$ is pure injective. This implies pure exact seq. $0 \longrightarrow M/Ker(f) \longrightarrow M$ splits. Since $M/Ker(f) \cong Img(f)$, is a direct summand of M. Therefore we got our claim.

Next, we find an equivalent condition for a pure Rickart ring to be a pure direct projective ring. A pure direct projective ring is described as a cyclic pure direct projective R-module over itself.

Proposition 5.1.12. *Every right Noetherian ring is a pure direct projective ring if it is a pure Rickart ring.*

Proof:- This follows from the fact that, in a Noetherian ring, every pure ideal is a direct summand.

Next, we discuss the direct sum and summand of Pure direct projective modules. In the following proposition, we prove that the direct summand of pure direct projective module is pure direct projective.

Proposition 5.1.13. *The direct summand of a pure direct projective module is a pure direct projective module.*

Proof:- Let N be a direct summand of M and N' is complement of N, A be a pure submodule of N and satisfy $N/A \cong B \leq^{\oplus} N$ for some $B \leq^{\oplus} M$. Now we prove $A \leq^{\oplus} N$. Since M is a pure direct projective, and $(N \bigoplus N')/A \cong (B \oplus N') \leq^{\oplus} N \oplus N'$ which implies there exist some T directs ummand of N, such that $M/A \cong T \leq^{\oplus} M$, and since $A \leq^{p} N$ by transitivity $A \leq^{p} M$, therefore A will be a direct summand of M. We already assumed that A is a pure submodule of N hence $A \leq^{\oplus} N$.

In Proposition 5.1.13, We know that every direct summand of a pure direct projective module is pure direct projective. Still, if we consider the direct sum of two pure direct projective modules, then it need not be pure direct projective. For example, consider $\mathbb{Z}$-module $M_1 = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $\mathbb{Z}$-module $M_2 = \prod_{n=1}^{\infty} \mathbb{Z}_2 / \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Here both M_1 and M_2 are pure direct projective but $M_1 \bigoplus M_2$ is not a pure direct projective module. Since for $f \in Hom_{\mathbb{Z}}(M_1, M_2)$, $Ker(f)$ is pure submodule of M_1 but $Ker(f)$ is not a direct summand M_1.

Definition 5.1.14. *Let M and N be pure direct projective modules, M is called relatively pure direct projective to N, if there exists homomorphism $f \in Hom(M, N)$ with $Img(f)$ is a direct summand of N and $Ker(f)$ is a pure submodule of M, then $Ker(f)$ is a direct summand of M.*

Next, we will discuss the direct sum of pure direct projective modules with respect to PIP property. A module is said to have PIP property if the intersection of two pure submodules is again pure. Let M be an R-module with the PIP property, then for every decomposition $M = K \bigoplus T$ and for every $f \in Hom(T, K)$, $Ker(f)$ is a pure submodule in M.

Proposition 5.1.15. *For a pure direct projective module, if $M = K \bigoplus T$ has the PIP property and T is a dual Rickart module, then K is relatively pure direct projective to T.*

Proof:- Follows from the definition.

In particular, every direct sum of pure direct projective modules is pure direct projective if modules are relatively pure direct projective to each other.

Proposition 5.1.16. *Let $M = \bigoplus_{i \in \mathbb{N}} M_i$. Then each M_i is a relatively pure direct projective module with respect to other if and only if M is a pure direct projective module.*

Proof:- If M is pure direct projective module then $\bigoplus_{i \in \mathbb{N}} M_i$ is pure direct projective. Hence each M_i is a relatively pure direct projective module with respect to others following from Proposition 5.1.6 and Definition 5.1.14. For the converse part, let for some pure submodule P of M such that $M/P \cong N \leq^{\oplus} M$, then N will be a direct summand of some M_k and P will be a pure submodule of some M_t. Since M_t is relatively pure direct projective to M_k, hence P is a direct summand of M. Also, as $k, t \in \mathbb{N}$ are arbitrary, therefore M is a pure direct projective module.

Proposition 5.1.17. *For a von-Neumann regular ring R, the following statements are equivalent:*

1. *R is a semi-simple ring;*

2. *All R-modules are relatively pure direct projective to any R-module.*

Proof:- (1) $\Rightarrow$ (2) Follows from the definition of semi-simple ring.

(2) $\Rightarrow$ (1) Let I be an ideal of a ring R. We are required to prove that I is a direct summand of R. From (2) all R-modules are relatively pure direct projective to any R-module. Then the R-module R is relatively Pure direct projective to R/I as an R-module. Then by definition for homomorphism $f : R \to R/I$ and $Ker(f) = I$ is pure in R since R is a von-Neumann regular ring. Then $Ker(f)$ is a direct summand of R. Hence R is a semi-simple ring as I is an arbitrary ideal of R.

5.2 Characterizations of rings using Pure Direct Projective Modules

In this section, we characterize rings using pure direct projective modules. At first, we will consider its endomorphism ring, and find equivalent conditions for pure direct projective modules and endoregular modules. Since every endoregular module is pure direct projective, but converse needs not to be true. For example $\mathbb{Z}$ module $\mathbb{Z}_{p^n}$ where p is prime and $n \in \mathbb{N}$, which is a pure direct projective module but not an endoregular module.

Proposition 5.2.1. *The following statements are equivalent for pure direct projective R-module M:*

1. *M is a pure Rickart and dual Rickart module;*
2. *$M \bigoplus M$ has SSP and PIP property;*
3. *M is an Endoregular module.*

Proof:- (1) $\Rightarrow$ (3) To prove M is endoregular module we have to prove that $Ker(f)$ and $Img(f)$ are direct summands of M for all $f \in End_R(M)$. Since M is dual Rickart, $Img(f)$ is a direct summand of M, also M is a pure Rickart module, hence $Ker(f)$ is pure for all $f \in End_R(M)$. So we get $Ker(f)$ is a direct summand of M, as M is a pure direct projective module.

(2) $\Rightarrow$ (3) Since $M \bigoplus M$ has SSP then $Img(g) \leq^{\oplus} M$ for each $g \in End_R(M)$ and due to PIP property $Ker(f)$ is pure for each $f \in End_R(M)$. This implies $Ker(f)$ is a direct summand of M, as M is a pure direct projective module.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2), It is clear from the definition of Endoregular module.

In the next proposition, we find when a purely Rickart module is a pure direct projective. We also discuss pure direct projective modules with respect to purely semisimple rings. A ring R is purely semisimple if and only if every pure submodule of an R module M is a direct summand of M.

Proposition 5.2.2. *Every purely Rickart module is a pure direct projective module over the purely semisimple ring.*

Proof:- Suppose M be a purely Rickart module over purely semisimple ring R, hence for a $P \leq M$ and satisfy $M/P \cong N \leq^{\oplus} M$, then P is a pure submodule of M. Since R is a purely semisimple ring hence P is a direct summand of M this implies M is a pure direct projective module.

Proposition 5.2.3. *The following statements are equivalent:-*

1. *R is a purely semisimple ring;*

2. *Every finitely generated R-module is a pure direct projective;*

3. *Every 2-generated module is a pure direct projective.*

Proof:- (1) $\Rightarrow$ (2) $\Rightarrow$ (3), every R-module M is pure direct projective.

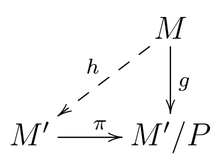
(3) $\Rightarrow$ (1) We must prove that every pure ideal, say I of R is a direct summand of R. Consider $R \bigoplus R/I$ which is pure direct projective. So from Proposition 5.1.4, there exists $f : R \longrightarrow R/I$ such that $Ker(f) = I$, which is pure in R, hence I is a direct summand of R. Therefore R is the purely semisimple ring.

The next corollary is based on the fact that every commutative ring is a pure semi-simple if and only if it is Artinian PID.

Corollary 5.2.4. *A commutative ring R is Artinian PID if and only if every finitely generated R-module is pure direct projective.*

Next, we define a module that is pure direct projective with respect to the other module and characterize this condition with respect to pure semisimple modules.

Definition 5.2.5. *Let M and M' be R-modules, then M is M' pure direct projective, if for a module homomorphism $g : M \longrightarrow M'/P$ where P is a pure submodule of M' and M'/P is isomorphic to a direct summand of M, $\exists$ a $h \in Hom_R(M, M')$ such that $\pi \circ h = g$, where π is natural epimorphism from M' to M'/P.*



This implies that the above diagram commutes.

Proposition 5.2.6. *For each pure submodule P of M, if M/P isomorphic to a direct summand of a module M and M/P is M-pure direct projective, then the module M is pure semisimple.*

Proof:- Since M/P is M pure direct projective, identity mapping from M/P to M/P has a lifting $g : M/P \longrightarrow M$. Hence short exact sequence $0 \longrightarrow P \longrightarrow M \longrightarrow M/P \longrightarrow 0$ splits, implies $P \leq^{\oplus} M$.

Proposition 5.2.7. *Let $M = M' \bigoplus M''$ be a pure direct projective and M' be a pure simple module. Then M' is M''-pure direct projective module.*

Proof:- Follows from the above definition.

Proposition 5.2.8. *Let M be a pure injective module over a pure hereditary ring R, then R-module M is a pure direct projective module.*

Proof:- Since R-module M is pure injective over the pure hereditary ring, every quotient module of pure injective is pure injective. For pure submodule P of M, M/P is pure injective hence $P \leq^{\oplus} M$. Therefore R-module M is pure direct projective.

Lemma 5.2.9. *Let T be a projective module and $T \bigoplus N$ be a pure direct projective module where N is an R-module if there exists an epimorphism $f : T \longrightarrow N$ such that $Ker(f)$ is pure, then N is a projective module.*

Proof:- Since f is an epimorphism and $Ker(f)$ is a pure submodule of T. Then from Corollary 5.1.7, $Ker(f)$ will be a direct summand of T. Hence epimorphism f splits and N is projective.

In the next proposition, we characterize von-Neumann regular rings in terms of pure direct projective modules.

Proposition 5.2.10. *For a hereditary ring R, the following statements are equivalent:*

1. *R is a von-Neumann regular ring;*

2. *Every pure direct projective R-module is a direct projective module;*

3. *Every pure projective R-module is a projective.*

Proof:- (1) $\Rightarrow$ (2) Let R be a von-Neumann regular ring this implies every submodule of M is pure. Now for any submodule P of M, $M/P \cong N \leq^{\oplus} M$, implies $P \leq^{\oplus} M$ as given M is pure direct projective and P is pure submodule. Hence M is a direct projective.

(2) $\Rightarrow$ (3) Let S be a projective module and every pure projective module can be written as $S \bigoplus N$ from [33, Proposition 2.1]. If we prove N is projective then our work is over. Since $S \bigoplus N$ is pure projective hence pure direct projective and by assumption, it is direct projective. Therefore, every epimorphism $f : S \longrightarrow N$ splits hence N is projective.

(3) $\Rightarrow$ (1) Follows from [33, Proposition 3.7] R is von Neumann regular ring.