CHAPTER 4

FIXED-TIME SYNCHRONIZATION OF DIFFERENT NEURAL NETWORKS

4.1 Some Results on Fixed-Time Synchronization of Memristor Neural Networks with Impulsive Effects

4.1.1 Introduction[1]

The dynamical behaviour of neural networks has been a subject of extensive study due to its significance in various applications such as pattern recognition, optimization, signal processing, and associative memories. In addition, since the pioneering work of Pecora and Carroll [32] in 1990, chaos synchronization of chaotic neural networks has been a fascinating problem. To date, researchers have explored and studied various types of synchronization in neural networks [127, 130, 131] due to their theoretical importance and practical applications in secure communication, tracking control and image encryption. In this chapter, we have considered the synchronization scheme in which drive and response systems are identical. Further, we have discussed two types of problems in the synchronization of neural networks within fixed-time one is memristor

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neural networks and another is complex-valued neural networks.

The term and concept of memristor, which stands for memory and resistor, was coined by Leon Chua [20] in 1971. It is well known that none of the three fundamental circuit elements viz. resistor, capacitor and inductor describe the relationship between charge and flux. This missing relation between charge and flux is covered by the fourth fundamental circuit element memristor. After the first physical demonstration of memristor at HP Lab in 2008, several models like the Strukov model and Simmon tunnel barrier model have been introduced for different applications [132, 133]. Having the property of non-volatility and being a nanoscale element, the memristor is the best electronic component to play a role as synapses to imitate neural networks better. The memristor can be used to replicate the synapsis in the human brain since it has the property of memorizing the direction of the past charge flow. Recently, the memristor has gained more interest because of its essential role in the next generation of computers and powerful human brain-like computers. Thus, the introduction of the memristor in place of the resistor in neural networks opens a new door in the study of the neural networks, known as memristor neural network (MNN), which has a vast field of applications such as in image processing, secure communication, fault diagnosis and pattern recognition. In the past two decades, researchers have made significant progress in their understanding of the dynamical behavior of MNNs, and a wide range of interesting results have been published in the references [134, 135, 136, 137, 138].

In the present subchapter, the fixed-time synchronization of MNNs with impulsive effects and time delays have been investigated. Using the concept of comparison principle and average impulsive interval, we have derived the sufficient conditions so that the concerned MNNs will achieve fixed-time synchronization under the influence of synchronizing impulses and desynchronizing impulses. Both, theoretically and numerically it has been shown that the estimated settling-time is less conservative and of high precision as compared to those results found in the existing fixed-time synchronization reported in [95, 96, 111, 139]. The fixed-time synchronization of MNNs for desynchronizing impulses have not been discussed in prior results reported in [96]. In this subchapter, an attempt has been made to find the condition under which MNNs with desynchronizing impulses can achieve fixed-time synchronization.

4.1.2 Model Description and Preliminaries

Let us consider the state equation of MNNs with impulsive effects as

$$\begin{cases} \dot{z}_p(t) = -d_p z_p(t) + \sum_{q=1}^{n} h_{pq}(z_p(t)) f_q(z_q(t)) \\ \qquad + \sum_{q=1}^{n} w_{pq}(z_p(t)) g_q(z_q(t-\tau_q(t)) + I_p, t \geq 0, \quad t \neq t_l, \\ \Delta z_p(t_l) = \eta_l z_p(t_l^-), t = t_l, l \in \mathbb{N}, \end{cases} \tag{4.1.1}$$

where $p = 1, 2, \ldots, n$, $z_p(t) \in \mathbb{R}$ represent the voltages of capacitors; $d_p > 0$ are the self-inhibition; $f_q(.)$ and $g_q(.)$ denote the activation functions of the q-th neuron ; $\tau_q(t)$ corresponds to the time-varying delay satisfying $0 \leq \tau_q(t) \leq \tau$; $h_{pq}(.)$ and $w_{pq}(.)$ represent memristor-based connection weights; I_p denotes the external input or bias. The sequence $\{t_1, t_2, t_3 \ldots\}$ is strictly increasing impulsive instants, η_l denotes the impulsive strength, $\Delta z_p(t_l) = z_p(t_l^+) - z_p(t_l^-)$, where $z_p(t_l^+) = \lim_{t\to t_l+0} z_p(t)$ and $z_p(t_l^-) = \lim_{t\to t_l-0} z_p(t)$. Suppose $z_p(t)$ is right continuous at $t = t_l$, i.e., $z_p(t_l^+) = z_p(t_l)$. The initial condition of system (4.1.1) is $z_p(s) = \phi_p(s)$, $s \in [-\tau, 0]$. The memristor-based connection weights in system (4.1.1) satisfy

$$h_{pq}(z_p(t)) = \begin{cases} \hat{h}_{pq}, & |z_p(t)| \leq T_p, \\ \check{h}_{pq}, & |z_p(t)| > T_p, \end{cases} \qquad w_{pq}(z_p(t)) = \begin{cases} \hat{w}_{pq}, & |z_p(t)| \leq T_p, \\ \check{w}_{pq}, & |z_p(t)| > T_p, \end{cases}$$

for $p, q = 1, 2, \ldots n$, where $\hat{h}_{pq}$, $\check{h}_{pq}$, $\hat{w}_{pq}$, $\check{w}_{pq}$ are known constants with respect to memristances, and T_p is the non-negative threshold level. By following the theory of differential inclusion and set valued map [140], it can be obtained that the system (4.1.1) is equivalent to the following differential inclusion:

$$\dot{z}_p(t) \in -d_p z_p(t)+\sum_{q=1}^{n} K(h_{pq}(z(t)))f_q(z_q(t))+\sum_{q=1}^{n} K(w_{pq}(z(t)))g_q(z_q(t-\tau_q(t))+I_p, t \neq t_k, \tag{4.1.2}$$

where

$$K(h_{pq}(z(t))) = \begin{cases} \hat{h}_{pq}, & |z(t)| < T_p, \\ \text{co}\{\hat{h}_{pq}, \check{h}_{pq}\}, & |z(t)| = T_p, \\ \check{h}_{pq}, & |z(t)| > T_p, \end{cases}$$

$$K(w_{pq}(z(t))) = \begin{cases} \hat{w}_{pq}, & |z(t)| < T_p, \\ \text{co}\{\hat{w}_{pq}, \check{w}_{pq}\}, & |z(t)| = T_p, \\ \check{w}_{pq}, & |z(t)| > T_p. \end{cases}$$

In above equation, $\text{co}\{a, b\}$ denotes the convex closure of a set. According to the measurable selection theorem, there exists measurable function $h^*_{pq}(z_p(t)) \in K(h_{pq}(z(t)))$ and $w^*_{pq}(z_p(t)) \in K(w_{pq}(z(t)))$ such that

$$\begin{cases} \dot{z}_p(t) = -d_p z_p(t) + \sum_{q=1}^{n} h^*_{pq}(z_p(t))f_q(z_q(t)) \\ \qquad + \sum_{q=1}^{n} w^*_{pq}(z_p(t))g_q(z_q(t-\tau_q(t)) + I_p, \quad t \neq t_l, \\ z_p(t_l) = (1+\eta_l)z_p(t_l^-), t = t_l, l \in \mathbb{N}. \end{cases} \tag{4.1.3}$$

The system (4.1.1) is considered as the drive system. By similar arguments, the corresponding response system with control input is given as

$$\begin{cases} \dot{\widetilde{z}}_p(t) = -d_p \widetilde{z}_p(t) + \sum_{q=1}^{n} h^*_{pq}(\widetilde{z}_p(t))f_q(\widetilde{z}_q(t)) \\ \qquad + \sum_{q=1}^{n} w^*_{pq}(\widetilde{z}_p(t))g_q(\widetilde{z}_q(t-\tau_q(t)) + I_p + u_p(t), \quad t \neq t_l, \\ \widetilde{z}_p(t_l) = (1+\eta_l)\widetilde{z}_p(t_l^-), t = t_l, l \in \mathbb{N}, \end{cases} \tag{4.1.4}$$

where $u_p(t)$ be the designed fixed-time synchronization control input to be designed later and the initial condition is denoted as $\widetilde{z}_p(s) = \psi_p(s)$, $s \in [-\tau, 0]$. Assume $e_p(t) = \widetilde{z}_p(t) - z_p(t)$ be the synchronization error and thus the synchronization error system can be described as

$$\begin{cases} \dot{e}_p(t) = -d_p e_p(t) + \sum_{q=1}^{n} h^*_{pq}(e_p(t)) f_q(e_q(t)) \\ \qquad\qquad + \sum_{q=1}^{n} w^*_{pq}(e_p(t)) g_q(e_q(t-\tau_q(t)) + u_p(t), \quad t \neq t_l, \\ e_p(t_l) = (1+\eta_l) e_p(t_l^-), t = t_l, l \in \mathbb{N}, \end{cases} \tag{4.1.5}$$

where the initial conditions of error system is assumed to be $\varphi_p(s) = \psi_p(s) - \phi_p(s)$ for $s \in [-\tau, 0]$. Further, we denote $h^*_{pq}(e_p(t)) f_q(e_q(t)) = h^*_{pq}(\widetilde{z}_p(t)) f_q(\widetilde{z}_q(t)) - h^*_{pq}(z_p(t)) f_q(z_q(t))$, and $w^*_{pq}(e_p(t)) g_q(e_q(t-\tau_q(t))) = w^*_{pq}(\widetilde{z}_p(t)) g_q(\widetilde{z}_q(t-\tau_q(t)) - w^*_{pq}(z_p(t)) g_q(z_q(t-\tau_q(t))$.

To obtain the main results, the following assumptions have been made throughout this subchapter.

Assumption 4.1.2.1. Suppose that the activation functions f_q and g_q are Lipschitz continuous. That is, for any $q = 1, 2, \ldots, n$, there exist positive real numbers L_q and M_q such that

$$|f_q(u) - f_q(v)| \leq L_q |u - v|, \quad |g_q(u) - g_q(v)| \leq M_q |u - v|,$$

for all u, $v \in \mathbb{R}$, $u \neq v$.

Assumption 4.1.2.2. Suppose that the activation functions f_q and g_q are bounded. That is, for any $q = 1, 2, \ldots, n$, there exist real numbers R_q and S_q such that

$$|f_q(u)| \leq R_q, \quad |g_q(u)| \leq S_q,$$

for all $u \in \mathbb{R}$.

Assumption 4.1.2.3. (See [96]) The impulse sequence $\{t_l, l \in \mathbb{N}\}$ belongs to $\Omega(\tau_{min}, \tau_{max}) \triangleq \{t_0 < t_1 < \ldots < t_l < \ldots, \lim_{t_l \to +\infty} = +\infty, \tau_{min} \leq t_l - t_{l-1} \leq \tau_{max}\}$, where τ_{min}, τ_{max} are positive constants satisfying $\tau_{min} \leq \tau_{max}$. Let $\omega(t, s)$ be the number of impulsive jumps on the time interval (s, t). Suppose that there exists a positive constant ρ such that

$$\frac{t-s}{\tau_{max}} - \rho \leq \omega(t,s) \leq \frac{t-s}{\tau_{min}} + \rho.$$

We also recall the following lemmas and definitions:

Lemma 4.1.2.1. (See [84]) Suppose $W : \mathbb{R}^n \to \mathbb{R}^+$ be a radially unbounded and continuous function which satisfies the following two conditions:

1. $W(e(t)) = 0 \Longleftrightarrow e(t) = 0$,
2. $W(e(t)) \leq - aW^{\alpha}(e(t)) - bW^{\beta}(e(t)) - cW(e(t))$,

where a, b, $c > 0$, $\alpha > 1$ and $0 < \beta < 1$.

Then the origin of error system (4.1.5) is FxTS, and the settling-time $T(e_0)$ is bounded by

$$T(e_0) \leq T_{max} = \frac{1}{c(1-\beta)} \ln(1 + \frac{c}{b}) + \frac{1}{c(\alpha - 1)} \ln(1 + \frac{c}{a}), \quad \forall e_0 \in \mathbb{R}^n.$$

Lemma 4.1.2.2. (See [141]) Suppose that the Assumptions 4.1.2.1 and 4.1.2.2 hold, then for any u, $v \in \mathbb{R}^n$, p, $q \in \mathbb{N}$, following inequalities holds

$$|K(h_{pq}(v))f_q(v_q) - K(h_{pq}(u)f_q(u_q)| \leq \vec{h}_{pq}L_q|v_q - u_q| + R_q|\hat{h}_{pq} - \check{h}_{pq}|,$$

$$|K(w_{pq}(v))f_q(v_q) - K(w_{pq}(u)f_q(u_q)| \leq \vec{w}_{pq}M_q|v_q - u_q| + S_q|\hat{w}_{pq} - \check{w}_{pq}|,$$

where $\vec{h}_{pq} = \max\{|\hat{h}_{pq}|, |\check{h}_{pq}|\}$, $\vec{w}_{pq} = \max\{|\hat{w}_{pq}|, |\check{w}_{pq}|\}$.

Definition 4.1.2.1. (See [84]) The drive-response systems (4.1.1) and (4.1.4) are said to be finite-time synchronized, if for any initial value e_0, there exits a constant $T(e_0) > 0$ and $e(t) = (e_1(t), e_2(t), \dots e_n(t))^T$, such that $\lim_{t \to T(e_0)} ||e(t)|| = 0, \text{and}\, ||e(t)|| = 0 \;\; \forall t \geq T(e_0)$, where $T(e_0)$ is called the settling-time.

Definition 4.1.2.2. (See [84]) The drive-response systems (4.1.1) and (4.1.4) are said to be fixed-time synchronized, if the following two conditions hold:

(1) The system (4.1.4) can synchronize with the system (4.1.1) in finite-time.

(2) The settling-time function $T(e_0)$ is bounded for any initial value e_0, i.e., there exits a fixed constant $T_{max} > 0$ such that $T(e_0) \leq T_{max}$.

Definition 4.1.2.3. (See [142]) The Dini's upper right-hand derivative of a continuous function $V(t) : \mathbb{R} \to \mathbb{R}$ is defined by

$$D^+(V(t)) = \lim_{h \to 0^+} \sup \frac{V(t+h) - V(t)}{h}.$$

4.1.3 Main Results

In order to establish some sufficient conditions to achieve the fixed-time synchronization between the drive-response systems (4.1.1) and (4.1.4), let us consider the fixed-time controller $u_p(t)$ as follows:

$$u_p(t) = -\sigma_p e_p(t) - \text{sign}(e_p(t)) \left[\varrho_p + l_1|e_p(t)|^{\alpha_1} + l_2|e_p(t)|^{\alpha_2} + \varsigma_p|e_p(t-\tau_p(t))|\right], \quad (4.1.6)$$

where σ_p, ϱ_p, ς_p are positive parameters to be designed later. l_1, l_2 are some positive constants, and the real numbers α_1, α_2 satisfy $\alpha_1 > 1$ and $0 < \alpha_2 < 1$.

Theorem 4.1.3.1. Under the Assumptions 4.1.2.1- 4.1.2.3, if the control gain parameters σ_p, ϱ_p and ς_p satisfy the following conditions:

$$\begin{cases} \sigma_p \geq -d_p + \sum_{q=1}^{n} \vec{h}_{qp} L_p, \\ \varsigma_p \geq \sum_{q=1}^{n} \vec{w}_{qp} M_p, \\ \varrho_p \geq \sum_{q=1}^{n} (R_q|\hat{h}_{pq} - \check{h}_{pq}| + S_q|\hat{w}_{pq} - \check{w}_{pq}|), \end{cases} \quad (4.1.7)$$

for $p, q = 1, 2, \ldots, n$, then the drive-response systems (4.1.1) and (4.1.4) can achieve fixed-time synchronization under the controller (4.1.6). Furthermore, the settling-time is estimated as

(i) $\widetilde{T} = T_1^* + T_2^*$, when $0 < \mu_l \triangleq |1+\eta_l| < 1$, $\forall\, l \in \mathbb{N}$, where $T_1^* = \frac{\ln\left[1+\frac{\mu_l^{\rho(1-\alpha_1)}\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{max}}\right)}{\bar{l}_1}\right]}{\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{max}}\right)(\alpha_1 - 1)}$ and $T_2^* = \frac{\ln\left[\frac{\bar{l}_2 \mu_l^{\rho(1-\alpha_2)}}{\mu_l^{-\rho(1-\alpha_2)}\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{min}}\right) + \bar{l}_2 \mu_l^{\rho(1-\alpha_2)}}\right]}{\left(\frac{\ln \mu_l}{\tau_{max}} - \bar{l}_3\right)(1-\alpha_2)}$.

(ii) $\breve{T} = \frac{\ln\left[1+\frac{\bar{l}_3}{\bar{l}_1}\right]}{\bar{l}_3(\alpha_1-1)} - \frac{\ln\left[\frac{\bar{l}_2}{\bar{l}_2+\bar{l}_3}\right]}{\bar{l}_3(1-\alpha_2)}$, when $\mu_l = 1$, $\forall\, l \in \mathbb{N}$.

Proof. Construct the Lyapunov function as

$$W(t) = \sum_{p=1}^{n} |e_p(t)|, \quad (4.1.8)$$

for $t \neq t_k$, and the derivative of $W(t)$ along the trajectory of the error system (4.1.5) can be calculated as

$$\begin{aligned}
D^{+}(W(t)) &= \sum_{p=1}^{n} \operatorname{sign}(e_p(t))\dot{e}_p(t) \\
&= \sum_{p=1}^{n} \operatorname{sign}(e_p(t))(-d_p e_p(t) + \sum_{q=1}^{n} h^{*}_{pq}(e_p(t)) f_q(e_q(t)) \\
&+ \sum_{q=1}^{n} w^{*}_{pq}(e_p(t)) g_q(e_q(t-\tau_q(t))) + u_p(t)) \\
&\leq -\sum_{p=1}^{n} d_p|e_p(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n} |h^{*}_{pq}(e_p(t)) f_q(e_q(t))| \\
&+ \sum_{p=1}^{n}\sum_{q=1}^{n} |w^{*}_{pq}(e_p(t)) g_q(e_q(t-\tau_q(t)))| + \sum_{p=1}^{n} \operatorname{sign}(e_p(t))(-\sigma_p e_p(t) \\
&- \operatorname{sign}(e_p(t))(\varrho_p + l_1|e_p(t)|^{\alpha_1} + l_2|e_p(t)|^{\alpha_2} + \varsigma_p|e_p(t-\tau_p(t))|)),
\end{aligned}$$

$$\begin{aligned}
i.e., \; D^{+}(W(t)) \leq & -\sum_{p=1}^{n} d_p|e_p(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n} |h^{*}_{pq}(e_p(t)) f_q(e_q(t))| \\
& + \sum_{p=1}^{n}\sum_{q=1}^{n} |w^{*}_{pq}(e_p(t)) g_q(e_q(t-\tau_q(t)))| - \sum_{p=1}^{n} \sigma_p|e_p(t)| - \sum_{p=1}^{n} \varrho_p \\
& - l_1\sum_{p=1}^{n} |e_p(t)|^{\alpha_1} - l_2\sum_{p=1}^{n} |e_p(t)|^{\alpha_2} - \sum_{p=1}^{n} \varsigma_p|e_p(t-\tau_p(t))|. \qquad (4.1.9)
\end{aligned}$$

According to Lemma 4.1.2.2, one has

$$\begin{cases}
|h^{*}_{pq}(e_p(t)) f_q(e_q(t))| \leq \vec{h}_{pq} L_q |e_q(t)| + R_q|\hat{h}_{pq} - \check{h}_{pq}|, \\
|w^{*}_{pq}(e_p(t)) g_q(e_q(t-\tau_q(t)))| \leq \vec{w}_{pq} M_q |e_q(t-\tau_q(t))| + S_q|\hat{w}_{pq} - \check{w}_{pq}|.
\end{cases} \qquad (4.1.10)$$

From Eq. (4.1.9) and Eq. (4.1.10), we have

$$
\begin{aligned}
D^+(W(t)) \le& \sum_{p=1}^{n}(-d_p-\sigma_p)|e_p(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\vec{h}_{pq}L_q|e_q(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\vec{w}_{pq}M_q|e_q(t-\tau_q(t))| \\
&+ \sum_{p=1}^{n}\sum_{q=1}^{n}R_q|\hat{h}_{pq}-\check{h}_{pq}| + \sum_{p=1}^{n}\sum_{q=1}^{n}S_q|\hat{w}_{pq}-\check{w}_{pq}| - \sum_{p=1}^{n}\varrho_p - l_1\sum_{p=1}^{n}|e_p(t)|^{\alpha_1} \\
&- l_2\sum_{p=1}^{n}|e_p(t)|^{\alpha_2} - \sum_{p=1}^{n}\varsigma_p|e_p(t-\tau_p(t))| \\
\le& \sum_{p=1}^{n}(-d_p-\sigma_p+\sum_{q=1}^{n}\vec{h}_{qp}L_p)|e_p(t)| + \sum_{p=1}^{n}(\sum_{q=1}^{n}\vec{w}_{qp}M_p-\varsigma_p)|e_p(t-\tau_p(t))| \\
&+ \sum_{p=1}^{n}(\sum_{q=1}^{n}(R_q|\hat{h}_{pq}-\check{h}_{pq}|+S_q|\hat{w}_{pq}-\check{w}_{pq}|)-\varrho_p) - l_1\sum_{p=1}^{n}|e_p(t)|^{\alpha_1} - l_2\sum_{p=1}^{n}|e_p(t)|^{\alpha_2}.
\end{aligned}
\tag{4.1.11}
$$

If $\sigma_p \ge -d_p+\sum_{q=1}^{n}\vec{h}_{qp}L_p$, $\varsigma_p \ge \sum_{q=1}^{n}\vec{w}_{qp}M_p$, $\varrho_p \ge \sum_{q=1}^{n}(R_q|\hat{h}_{pq}-\check{h}_{pq}|+S_q|\hat{w}_{pq}-\check{w}_{pq}|)$,

$p,q=1,2,\ldots,n$,

one can obtain

$$
D^+(W(t)) \le -l_1\sum_{p=1}^{n}|e_p(t)|^{\alpha_1} - l_2\sum_{p=1}^{n}|e_p(t)|^{\alpha_2} - \sum_{p=1}^{n}c_p|e_p(t)|, \tag{4.1.12}
$$

where $c_p = d_p+\sigma_p-\sum_{q=1}^{n}\vec{h}_{qp}L_p > 0$.

Let $l_3 = \min_p\{c_p\}$, then we have

$$
D^+(W(t)) \le -l_1\sum_{p=1}^{n}|e_p(t)|^{\alpha_1} - l_2\sum_{p=1}^{n}|e_p(t)|^{\alpha_2} - l_3\sum_{p=1}^{n}|e_p(t)|. \tag{4.1.13}
$$

In view of Lemma 3.2.1, we obtain

$$
\sum_{p=1}^{n}|e_p(t)|^{\alpha_1} \ge n^{1-\alpha_1}\left(\sum_{p=1}^{n}|e_p(t)|\right)^{\alpha_1}, \ \sum_{p=1}^{n}|e_p(t)|^{\alpha_2} \ge \left(\sum_{p=1}^{n}|e_p(t)|\right)^{\alpha_2}. \tag{4.1.14}
$$

Now, the inequalities (4.1.13) and (4.1.14) yields

$$
\begin{aligned}
D^{+}(W(t)) &\leq -l_1 n^{1-\alpha_1}\left(\sum_{p=1}^{n}|e_p(t)|\right)^{\alpha_1} - l_2\left(\sum_{p=1}^{n}|e_p(t)|\right)^{\alpha_2} - l_3\sum_{p=1}^{n}|e_p(t)| \\
&= -\bar{l}_1 W^{\alpha_1}(t) - \bar{l}_2 W^{\alpha_2}(t) - \bar{l}_3 W(t), \qquad (4.1.15)
\end{aligned}
$$

where $\bar{l}_1 = l_1 n^{1-\alpha_1}$, $\bar{l}_2 = l_2$, $\bar{l}_3 = l_3$.

When $t = t_l$, it follows from Eq. (4.1.8) that

$$
\begin{aligned}
W(t_l) &= \sum_{p=1}^{n}|e_p(t_l)| = \sum_{p=1}^{n}|(1+\eta_l)e_p(t_l^-)| \\
&\leq |1+\eta_l|W(t_l^-) = \mu_l W(t_l^-), \qquad (4.1.16)
\end{aligned}
$$

where $\mu_l = |1+\eta_l|$, and we take $0 < \mu_l \leq 1$.

Further, we construct the following comparison system to study the fixed-time synchronization of error system (4.1.5) under the controller (4.1.6):

$$
\begin{cases}
\dot{V}(t) = \begin{cases}
-\bar{l}_1 V^{\alpha_1}(t) - \bar{l}_3 V(t), V(t) \geq 1, t \neq t_l, \\
-\bar{l}_2 V^{\alpha_2}(t) - \bar{l}_3 V(t), 0 < V(t) < 1, t \neq t_l, \\
0,\ t \neq t_l,
\end{cases} \\
V(t_l) = \mu_l V(t_l^-), t = t_l, \\
V(0) = V_0 = \sum_{p=1}^{n}|e_p(0)|.
\end{cases} \qquad (4.1.17)
$$

From Eq. (4.1.15), Eq. (4.1.16) with Eq. (4.1.17), we get the inequality $0 \leq W(t) \leq V(t)$. That is, if there exists $\widetilde{T} > 0$, for $t \geq \widetilde{T}$ such that $V(t) \equiv 0$, then $W(t) \equiv 0$ for $t \geq \widetilde{T}$. Therefore, to prove the system (4.1.5) is fixed-time synchronized, it is needed to show that the zero solution of system (4.1.17) yields fixed-time synchronization.

When $V(t) \geq 1$, the transformation $\Psi(t) = V^{1-\alpha_1}(t)$ is considered. Since $\alpha_1 > 1$ and from Eq. (4.1.17), we get that $\Psi(t) \to 0^+$ as $V(t) \to +\infty$ and $\Psi(t) \to 1^+$ as $V(t) \to 1^+$. Therefore, we have

$$\begin{cases} \dot{\Psi}(t) = \bar{l}_1(\alpha_1 - 1) + \bar{l}_3(\alpha_1 - 1)\Psi(t), t \neq t_l, \quad 0 < \Psi(t) \leq 1, \\ \Psi(t_l) = \bar{\mu}_l \Psi(t_l^-), t = t_l, \\ \Psi(0) = V_0^{1-\alpha_1}, \end{cases} \tag{4.1.18}$$

where $\overline{\mu}_l = \mu_l^{1-\alpha_1}$. It is not difficult to get that $V(t) \to 1^+$ is equivalent to $\Psi(t) \to 1^-$. Let us take the transformation $\Psi(t) = V^{1-\alpha_2}(t)$, where $0 < \alpha_2 < 1$ for $0 < V(t) < 1$, so that we have

$$\begin{cases} \dot{\Psi}(t) = -\bar{l}_2(1 - \alpha_2) - \bar{l}_3(1 - \alpha_2)\Psi(t), t \neq t_l, \quad 0 \leq \Psi(t) < 1, \\ \Psi(t_l) = \tilde{\mu}_l \Psi(t_l^-), t = t_l, \\ \Psi(0) = 1, \end{cases} \tag{4.1.19}$$

where $\tilde{\mu}_l = \mu_l^{1-\alpha_2}$. It follows that $V(t) \to 0^+$ is equivalent to $\Psi(t) \to 0^+$.

The fixed-time synchronization of Eq. (4.1.17) is decomposed into the following two processes: (i) the trajectories of (4.1.18) approach 1 in a fixed-time T_1^*; (ii) the trajectories of (4.1.19) approach 0 from 1 in a fixed-time T_2^*. Then we can get that $V(t) \to 0$ in the fixed-time $\widetilde{T} = T_1^* + T_2^*$ for any initial value V_0. Next we divide the proof into following two cases: $0 < \mu_l < 1$ and $\mu_l = 1$.

Case 1: Let $0 < \mu_l < 1$, then, $\bar{\mu}_l > 1$ and $0 < \tilde{\mu} < 1$. Thus, we have the conditions $V(t_l^-) \geq 1$ and $V(t_l) \geq 1$. Now, from Eq. (4.1.18) and from the formula of variation of parameters, we have

$$\Psi(t) = e^{\bar{l}_3(\alpha_1-1)t}\bar{\mu}_l^{\omega(t,0)}\Psi(0) + \bar{l}_1(\alpha_1-1)\int_0^t e^{\bar{l}_3(\alpha_1-1)(t-s)}\bar{\mu}_l^{\omega(t,s)}ds. \tag{4.1.20}$$

Since $\Psi(0) = \bar{\mu}_l^{\omega(0,0)}\Psi(0) < 1$ and $\lim_{t\to\infty}\Psi(t) = \infty$, there exists T_1^* such that $\lim_{t\to T_1^*}\Psi(t) = 1$ and $0 < \Psi(t) < 1$ for $0 < t < T_1^*$. According to Eq. (4.1.20), we have

$$e^{\bar{l}_3(\alpha_1-1)t}\bar{\mu}_l^{\omega(t,0)}\Psi(0) + \bar{l}_1(\alpha_1-1)\int_0^t e^{\bar{l}_3(\alpha_1-1)(t-s)}\bar{\mu}_l^{\omega(t,s)}ds = 1,$$

which implies that

$$\bar{l}_1(\alpha_1-1)\int_0^t e^{\bar{l}_3(\alpha_1-1)(t-s)}\bar{\mu}_l^{\omega(t,s)}ds \leq 1.$$

It follows from Assumption 4.1.2.3 that $\bar{\mu}_l^{\frac{t-s}{\tau_{max}}-\rho} \leq \bar{\mu}_l^{\omega(t,s)} \leq \bar{\mu}_l^{\frac{t-s}{\tau_{min}}+\rho}$, we get

$$e^{\bar{l}_3(\alpha_1-1)t}\int_0^t e^{-\bar{l}_3(\alpha_1-1)s}\bar{\mu}_l^{\frac{t-s}{\tau_{max}}-\rho}ds \leq \frac{1}{\bar{l}_1(\alpha_1-1)}.$$

After solving the above equation, we obtain

$$t \leq \frac{\ln\left[1 + \frac{\bar{\mu}_l^{\rho}\left(\bar{l}_3(\alpha_1-1)+\frac{\ln\bar{\mu}_l}{\tau_{max}}\right)}{\bar{l}_1(\alpha_1-1)}\right]}{\left(\bar{l}_3(\alpha_1-1)+\frac{\ln\bar{\mu}_l}{\tau_{max}}\right)} = T_1^*.$$

Substituting $\bar{\mu}_l = \mu_l^{1-\alpha_1}$ into the above inequality, we have

$$T_1^* = \frac{\ln\left[1 + \frac{\mu_l^{\rho(1-\alpha_1)}\left(\bar{l}_3-\frac{\ln\mu_l}{\tau_{max}}\right)}{\bar{l}_1}\right]}{\left(\bar{l}_3-\frac{\ln\mu_l}{\tau_{max}}\right)(\alpha_1-1)}. \tag{4.1.21}$$

Similar to above analysis, we estimate the fixed-time T_2^* such that the system (4.1.19) approach 0 from 1. From Eq. (4.1.20), we find that

$$\Psi(t) = e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\omega(t,0)}\Psi(0) - \bar{l}_2(1-\alpha_2)\int_0^t e^{-\bar{l}_3(1-\alpha_2)(t-s)}\tilde{\mu}_l^{\omega(t,s)}. \tag{4.1.22}$$

Since $0 < \tilde{\mu}_l < 1$, from Assumption 4.1.2.3, we have $\tilde{\mu}_l^{\frac{t-s}{\tau_{min}}+\rho} \le \tilde{\mu}_l^{\omega(t,s)} \le \tilde{\mu}_l^{\frac{t-s}{\tau_{max}}-\rho}$. Now, considering the fixed-time T_2^*, $\Psi(t)$ changes from 1 to 0. Also, we know that, $\Psi(0) = 1$ and $\Psi(T_2^*) = 0$. From Eq. (4.1.22), we therefore have

$$\begin{aligned}
\Psi(t) &= e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\omega(t,0)} - \bar{l}_2(1-\alpha_2)\int_0^t e^{-\bar{l}_3(1-\alpha_2)(t-s)}\tilde{\mu}_l^{\omega(t,s)}ds \\
&\le e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\frac{t}{\tau_{max}}-\rho} - \bar{l}_2(1-\alpha_2)e^{-\bar{l}_3(1-\alpha_2)t}\int_0^t e^{\bar{l}_3(1-\alpha_2)s}\tilde{\mu}_l^{\frac{t-s}{\tau_{mix}}+\rho}ds \\
&= e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\frac{t}{\tau_{max}}-\rho} - \bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}\left[\frac{1-e^{-\left[\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}\right]t}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}}\right] \\
&\le \left[\tilde{\mu}_l^{-\rho} + \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}}\right]e^{-\left[\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}\right]t} - \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}} \triangleq H(t).
\end{aligned}$$

Since $H(0) = \tilde{\mu}_l^{-\rho} > 0$, $H(+\infty) = -\frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}} < 0$, and $\dot{H}(t) < 0$, there exists a unique T_2^* satisfying $H(T_2^*) = 0$, so that we can get

$$H(T_2^*) = \left[\tilde{\mu}_l^{-\rho} + \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}}\right]e^{-\left[\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}\right]t} - \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}} = 0,$$

which implies that

$$T_2^* = \frac{\ln\left[\frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}}{\tilde{\mu}_l^{-\rho}\left(\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}\right)+\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{\rho}}\right]}{\left(\frac{\ln\tilde{\mu}_l}{\tau_{max}}-\bar{l}_3\right)(1-\alpha_2)}.$$

Substituting $\tilde{\mu}_l = \mu_l^{1-\alpha_2}$ into the above equation leads to

$$T_2^* = \frac{\ln\left[\frac{\bar{l}_2\mu_l^{\rho(1-\alpha_2)}}{\mu_l^{-\rho(1-\alpha_2)}\left(\bar{l}_3-\frac{\ln\mu_l}{\tau_{min}}\right)+\bar{l}_2\mu_l^{\rho(1-\alpha_2)}}\right]}{\left(\frac{\ln\mu_l}{\tau_{max}}-\bar{l}_3\right)(1-\alpha_2)}. \tag{4.1.23}$$

Thus, when it comes to the second condition $V(t_l^-) \ge 1$ and $V(t_l) < 1$, this implies that the time interval of $V(t) > 1$ converges to 1 is obviously smaller than T_1^* and the time interval of $0 < V(t) < 1$ from 1 converges to 0 is obviously smaller than T_2^*. Therefore, we may conclude that the error system (4.1.5) can achieve synchronization in fixed-time $T_1^* + T_2^*$. In others words, when $0 < \mu_l < 1$, the systems (4.1.4) and (4.1.1) can be synchronized within the fixed-time $\widetilde{T} = T_1^* + T_2^*$,

where

$$\widetilde{T} = \frac{\ln\left[1 + \frac{\mu_l^{\rho(1-\alpha_1)}\left(\bar{l}_3 - \frac{\ln\mu_l}{\tau_{max}}\right)}{\bar{l}_1}\right]}{\left(\bar{l}_3 - \frac{\ln\mu_l}{\tau_{max}}\right)(\alpha_1 - 1)} + \frac{\ln\left[\frac{\bar{l}_2\mu_l^{\rho(1-\alpha_2)}}{\mu_l^{-\rho(1-\alpha_2)}\left(\bar{l}_3 - \frac{\ln\mu_l}{\tau_{min}}\right) + \bar{l}_2\mu_l^{\rho(1-\alpha_2)}}\right]}{\left(\frac{\ln\mu_l}{\tau_{max}} - \bar{l}_3\right)(1-\alpha_2)}. \tag{4.1.24}$$

Case 2: Consider $\mu_l = 1$. Then it can be obtained that $\tilde{\mu}_l = \bar{\mu}_l = 1$ and using the same formula as (4.1.20), one can get

$$\Psi(t) = e^{\bar{l}_3(\alpha_1-1)t}\Psi(0) + \bar{l}_1(\alpha_1 - 1)e^{\bar{l}_3(\alpha_1-1)t}\int_0^t e^{-\bar{l}_3(\alpha_1-1)s}ds.$$

Thus, we have

$$e^{\bar{l}_3(\alpha_1-1)t}\int_0^t e^{-\bar{l}_3(\alpha_1-1)s}ds \le \frac{1}{\bar{l}_1(\alpha_1-1)},$$

$$i.e.,\ t \le \frac{\ln\left[1 + \frac{\bar{l}_3}{\bar{l}_1}\right]}{\bar{l}_3(\alpha_1-1)} = T_1^*.$$

Since $\Psi(0) = 1$, $\Psi(T_2^*) = 0$ and $\tilde{\mu}_l = 1$, one can obtain from (4.1.22) that

$$\Psi(t) = e^{-\bar{l}_3(1-\alpha_2)t} - \bar{l}_2(1-\alpha_2)e^{-\bar{l}_3(1-\alpha_2)t}\int_0^t e^{\bar{l}_3(1-\alpha_2)s}ds,$$

$$\Psi(T_2^*) = e^{-\bar{l}_3(1-\alpha_2)t} - \bar{l}_2(1-\alpha_2)e^{-\bar{l}_3(1-\alpha_2)t}\int_0^t e^{\bar{l}_3(1-\alpha_2)s}ds = 0.$$

Solving the above equation, we get

$$T_2^* = \frac{\ln\left[\frac{\bar{l}_2}{\bar{l}_2+\bar{l}_3}\right]}{-\bar{l}_3(1-\alpha_2)}.$$

Hence, when $\mu_l = 1$, the drive-response systems (4.1.1) and (4.1.4) can be synchronized within the fixed-time $\breve{T}$ given by

$$\breve{T} = T_1^* + T_2^* = \frac{\ln\left[1+\frac{\bar{l}_3}{\bar{l}_1}\right]}{\bar{l}_3(\alpha_1-1)} - \frac{\ln\left[\frac{\bar{l}_2}{\bar{l}_2+\bar{l}_3}\right]}{\bar{l}_3(1-\alpha_2)}. \tag{4.1.25}$$

Therefore, the proof is completed.

Remark 4.1.3.1. Theorem 4.1.3.1 establishes a novel fixed-time synchronization on MNNs constituting time-varying delay and synchronizing impulsive effects. As the continuous part of Lyapunov inequality in the above mentioned theorem, defined as $\dot{W}(e(t)) \leq -aW^{\alpha}(e(t)) - bW^{\beta}(e(t)) - cW(e(t))$, constitutes one more term than in the previous article [96], where the continuous part of Lyapunov inequality is given as $\dot{W}(e(t)) \leq -aW^{\alpha}(e(t)) - bW^{\beta}(e(t))$, the convergence rate of the system (4.1.5) under Theorem 4.1.3.1 is faster than that of [96]. Also, a less conservative and high precision estimate of settling-time is provided in the Theorem 4.1.3.1. The above mentioned statements are numerically verified in the numerical section 4.1.4 of the subchapter.

Remark 4.1.3.2 It is well known that, impulses are of two types: synchronizing impulses and desynchronizing impulses. One can easily notice that the impulse used in Theorem 4.1.3.1 is of synchronizing kind, which is beneficial for synchronization of systems. Certainly, a question arises, what happens if we consider desynchronizing impulses! Can the systems (4.1.1) and (4.1.4) be synchronized in fixed-time in that case too! The answer of this query is YES, which is verified by the next theorem, Theorem 4.1.3.2.

Theorem 4.1.3.2. Under the assumptions 4.1.2.1-4.1.2.3 to hold, if the parameters σ_p, ϱ_p and ς_p satisfy Eq. (4.1.7) in Theorem 4.1.3.1 and the following inequality holds:

$$\bar{l}_3 > \frac{\ln \mu_l}{\tau_{min}}, \tag{4.1.26}$$

then the drive-response systems (4.1.1) and (4.1.4) can achieve fixed-time synchronization under the controllers (4.1.6) for desynchronizing impulsive effects. Furthermore, the settling-time is estimated as $\widehat{T} = T_1^* + T_2^*$, when $0 < \mu_l \triangleq |1 + \eta_l| > 1$, $\forall\ l \in \mathbb{N}$, where $T_1^* = \frac{\ln\left[1 + \frac{\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{min}}\right)}{\bar{l}_1 \mu_l^{\rho(1-\alpha_1)}}\right]}{\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{min}}\right)(\alpha_1 - 1)}$ and $T_2^* = \frac{\ln\left[\frac{\bar{l}_2 \mu_l^{-\rho(1-\alpha_2)}}{\mu_l^{\rho(1-\alpha_2)}\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{max}}\right) + \bar{l}_2 \mu^{-\rho(1-\alpha_2)}}\right]}{\left(\frac{\ln \mu_l}{\tau_{min}} - \bar{l}_3\right)(1 - \alpha_2)}$.

Proof. The proof of this theorem for $t \neq t_l$ is similar as the proof of Theorem 4.1.3.1, so this part is omitted here.

Now, when $t = t_l$, it follows from Eq. (4.1.8) that

$$\begin{aligned} W(t_l) &= \sum_{p=1}^{n} |e_p(t_l)| = \sum_{p=1}^{n} |(1 + \eta_l) e_p(t_l^-)| \\ &\leq |1 + \eta_l| W(t_l^-) = \mu_l W(t_l^-), \end{aligned} \tag{4.1.27}$$

where $\mu_l = |1 + \eta_l|$, and we take $\mu_l > 1$. Now, the following comparison system is constructed to study the fixed-time synchronization of error system (4.1.5) under the controller (4.1.6) for desynchronizing impulsive effects,

$$\begin{cases} \dot{V}(t) = \begin{cases} -\bar{l}_1 V^{\alpha_1}(t) - \bar{l}_3 V(t), V(t) \geq 1, t \neq t_l, \\ -\bar{l}_2 V^{\alpha_2}(t) - \bar{l}_3 V(t), 0 < V(t) < 1, t \neq t_l, \\ 0,\ t \neq t_l, \end{cases} \\ V(t_l) = \mu_l V(t_l^-), t = t_l, \\ V(0) = \sum_{p=1}^{n} |e_p(0)|. \end{cases} \tag{4.1.28}$$

Combining Eq. (4.1.15), Eq. (4.1.27) with Eq. (4.1.28), we get the inequality $0 \leq W(t) \leq V(t)$. That is, there exists $\widehat{T} > 0$, such that $V(t) \equiv 0$ for $t \geq \widehat{T}$, then $W(t) \equiv 0$ for $t \geq \widehat{T}$. Therefore, to prove that the system (4.1.5) is fixed-time synchronized, we need to show that the zero solution of system (4.1.28) yields fixed-time synchronization. When $V(t) \geq 1$, let us take the transformation $\Psi(t) = V^{1-\alpha_1}(t)$. Since $\alpha_1 > 1$ and from (4.1.28), we have $\Psi(t) \to 1^+$ as $V(t) \to 1^+$ and $\Psi(t) \to 0^+$ as $V(t) \to +\infty$. Therefore, we find

$$\begin{cases} \dot{\Psi}(t) = \bar{l}_1(\alpha_1 - 1) + \bar{l}_3(\alpha_1 - 1)\Psi(t), t \neq t_l, \quad 0 < \Psi(t) \leq 1, \\ \Psi(t_l) = \bar{\mu}_l \Psi(t_l^-), \, t = t_l, \\ \Psi(0) = V_0^{1-\alpha_1}, \end{cases} \tag{4.1.29}$$

where $\bar{\mu}_l = \mu_l^{1-\alpha_1}$. It follows that $V(t) \to 1^+$ is equivalent to $\Psi(t) \to 1^+$. Now, considering the transformation $\Psi(t) = V^{1-\alpha_2}(t)$, where $0 < \alpha_2 < 1$ for $0 < V(t) < 1$, we obtain

$$\begin{cases} \dot{\Psi}(t) = -\bar{l}_2(1 - \alpha_2) - \bar{l}_3(1 - \alpha_2)\Psi(t), \, t \neq t_l, \quad 0 \leq \Psi(t) < 1, \\ \Psi(t_l) = \tilde{\mu}_l \Psi(t_l^-), \, t = t_l, \\ \Psi(0) = 1, \end{cases} \tag{4.1.30}$$

where $\tilde{\mu}_l = \mu_l^{1-\alpha_2}$. It is straight forward that $V(t) \to 1$ is equivalent to $\Psi(t) \to 1$.

Now from the above discussion, the fixed-time synchronization of Eq. (4.1.28) has been transformed into the following two processes: (i) the trajectories of Eq. (4.1.29) approaches 1 in a fixed-time T_1^*; (ii) the trajectories of Eq. (4.1.30) approaches 0 from 1 in fixed-time T_2^*. Then we can find that, $V(t) \to 0$ in the fixed-time $\widehat{T} = T_1^* + T_2^*$ for any initial value V_0. The above two cases will be considered for impulsive strength $\eta_l > 1$.

It follows from Eq. (4.1.29) and from the formula of variation of parameters that

$$\varPsi(t) = e^{\bar{l}_3(\alpha_1-1)t}\bar{\mu}_l^{\omega(t,0)}\varPsi(0) + \bar{l}_1(\alpha_1-1)\int_0^t e^{\bar{l}_3(\alpha_1-1)(t-s)}\bar{\mu}_l^{\omega(t,s)}ds. \tag{4.1.31}$$

Since $\varPsi(0) = \bar{\mu}_l^{\omega(0,0)}\varPsi(0) < 1$ and $\lim_{t\to\infty}\varPsi(t) = \infty$, when $0 < \bar{\mu}_l < 1$, there exists T_1^* such that $\lim_{t\to T_1^*}\varPsi(t) = 1$ and $0 < \varPsi(t) < 1$ for $0 < t < T_1^*$ that is,

$$e^{\bar{l}_3(\alpha_1-1)t}\bar{\mu}_l^{\omega(t,0)}\varPsi(0) + \bar{l}_1(\alpha_1-1)\int_0^t e^{\bar{l}_3(\alpha_1-1)(t-s)}\bar{\mu}_l^{\omega(t,s)}ds = 1.$$

Furthermore,

$$\bar{l}_1(\alpha_1-1)\int_0^t e^{\bar{l}_3(\alpha_1-1)(t-s)}\bar{\mu}_l^{\omega(t,s)}ds \leq 1.$$

According to Assumption 4.1.2.3, $\bar{\mu}_l^{\frac{t-s}{\tau_{min}}+\rho} \leq \bar{\mu}_l^{\omega(t,s)} \leq \bar{\mu}_l^{\frac{t-s}{\tau_{max}}-\rho}$, so that

$$e^{\bar{l}_3(\alpha_1-1)t}\int_0^t e^{-\bar{l}_3(\alpha_1-1)s}\bar{\mu}_l^{\frac{t-s}{\tau_{min}}+\rho}ds \leq \frac{1}{\bar{l}_1(\alpha_1-1)}.$$

After solving above equation, we obtain

$$t \leq \frac{\ln\left[1 + \frac{\left(\bar{l}_3(\alpha_1-1)+\frac{\ln\bar{\mu}_l}{\tau_{min}}\right)}{\bar{l}_1(\alpha_1-1)\bar{\mu}_l^{\rho}}\right]}{\left(\bar{l}_3(\alpha_1-1)+\frac{\ln\bar{\mu}_l}{\tau_{min}}\right)} = T_1^*.$$

Substituting $\bar{\mu}_l = \mu_l^{1-\alpha_1}$, we get

$$T_1^* = \frac{\ln\left[1 + \frac{\left(\bar{l}_3-\frac{\ln\mu_l}{\tau_{min}}\right)}{\bar{l}_1\mu_l^{\rho(1-\alpha_1)}}\right]}{\left(\bar{l}_3-\frac{\ln\mu_l}{\tau_{min}}\right)(\alpha_1-1)}. \tag{4.1.32}$$

In similar way, one finds from Eq. (4.1.30) that

$$\Psi(t) = e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\omega(t,0)}\Psi(0) - \bar{l}_2(1-\alpha_2)\int\limits_0^t e^{-\bar{l}_3(1-\alpha_2)(t-s)}\tilde{\mu}_l^{\omega(t,s)}. \tag{4.1.33}$$

Since $1 < \tilde{\mu}_l < \infty$ and from Assumption 4.1.2.3, we have $\tilde{\mu}_l^{\frac{t-s}{\tau_{max}}-\rho} \leq \tilde{\mu}_l^{\omega(t,s)} \leq \tilde{\mu}_l^{\frac{t-s}{\tau_{min}}+\rho}$. Now, considering the fixed-time T_2^*, $\Psi(t)$ changes from 1 to 0, we know that, $\Psi(0) = 1$ and $\Psi(T_2^*) = 0$.

From Eq. (4.1.33), we get

$$\begin{aligned}
\Psi(t) &= e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\omega(t,0)} - \bar{l}_2(1-\alpha_2)\int\limits_0^t e^{-\bar{l}_3(1-\alpha_2)(t-s)}\tilde{\mu}_l^{\omega(t,s)}ds \\
&\leq e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\frac{t}{\tau_{min}}+\rho} - \bar{l}_2(1-\alpha_2)e^{-\bar{l}_3(1-\alpha_2)t}\int\limits_0^t e^{\bar{l}_3(1-\alpha_2)s}\tilde{\mu}_l^{\frac{t-s}{\tau_{max}}-\rho}ds \\
&= e^{-\bar{l}_3(1-\alpha_2)t}\tilde{\mu}_l^{\frac{t}{\tau_{min}}+\rho} - \bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}\left[\frac{1-e^{-\left[\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}\right]t}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}}\right] \\
&\leq \left[\tilde{\mu}_l^{\rho} + \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}}\right]e^{-\left[\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}\right]t} - \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}} \triangleq H(t).
\end{aligned}$$

Since $H(0) = \tilde{\mu}_l^{\rho} > 0$, $H(+\infty) = -\frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}} < 0$ and $\dot{H}(t) < 0$, there exists unique T_2^* satisfying $H(T_2^*) = 0$, so that we can get

$$H(T_2^*) = \left[\tilde{\mu}_l^{\rho} + \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}}\right]e^{-\left[\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{min}}\right]t} - \frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}}{\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}} = 0,$$

which implies that

$$T_2^* = \frac{\ln\left[\frac{\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}}{\tilde{\mu}_l^{\rho}\left(\bar{l}_3(1-\alpha_2)-\frac{\ln\tilde{\mu}_l}{\tau_{max}}\right)+\bar{l}_2(1-\alpha_2)\tilde{\mu}_l^{-\rho}}\right]}{\left(\frac{\ln\tilde{\mu}_l}{\tau_{min}} - \bar{l}_3(1-\alpha_2)\right)}.$$

Substituting $\tilde{\mu}_l = \mu_l^{1-\alpha_2}$ into the above equation, we obtain

$$T_2^* = \frac{\ln\left[\frac{\bar{l}_2 \mu_l^{-\rho(1-\alpha_2)}}{\mu_l^{\rho(1-\alpha_2)}\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{max}}\right) + \bar{l}_2 \mu^{-\rho(1-\alpha_2)}}\right]}{\left(\frac{\ln \mu_l}{\tau_{min}} - \bar{l}_3\right)(1-\alpha_2)}. \tag{4.1.34}$$

This implies that the time interval of $V(t) > 1$ converges to 1 is obviously smaller than T_1^* and the time interval of $0 < V(t) < 1$ from 1 converges to 0 is obviously smaller than T_2^*. Therefore, we may conclude that the error system (4.1.5) can achieve synchronization in fixed-time $T_1^* + T_2^*$. In other words, when $\mu_l > 1$, the systems (4.1.1) and (4.1.4) can be synchronized within the fixed-time $\widehat{T} = T_1^* + T_2^*$ given by

$$\widehat{T} = T_1^* + T_2^* = \frac{\ln\left[1 + \frac{\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{min}}\right)}{\bar{l}_1 \mu_l^{\rho(1-\alpha_1)}}\right]}{\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{min}}\right)(\alpha_1 - 1)} + \frac{\ln\left[\frac{\bar{l}_2 \mu_l^{-\rho(1-\alpha_2)}}{\mu_l^{\rho(1-\alpha_2)}\left(\bar{l}_3 - \frac{\ln \mu_l}{\tau_{max}}\right) + \bar{l}_2 \mu^{-\rho(1-\alpha_2)}}\right]}{\left(\frac{\ln \mu_l}{\tau_{min}} - \bar{l}_3\right)(1-\alpha_2)}. \tag{4.1.35}$$

Therefore, the proof is completed.

Remark 4.1.3.3 Contrary to the claim mentioned in the previous works reported in [111] that the fixed-time synchronization of system (4.1.5) cannot be guaranteed when the impulsive strength $\mu_l > 1$, we have proved in Theorem 4.1.3.2 that if the condition $\bar{l}_3 > \frac{\ln \mu_l}{\tau_{min}}$ holds then the fixed-time synchronization of MNNs can be achieved under the influence of desynchronizing impulsive effects and one can estimate the corresponding settling time $\widehat{T}$ as given above.

4.1.4 Numerical Simulations and Discussions

Example 4.1.4.1. Consider the following MNNs with impulsive effects for two nodes

$$\begin{cases} \dot{z}_p(t) = -d_p z_p(t) + \sum_{q=1}^{2} h_{pq}(z_p(t)) f_q(z_q(t)) + \sum_{q=1}^{2} w_{pq}(z_p(t)) g_q(z_q(t-\tau_q(t)), \quad t \neq t_l, \\ z_p(t_l) = \mu_l z_p(t_l^-), t = t_l, l \in \mathbb{N}, \end{cases} \tag{4.1.36}$$

where $p, q = 1, 2$, $d_1 = d_2 = 1$ and the memristive connection weights are taken as

$$h_{11}(z_1) = \begin{cases} 1.2, & |z_1| \leq 1, \\ 0.9, & |z_1| > 1, \end{cases} \quad h_{12}(z_1) = \begin{cases} -0.2, & |z_1| \leq 1, \\ -0.5, & |z_1| > 1, \end{cases}$$

$$h_{21}(z_2) = \begin{cases} -1.5, & |z_2| \leq 1, \\ -2.0, & |z_2| > 1, \end{cases} \quad h_{22}(z_2) = \begin{cases} 0.4, & |z_1| \leq 1, \\ 1.2, & |z_2| > 1, \end{cases}$$

$$w_{11}(z_1) = \begin{cases} -1.5, & |z_1| \leq 1, \\ -0.7, & |z_1| > 1, \end{cases} \quad w_{12}(z_1) = \begin{cases} -0.5, & |z_1| \leq 1, \\ 1, & |z_1| > 1, \end{cases}$$

$$w_{21}(z_2) = \begin{cases} 0.3, & |z_1| \leq 1, \\ 1.5, & |z_1| > 1, \end{cases} \quad w_{22}(z_2) = \begin{cases} -2.1, & |z_1| \leq 1, \\ -1.1, & |z_1| > 1, \end{cases}$$

The neuron activation functions and time-varying delays are defined as $f_p(z) = \tanh(z)$, $g_p(z) = \frac{1}{2}(|z+1| - |z-1|)$ and $\tau_p(t) = \frac{e^t}{1+e^t}$, $p = 1, 2$, then we have $L_p = M_p = R_p = S_p = 1$. The initial values of MNN (4.1.36) are assumed as $\phi_1(s) = 0.6$, $\phi_2(s) = 0.2$, $s \in [-1, 0]$.

The corresponding response system with impulsive effects is given by

$$\begin{cases} \dot{\widetilde{z}}_p(t) = -d_p \widetilde{z}_p(t) + \sum_{q=1}^{2} h^*_{pq}(\widetilde{z}_p(t)) f_q(\widetilde{z}_q(t)) + \sum_{q=1}^{2} w^*_{pq}(\widetilde{z}_p(t)) g_q(\widetilde{z}_q(t-\tau_q(t)) + u_p(t), \quad t \neq t_l, \\ \widetilde{z}_p(t_l) = \mu_l \widetilde{z}_p(t_l^-), t = t_l, l \in \mathbb{N}, \end{cases} \tag{4.1.37}$$

for $p, q = 1, 2$, where $u_p(t)$ is the controller. The initial values of MNN (4.1.36) are taken as $\psi_1(s) = -0.5$, $\psi_2(s) = 0.8$, $s \in [-1, 0]$. The evolutions of drive-response systems (4.1.1) and (4.1.4) under the controller are shown in Fig. 4.1.1 and Fig. 4.1.2.

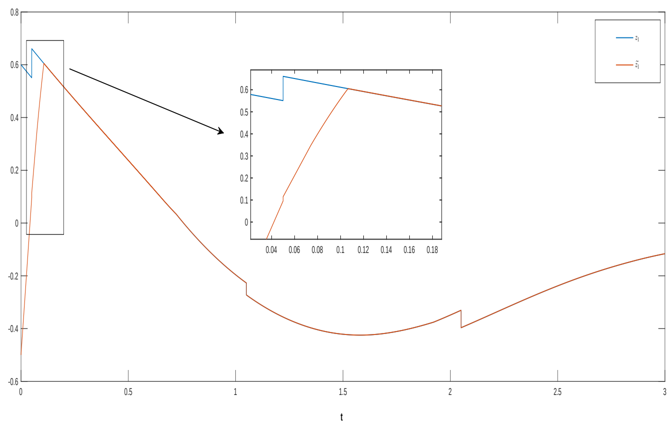


Figure 4.1.1: Synchronization curves of z_1 and $\widetilde{z}_1$.

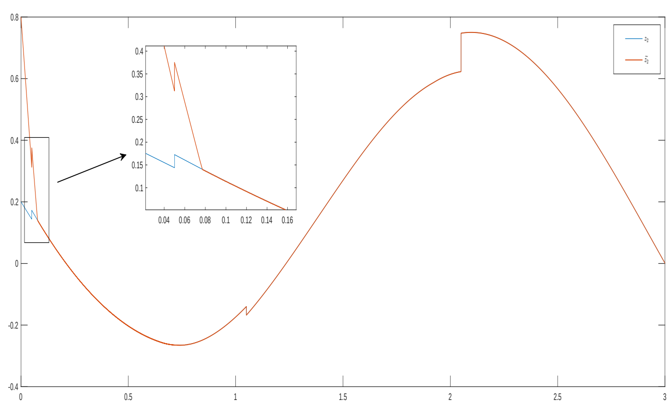


Figure 4.1.2: Synchronization curves of z_2 and $\widetilde{z}_2$.

According to the Theorem 4.1.31, the control gains σ_p, ϱ_p and ς_p should satisfy

$$\sigma_1 \geq -d_1 + \sum_{q=1}^{2} \vec{h}_{q1} L_1 = 2.2, \;\; \sigma_2 \geq -d_2 + \sum_{q=1}^{2} \vec{h}_{q2} L_2 = 0.7,$$

$$\varsigma_1 \geq \sum_{q=1}^{2} \vec{w}_{q1} M_1 = 3, \;\; \varsigma_2 \geq \sum_{q=1}^{2} \vec{w}_{q2} M_2 = 3.1,$$

$$\varrho_1 \geq \sum_{q=1}^{2} (R_q |\hat{h}_{1q} - \check{h}_{1q}| + S_q |\hat{w}_{1q} - \check{w}_{1q}|) = 2.9,$$
$$\varrho_2 \geq \sum_{q=1}^{2} (R_q |\hat{h}_{2q} - \check{h}_{2q}| + S_q |\hat{w}_{2q} - \check{w}_{2q}|) = 3.5$$

Let us choose $\sigma_1 = 2.5$ $\sigma_2 = 1$, $\varsigma_1 = 3.1$, $\varsigma_2 = 3.2$, $\varrho_1 = 3$, $\varrho_2 = 3.6$ and also $l_1 = 3$, $l_2 = 2$, $l_3 = 1$ $\alpha_1 = 1.5$, $\alpha_2 = 0.5$, then we have $\bar{l}_1 = 2.1$, $\bar{l}_2 = 2$, $\bar{l}_3 = 1$.

The impulsive strength is taken as $\mu_l = 0.8$, indicating that the impulses are synchronizing and the impulsive sequences $\{t_l, l \in \mathbb{N}\}$ are chosen which satisfy $\rho = 2$, $\tau = 1$. According to Assumption 4.1.2.3 and [88], we can obtain that $\tau_{min} = 1.2$, $\tau_{max} = 0.8$. It can be shown that all the conditions (4.1.7) of Theorem 4.1.3.1 are satisfied. Thus, the drive-response systems (4.1.1) and (4.1.4) can be achieved fixed-time synchronization under the controllers (4.1.6) for synchronizing impulses and the settling-time can be estimated as $\widetilde{T} = 1.874$. The evolutions of the synchronization error system (4.1.5) for synchronizing impulses under the controllers (4.1.6) are shown in Fig. 4.1.3. Also it follows from Theorem 4.1.3.1, if the impulsive strength is taken as $\mu_l = 1$ that means the impulses are inactive, and the same impulsive sequences and parameters are chosen which are already defined above for synchronizing impulsive case, then the drive-response systems (4.1.1) and (4.1.4) can achieve fixed-time synchronization under the controllers (4.1.6) and the settling-time can be estimated as $\breve{T} = 1.589$. The evolution of the synchronization error system (4.1.5) under the controller (4.1.6) for inactive impulses are shown in Fig. 4.1.4. This verifies, through the above numerical simulations, that the estimated settling-time in Theorem 4.1.3.1 is much lesser then the existing results reported in [95, 96, 139].

Example 4.1.4.2. Considering the same systems in Example 4.1.4.1, the results obtained in Theorem 4.1.3.2 can be verified. Let $\mu_l = 1.2$ is the impulsive strength of the impulsive sequence $\{t_l, l \in \mathbb{N}\}$ with average impulsive interval $\tau_{min} = 1.2$, $\tau_{max} = 0.8$, and the positive constant $\rho = 2$ such that the sufficient condition $\bar{l}_3 > \frac{\ln \mu_l}{\tau_{min}}$ is satisfied. Therefore, the drive-response systems (4.1.1) and (4.1.4) can be achieved fixed-time synchronization under the controller (4.1.6) for desynchronizing impulses and the settling-time can be estimated as $\widehat{T} = 1.973$. The evolutions of the synchronization error system (4.1.5) for desynchronizing impulses under the controller (4.1.6) are shown in Fig. 4.1.5.

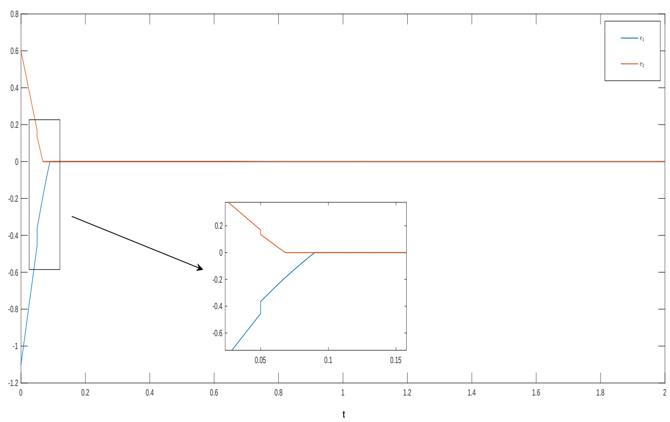


Figure 4.1.3: The trajectories of error system (4.1.5) for synchronizing impulses ($\mu_l = 0.8$).

4.1.5 Conclusion

In this subchapter, the problem of fixed-time synchronization is studied for MNNs with time-varying delay and impulsive effects. Synchronizing and desynchronizing impulsive effects have been considered separately to investigate the MNNs to achieve the fixed-time synchronization within the settling time. The results for desynchronizing impulses

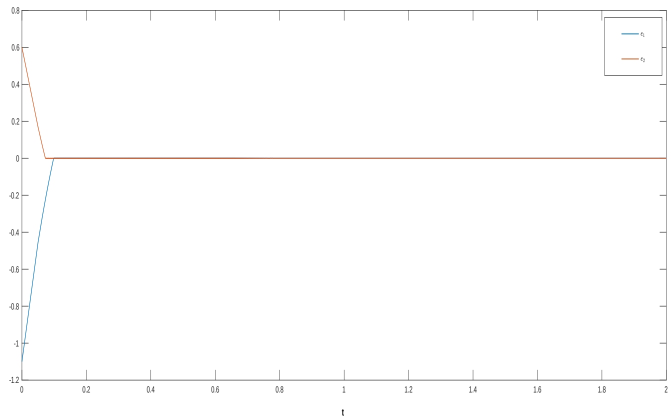


Figure 4.1.4: The trajectories of error system (4.1.5) for inactive impulses ($\mu_l = 1$).

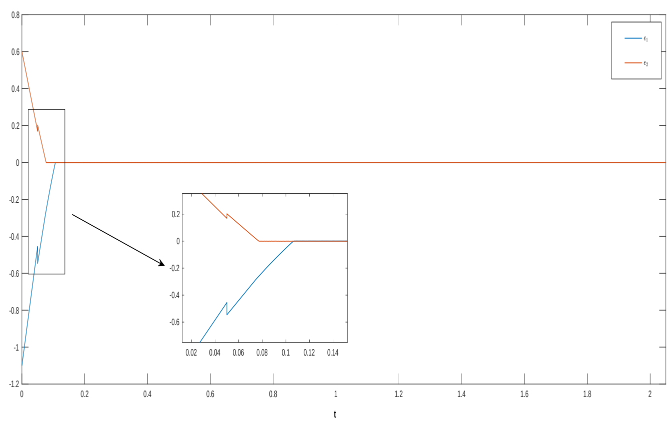


Figure 4.1.5: The trajectories of error system (4.1.5) for desynchronizing impulses ($\mu_l =$ 1.2).

have not been established in any previous works reported in [95, 96, 139, 111]. The main highlight of the present work is that it provides the less conservative results through numerical simulation and theoretical derivation for the case of synchronizing impulses and desynchronizing impulses. Finally, numerical examples are provided to validate the effectiveness of our proposed theoretical results. In this present subchapter, the impulsive strength is considered as a constant. If the impulsive strength can be considered as a function of time, which is more general, then investigating the dynamical behaviour of MNNs is a more challenging task which would be thoroughly investigated in the future research work.

4.2 Fixed-Time Synchronization of Complex-Valued Inertial Neural Networks with Time Delays and Impulsive Effects

4.2.1 Introduction[2]

Complex-valued signals play an important role in many applications, which cannot be handled by real-valued neural networks. Therefore, it is essential to investigate the dynamical properties of complex-valued neural networks (CVNNs) in many research areas. Unlike real-valued neural networks, CVNNs consist of complex-valued states, activation functions, connection weights, and bias parameters. The main aim of studying CVNNs is not only to investigate new dynamic performance but also to address some issues that real-valued neural networks are unable to resolve. For instance, single CVNNs can solve the XOR problem and the detection symmetry problem, which cannot be solved by single-neuron real-valued neural networks. To date, significant research work has been proposed for CVNNs, involving stability, synchronization, and passivity, which are based on the direct method [143, 144, 145, 146] and the separation method [147, 148, 149]. Using the separation method, the authors intended to change the system into a real-valued one, covering various aspects of the study of many CVNNs. Based on the separation method, the authors developed synchronization criteria for delayed CVNNs and also designed two different controllers for the separated subsystems [147].

In view of the above discussion, different from the previous works about infinite-time synchronization with impulses in [150, 151, 152], this present subchapter investigates the fixed-time synchronization of complex-valued inertial neural networks (CVINNs) with synchronizing and desynchronizing impulsive effects. To achieve this, the original CVINNs are first transformed into first-order CVINNs using variable transformation.

[2]The content of this subchapter is under review in an International Journal

Then, based on the average impulsive interval and the comparison principle method, one sufficient condition, $\rho > \frac{\ln \xi}{\tau_a}$ is derived to ensure the fixed-time synchronization of CVINNs subjeted to desynchronizing impulses, which is different from the existing work in [153] where the class of impulsive sequence is taken to be of some specific type. The settling-time, which depends on the parameters of the impulsive sequences, is also estimated.

4.2.2 Model Formulation and Preliminaries

Let us consider the following model of CVINN with impulsive effects described as the drive system:

$$\begin{cases} \ddot{u}_r(t) = -a_r\dot{u}_r(t) - b_r u_r(t) + \sum_{s=1}^{q} c_{rs} f_s(u_s(t)) \\ \qquad + \sum_{s=1}^{q} d_{rs} g_s(u_s(t-\tau_s)), r \in I, \qquad t \neq t_l, \\ u_r(t_l) = \mu u_r(t_l^-), \dot{u}_r(t_l) = \mu \dot{u}_r(t_l^-),\ l \in \mathbb{N}, \end{cases} \tag{4.2.1}$$

where $r \in I$, $u_r(t) \in \mathbb{C}$ is the state of the rth neurons at time t and its second derivative $\ddot{u}_r(t)$ is called an inertial term of system (4.2.1), $a_r > 0$ and $b_r > 0$ are the self feedback connection weight, $c_{rs} \in \mathbb{C}$ and $d_{rs} \in \mathbb{C}$ are complex-valued connection weight of the neurons, $f_s(u_s(t)) \in \mathbb{C}$ and $g_s(u_s(t-\tau_s)) \in \mathbb{C}$ are the complex-valued activation functions, and $\tau_s \geq 0$ denotes the time delay. The sequence $\{t_1, t_2, \ldots, t_l, \ldots\}$ is a strictly increasing impulsive moments such that $\lim_{l\to\infty} t_l = \infty$, $\mu \in \mathbb{R}^+$ represents the impulsive strength. Assume that $u_r(t)$ is right continuous at $t = t_l$, i.e., $u_r(t_l^+) = \lim_{t\to t_l+0} u_r(t) = u_r(t_l)$ and $u_r(t_l^-) = \lim_{t\to t_l-0} u_r(t)$. Therefore, the solution of system (4.2.1) are piecewise right continuous at $t = t_l$ for all $l \in \mathbb{N}$. The initial conditions of the system (4.2.1) are assumed as $u_r(m) = \phi_r(m)$, $\dot{u}_r(m) = \psi_r(m)$, $-\tau \leq m \leq 0$, where $\tau = \max\{\tau_s\}$ and $\phi_r(m)$, $\psi_r(m)$ are continuous functions.

Introducing the following variable transformation

$$v_r(t) = \alpha_r \dot{u}_r(t) + \beta_r u_r(t), \tag{4.2.2}$$

where α_r, β_r are nonzero positive scalars in $\mathbb{R}$, then the system (4.2.1) can be rewritten as

$$\begin{cases} \dot{u}_r(t) = -\frac{\beta_r}{\alpha_r} u_r(t) + \frac{1}{\alpha_r} v_r(t), & t \neq t_l, \\ \dot{v}_r(t) = -\pi_r v_r(t) + \gamma_r u_r(t) + \sum_{s=1}^{q} \alpha_r c_{rs} f_s(u_s(t)) & \\ \qquad + \sum_{s=1}^{q} \alpha_r d_{rs} g_s(u_s(t-\tau_s)), & t \neq t_l, \\ u_r(t_l) = \mu u_r(t_l^-), & \\ v_r(t_l) = \eta v_r(t_l^-), l \in \mathbb{N}, & \end{cases} \tag{4.2.3}$$

where $\pi_r = a_r - \frac{\beta_r}{\alpha_r}$, $\gamma_r = a_r\beta_r - \frac{\beta_r^2}{\alpha_r} - \alpha_r b_r$, and $\eta \in \mathbb{R}^+$ be the impulsive strength. The corresponding response system is considered as

$$\begin{cases} \ddot{\widetilde{u}}_r(t) = -a_r \dot{\widetilde{u}}_r(t) - b_r \widetilde{u}_r(t) + \sum_{s=1}^{q} c_{rs} f_s(\widetilde{u}_s(t)) & \\ \qquad + \sum_{s=1}^{q} d_{rs} g_s(\widetilde{u}_s(t-\tau_s)) + R_r(t), r \in I, & t \neq t_l, \\ \widetilde{u}_r(t_l) = \mu \widetilde{u}_r(t_l^-), \dot{\widetilde{u}}_r(t_l) = \mu \dot{\widetilde{u}}_r(t_l^-), \; l \in \mathbb{N}, & \end{cases} \tag{4.2.4}$$

where $R_r(t)$ be the controller which will be designed later. The initial conditions are defined as $\widetilde{u}_r(m) = \widetilde{\phi}_r(m)$, $\dot{\widetilde{u}}_r(m) = \widetilde{\psi}_r(m)$, $-\tau \leq m \leq 0$ and $\widetilde{\phi}_r(m)$, $\widetilde{\psi}_r(m)$ are continuous functions.

Assume that $\widetilde{v}_r(t) = \alpha_r \dot{\widetilde{u}}_r(t) + \beta_r \widetilde{u}_r(t)$, $U_r(t) = \alpha_r R_r(t)$, and in similar way as above, the system (4.2.4) can be rewritten as

$$\begin{cases} \dot{\widetilde{u}}_r(t) = -\frac{\beta_r}{\alpha_r}\widetilde{u}_r(t) + \frac{1}{\alpha_r}\widetilde{v}_r(t), & t \neq t_l, \\ \dot{\widetilde{v}}_r(t) = -\pi_r\widetilde{v}_r(t) + \gamma_r\widetilde{u}_r(t) + \sum_{s=1}^{q}\alpha_r c_{rs} f_s(\widetilde{u}_s(t)) & \\ \qquad + \sum_{s=1}^{q}\alpha_r d_{rs} g_s(\widetilde{u}_s(t-\tau_s)) + U_r(t), & t \neq t_l, \\ \widetilde{u}_r(t_l) = \mu\widetilde{u}_r(t_l^-), & \\ \widetilde{v}_r(t_l) = \eta\widetilde{v}_r(t_l^-), l \in \mathbb{N}. & \end{cases} \tag{4.2.5}$$

Assumption 4.2.2.1. For any $u_r = x_r^{'} + iy_r^{'}$ and $v_r = x_r^{''} + iy_r^{''}$, $f_r(.)$ and $g_r(.)$ can be separated into real and imaginary parts as $f_r(u_r) = f_r^R(x_r^{'}, y_r^{'}) + if_r^I(x_r^{'}, y_r^{'})$ and $g_r(u_r) = g_r^R(x_r^{'}, y_r^{'}) + ig_r^I(x_r^{'}, y_r^{'})$. Assume that there exist positive Lipschitz constants $l_r^{\omega\omega}$, $p_r^{\omega\omega}$ ($\omega = R, I$) such that

$$|f_r^{\omega}(x_r^{'}, y_r^{'}) - f_r^{\omega}(x_r^{''}, y_r^{''})| \leq l_r^{\omega R}|x_r^{'} - x_r^{''}| + l_r^{\omega I}|y_r^{'} - y_r^{''}|,$$
$$|g_r^{\omega}(x_r^{'}, y_r^{'}) - g_r^{\omega}(x_r^{''}, y_r^{''})| \leq p_r^{\omega R}|x_r^{'} - x_r^{''}| + p_r^{\omega I}|y_r^{'} - y_r^{''}|.$$

Remark 4.2.2.1. The uniqueness and existence of the solution to the systems (4.2.1) and (4.2.4) can be ensured since the activation functions satisfy Assumption 4.2.2.1. In contrast to Assumption 4.2.2.1, many existing articles [150, 151, 154] have chosen more general type of assumption that describes the non-decreasing activation. In this subchapter, the activation function is based on Assumption 4.2.2.1.

The synchronization error be $\dot{\hat{u}}_r(t) = \widetilde{u}_r(t) - u_r(t)$, so that the corresponding error system of Eq. (4.2.1) and Eq. (4.2.4) can be transformed into

$$\begin{cases} \ddot{\hat{u}}_r(t) = -a_r\dot{\hat{u}}_r(t) - b_r\hat{u}_r(t) + \sum_{s=1}^{q} c_{rs}F_s(\hat{u}_s(t)) & \\ \qquad + \sum_{s=1}^{q} d_{rs}G_s(\hat{u}_s(t-\tau_s)) + R_r(t), r \in I, & t \neq t_l, \\ \hat{u}_r(t_l) = \mu\hat{u}_r(t_l^-), \dot{\hat{u}}_r(t_l) = \mu\dot{\hat{u}}_r(t_l^-), \ l \in \mathbb{N}, & \end{cases} \tag{4.2.6}$$

where $F_r(\hat{u}_r(t)) = f_r(\widetilde{u}_r(t)) - f_r(u_r(t))$, $G_s(\hat{u}_s(t-\tau_s)) = g_s(\widetilde{u}_s(t-\tau_s)) - g_s(u_s(t-\tau_s))$.

Assuming $\hat{v}_r(t) = \widetilde{v}_r(t) - v_r(t)$, another form of the error system can be obtained from Eq. (4.2.3) and Eq. (4.2.5) as

$$\begin{cases} \dot{\hat{u}}_r(t) = -\frac{\beta_r}{\alpha_r}\hat{u}_r(t) + \frac{1}{\alpha_r}\hat{v}_r(t), & t \neq t_l, \\ \dot{\hat{v}}_r(t) = -\pi_r\hat{v}_r(t) + \gamma_r\hat{u}_r(t) + \sum_{s=1}^{q}\alpha_r c_{rs}F_s(\hat{u}_s(t)) & \\ \qquad + \sum_{s=1}^{q}\alpha_r d_{rs}G_s(\hat{u}_s(t-\tau_s)) + U_r(t), & t \neq t_l, \\ \hat{u}_r(t_l) = \mu\hat{u}_r(t_l^-), & \\ \hat{v}_r(t_l) = \mu\hat{v}_r(t_l^-), l \in \mathbb{N}. & \end{cases} \tag{4.2.7}$$

Let $\hat{u}_r(t) = x_r(t) + iy_r(t)$, $\hat{v}_r(t) = \hat{x}_r(t) + i\hat{y}_r(t)$, then the real and imaginary parts of the error system (4.2.7) can be separated as

$$\begin{cases} \dot{x}_r(t) = -\frac{\beta_r}{\alpha_r}x_r(t) + \frac{1}{\alpha_r}\hat{x}_r(t), & t \neq t_l, \\ \dot{y}_r(t) = -\frac{\beta_r}{\alpha_r}y_r(t) + \frac{1}{\alpha_r}\hat{y}_r(t), & t \neq t_l, \\ \dot{\hat{x}}_r(t) = -\pi_r\hat{x}_r(t) + \gamma_r x_r(t) + \sum_{s=1}^{q}\alpha_r c_{rs}^R F_s^R(x_s(t), y_s(t)) & \\ \qquad - \sum_{s=1}^{q}\alpha_r c_{rs}^I F_s^I(x_s(t), y_s(t)) + \sum_{s=1}^{q}\alpha_r d_{rs}^R G_s^R(x_s(t-\tau_s), y_s(t-\tau_s)) & \\ \qquad + \sum_{s=1}^{q}\alpha_r d_{rs}^I G_s^I(x_s(t-\tau_s), y_s(t-\tau_s)) + U_r^R(t), & t \neq t_l, \\ \dot{\hat{y}}_r(t) = -\pi_r\hat{y}_r(t) + \gamma_r y_r(t) + \sum_{s=1}^{q}\alpha_r c_{rs}^R F_s^I(x_s(t), y_s(t)) & \\ \qquad + \sum_{s=1}^{q}\alpha_r c_{rs}^I F_s^R(x_s(t), y_s(t)) + \sum_{s=1}^{q}\alpha_r d_{rs}^R G_s^I(x_s(t-\tau_s), y_s(t-\tau_s)) & \\ \qquad + \sum_{s=1}^{q}\alpha_r d_{rs}^I G_s^R(x_s(t-\tau_s), y_s(t-\tau_s)) + U_r^I(t), & t \neq t_l, \\ x_r(t_l) = \mu x_r(t_l^-),\ y_r(t_l) = \mu y_r(t_l^-), & \\ \hat{x}_r(t_l) = \eta\hat{x}_r(t_l^-),\ \hat{y}_r(t_l) = \eta\hat{y}_r(t_l^-),\ l \in \mathbb{N}. & \end{cases} \tag{4.2.8}$$

For the main results of this subchapter, following essential definitions are required for further analysis.

Definition 4.2.2.1. (See [155]) The drive-response systems (4.2.3) and (4.2.5) are called to have finite-time synchronization, if for any initial value $\phi(0)$, there exits a constant settling-time function $T(\phi(0)) \geq 0$, such that $\lim_{t \to T(\phi(0))} ||\phi(t)|| = 0$, and $||\phi(t)|| = 0$ for all $t \geq T(\phi(0))$,where $\phi(t) = (\hat{u}_1(t), \hat{u}_2(t) \ldots \hat{u}_q(t), \hat{v}_1(t), \hat{v}_2(t), \ldots \hat{v}_q(t))^T$.

Definition 4.2.2.2. (See [155]) The drive-response systems (4.2.3) and (4.2.5) are called to have fixed-time synchronization, if following two conditions hold.

(1) The drive-response system (4.2.3) and (4.2.5) is finite-time synchronization.

(2) The settling-time function $T(\phi(0))$ is bounded for any initial value $\phi(0)$, i.e., there exists a $T_{max} > 0$ such that $T(\phi(0)) \leq T_{max}$.

4.2.3 Main Results

In this subsection, based on Lyapunov stability theory and comparison principle approach, it is aimed to derive some sufficient criteria for fixed-time synchronization of inertial complex-valued drive-response systems (4.2.3) and (4.2.5) for synchronizing and desynchronizing impulses. The feedback controllers for CVINNs (4.2.5) are designed as follows:

$$\begin{cases} R_r(t) = \frac{1}{\alpha_r}(U_r^R(t) + iU_r^I(t)), \\ U_r^R(t) = -\theta_{1r}\hat{x}_r(t) - \iota_{1r}\text{sign}(\hat{x}_r(t))x_r(t) - \text{sign}(\hat{x}_r(t))(\lambda_{1r}|x_r(t)|^{m_1} + \lambda_{2r}|\hat{x}_r(t)|^{m_2}) \\ \qquad -\text{sign}(\hat{x}_r(t))(\delta_{1r}|x_r(t)|^{n_1} + \delta_{2r}|\hat{x}_r(t)|^{n_2}) - \theta_{2r}\text{sign}(\hat{x}_r(t))\sum_{s=1}^{q}|x_s(t-\tau_s)|, \\ U_r^I(t) = -\theta_{3r}\hat{y}_r(t) - \iota_{2r}\text{sign}(\hat{y}_r(t))y_r(t) - \text{sign}(\hat{y}_r(t))(\lambda_{3r}|y_r(t)|^{m_3} + \lambda_{4r}|\hat{y}_r(t)|^{m_4}) \\ \qquad -\text{sign}(\hat{y}_r(t))(\delta_{3r}|y_r(t)|^{n_3} + \delta_{4r}|\hat{y}_r(t)|^{n_4}) - \theta_{4r}\text{sign}(\hat{y}_r(t))\sum_{s=1}^{q}|y_s(t-\tau_s)|, \end{cases} \tag{4.2.9}$$

where $0 < m_\upsilon < 1$, $n_\upsilon > 1$ and $\theta_{\upsilon r}$, $\delta_{\upsilon r}$, $\iota_{\upsilon r}$ $\lambda_{\upsilon r}$, $\upsilon \in S$, $r \in I$ are some positive parameters to be determined later on.

Theorem 4.2.3.1. Suppose that Assumption 4.2.2.1 holds, then the drive-response systems (4.2.3) and (4.2.5) achieve fixed-time synchronization if the following conditions are satisfied:

$$\begin{cases} \iota_{1r} \geq -\frac{\beta_r}{\alpha_r} + |\gamma_r| + \sum_{s=1}^{q}|\alpha_r||c_{rs}^R|l_r^{RR} + \sum_{s=1}^{q}|\alpha_s||c_{rs}^I|l_r^{IR} \\ \qquad + \sum_{s=1}^{q}|\alpha_s||c_{rs}^R|l_r^{IR} + \sum_{s=1}^{q}|\alpha_s||c_{rs}^I|l_r^{RR}, \\ \iota_{2r} \geq -\frac{\beta_r}{\alpha_r} + |\gamma_r| + \sum_{s=1}^{q}|\alpha_s||c_{rs}^R|l_r^{RI} + \sum_{s=1}^{q}|\alpha_s||c_{rs}^I|l_r^{II} \\ \qquad + \sum_{s=1}^{q}|\alpha_s||c_{rs}^R|l_r^{II} + \sum_{s=1}^{q}|\alpha_s||c_{rs}^I|l_r^{RI}, \\ \theta_{1r} \geq \frac{1}{|\alpha_r|} - \pi_r, \\ \theta_{3r} \geq \frac{1}{|\alpha_r|} - \pi_r, \\ \theta_{2s} \geq \sum_{r=1}^{q}|\alpha_s|(|d_{rs}^R|p_r^{RR} + |d_{rs}^I|p_r^{IR} + |d_{rs}^R|p_r^{IR} + |d_{rs}^I|p_r^{RR}), \\ \theta_{4s} \geq \sum_{r=1}^{q}|\alpha_s|(|d_{rs}^R|p_r^{RI} + |d_{rs}^I|p_r^{IR} + |d_{rs}^R|p_r^{II} + |d_{rs}^I|p_r^{RI}). \end{cases} \tag{4.2.10}$$

Moreover, the estimated settling-time function T_{max} satisfies $T_{max} = \frac{\ln\left[1+\frac{\xi^{N_0(1-n^*)}\left(\rho-\frac{\ln\xi}{T_a}\right)}{\epsilon_2}\right]}{\left(\rho-\frac{\ln\xi}{T_a}\right)(n^*-1)} +$

$\frac{\ln\left[\frac{\epsilon_1 \xi^{N_0(1-m^*)}}{\xi^{-N_0(1-m^*)}\left(\rho-\frac{\ln\xi}{Ta}\right)+\epsilon_1\xi^{N_0(1-m^*)}}\right]}{\left(\frac{\ln\xi}{Ta}-\rho\right)(1-m^*)}$ for $0 < \xi < 1$ and $T_{max} = \frac{\ln\left[1+\frac{\rho}{\epsilon_2}\right]}{\rho(n^*-1)} - \frac{\ln\left[\frac{\epsilon_1}{\epsilon_1+\rho}\right]}{\rho(1-m^*)}$ for $\xi = 1$, where $\xi = \max\{\mu, \eta\} \leq 1$, $\rho = \min_{r\in I}\{\rho_r, \varrho_r, \sigma_r, \varsigma_r\}$, $m^* = \max_{\upsilon\in S}\{m_\upsilon\}$, $n^* = \max_{\upsilon\in S}\{n_\upsilon\}$ and N_0 is a positive integer.

Proof. Consider the Lyapunov function as

$$V(t) = \sum_{\upsilon=1}^{4} V_\upsilon(t) = \sum_{r=1}^{q} |x_r(t)| + \sum_{r=1}^{q} |y_r(t)| + \sum_{r=1}^{q} |\hat{x}_r(t)| + \sum_{r=1}^{q} |\hat{y}_r(t)|. \tag{4.2.11}$$

When $t \neq t_l$, the upper right Dini-derivative of $V(t)$ along the trajectories of Eq. (4.2.8) can be obtained as

$$\begin{aligned} D^+(V(t)) \mid_{(4.2.8)} &= \sum_{r=1}^{q} \text{sign}(x_r(t))\Big(-\frac{\beta_r}{\alpha_r}x_r(t) + \frac{1}{\alpha_r}\hat{x}_r(t)\Big) \\ &+ \sum_{r=1}^{q} \text{sign}(y_r(t))\Big(-\frac{\beta_r}{\alpha_r}y_r(t) + \frac{1}{\alpha_r}\hat{y}_r(t)\Big) \\ &+ \sum_{r=1}^{q} \text{sign}(\hat{x}_r(t))\Big(-\pi_r\hat{x}_r(t) + \gamma_r x_r(t) + \sum_{s=1}^{q} \alpha_r d_{rs}^R G_s^R(x_s(t-\tau_s), y_s(t-\tau_s)) \\ &+ \sum_{s=1}^{q} \alpha_r d_{rs}^I G_s^I(x_s(t-\tau_s), y_s(t-\tau_s)) + U_r^R(t)\Big) + \sum_{s=1}^{q} \alpha_r c_{rs}^R F_s^R(x_s(t), y_s(t)) \\ &- \sum_{s=1}^{q} \alpha_r c_{rs}^I F_s^I(x_s(t), y_s(t)) + \sum_{r=1}^{q} \text{sign}(\hat{y}_r(t))\Big(-\pi_r\hat{y}_r(t) + \gamma_r y_r(t) \\ &+ \sum_{s=1}^{q} \alpha_r c_{rs}^R F_s^I(x_s(t), y_s(t)) + \sum_{s=1}^{q} \alpha_r c_{rs}^I F_s^R(x_s(t), y_s(t)) \\ &+ \sum_{s=1}^{q} \alpha_r d_{rs}^R G_s^I(x_s(t-\tau_s), y_s(t-\tau_s)) \\ &+ \sum_{s=1}^{q} \alpha_r d_{rs}^I G_s^R(x_s(t-\tau_s), y_s(t-\tau_s)) + U_r^I(t)\Big). \end{aligned} \tag{4.2.12}$$

It follows from Assumption 4.2.2.1 that

$$
\begin{aligned}
&| F_s^{\omega}(x_s(t), y_s(t)) | \leq l_s^{\omega R} | x_s(t) | + l_s^{\omega I} | y_s(t) |, \\
&| G_s^{\omega}(x_s(t-\tau_s), y_s(t-\tau_s)) | \leq p_s^{\omega R} | x_s(t-\tau_s) | + p_s^{\omega I} | y_s(t-\tau_s) | .
\end{aligned} \tag{4.2.13}
$$

Using Eq. (4.2.9) and inequality (4.2.13), one can estimate from Eq. (4.2.12) as follows:

$$
\begin{aligned}
D^+V(t) \,|_{(4.2.8)} \leq & \sum_{r=1}^{q}\Big(-\frac{\beta_r}{\alpha_r} + |\gamma_r| - \iota_{1r} + \sum_{s=1}^{q} |\alpha_s||c_{rs}^R| l_r^{RR} + \sum_{s=1}^{q} |\alpha_s||c_{rs}^I| l_r^{IR} \\
& + \sum_{s=1}^{q} |\alpha_s||c_{rs}^R| l_r^{IR} + \sum_{s=1}^{q} |\alpha_s||c_{rs}^I| l_r^{RR}\Big)|x_r(t)| \\
& + \sum_{r=1}^{q}\Big(-\frac{\beta_r}{\alpha_r} + |\gamma_r| - \iota_{2r} + \sum_{s=1}^{q} |\alpha_s||c_{rs}^R| l_r^{RI} + \sum_{s=1}^{q} |\alpha_s||c_{rs}^I| l_r^{II} \\
& + \sum_{s=1}^{q} |\alpha_s||c_{rs}^R| l_r^{II} + \sum_{s=1}^{q} |\alpha_s||c_{rs}^I| l_r^{RI}\Big)|y_r(t)| + \sum_{r=1}^{q}\Big(\frac{1}{|\alpha_r|} - \pi_r - \theta_{1r}\Big)|\hat{x}_r(t)| \\
& + \sum_{r=1}^{q}\Big(\frac{1}{|\alpha_r|} - \pi_r - \theta_{3r}\Big)|\hat{y}_r(t)| + \sum_{r=1}^{q}\sum_{s=1}^{q}\Big(|\alpha_s|\Big(|d_{rs}^R| p_r^{RR} + |d_{rs}^I| p_r^{IR} \\
& + |d_{rs}^R| p_r^{IR} + |d_{rs}^I| p_r^{RR}\Big) - \theta_{2r}\Big)|x_r(t-\tau_s)| + \sum_{r=1}^{q}\sum_{s=1}^{q}\Big(|\alpha_s|\Big(|d_{rs}^R| p_r^{RI} \\
& + |d_{rs}^I| p_r^{IR} + |d_{rs}^R| p_r^{II} + |d_{rs}^I| p_r^{RI}\Big) - \theta_{4r}\Big)|y_r(t-\tau_s)| \\
& - \sum_{r=1}^{q}\Big(\lambda_{1r}|x_r(t)|^{m_1} + \lambda_{2r}|\hat{x}_r(t)|^{m_2} + \lambda_{3r}|y_r(t)|^{m_3} + \lambda_{4r}|\hat{y}_r(t)|^{m_4}\Big) \\
& - \sum_{r=1}^{q}\Big(\delta_{1r}|x_r(t)|^{n_1} + \delta_{2r}|\hat{x}_r(t)|^{n_2} + \delta_{3r}|y_r(t)|^{n_3} + \delta_{4r}|\hat{y}_r(t)|^{n_4}\Big).
\end{aligned} \tag{4.2.14}
$$

In view of Eq. (4.2.10), we can find that

$$
\begin{aligned}
D^+V(t) \leq & -\sum_{r=1}^{q} \rho_r |x_r(t)| - \sum_{r=1}^{q} \varrho_r |y_r(t)| - \sum_{r=1}^{q} \sigma_r |\hat{x}_r(t)| - \sum_{r=1}^{q} \varsigma_r |\hat{y}_r(t)| \\
& - \sum_{r=1}^{q}\Big(\lambda_{1r}|x_r(t)|^{m_1} + \lambda_{2r}|\hat{x}_r(t)|^{m_2} + \lambda_{3r}|y_r(t)|^{m_3} + \lambda_{4r}|\hat{y}_r(t)|^{m_4}\Big) - \sum_{r=1}^{q}\Big(\delta_{1r}|x_r(t)|^{n_1} \\
& + \delta_{2r}|\hat{x}_r(t)|^{n_2} + \delta_{3r}|y_r(t)|^{n_3} + \delta_{4r}|\hat{y}_r(t)|^{n_4}\Big),
\end{aligned} \tag{4.2.15}
$$

where $\rho_r = \frac{\beta_r}{\alpha_r} - |\gamma_r| + \iota_{1r} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^R|l_r^{RR} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^I|l_r^{IR} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^R|l_r^{IR} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^I|l_r^{RR}$, $\varrho_r = \frac{\beta_r}{\alpha_r} - |\gamma_r| + \iota_{2r} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^R|l_r^{RI} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^I|l_r^{II} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^R|l_r^{II} - \sum_{s=1}^{q} |\alpha_s||c_{rs}^I|l_r^{RI}$, $\sigma_r = \pi_r + \theta_{1r} - \frac{1}{|\alpha_r|}$, $\varsigma_r = \pi_r + \theta_{2r} - \frac{1}{|\alpha_r|}$.

Let $\bar{\lambda} = \min_{\upsilon \in S,\, r \in I}\{\lambda_{\upsilon r}\}$, $\bar{\delta} = \min_{\upsilon \in S,\, r \in I}\{\delta_{\upsilon r}\}$, $\rho = \min_{r \in I}\{\rho_r, \varrho_r, \sigma_r, \varsigma_r\}$, then using Lemma 3.2.1, it follows from inequality (4.2.15) that

$$\begin{aligned}
D^+V(t) \le & -\rho\Big(\sum_{r=1}^{q} |x_r(t)| + \sum_{r=1}^{q} |\hat{x}_r(t)| + \sum_{r=1}^{q} |y_r(t)| + \sum_{r=1}^{q} |\hat{y}_r(t)|\Big) \\
& - \bar{\lambda}\Big(\sum_{r=1}^{q} |x_r(t)|^{m_1} + \sum_{r=1}^{q} |\hat{x}_r(t)|^{m_2} + \sum_{r=1}^{q} |y_r(t)|^{m_3} + \sum_{r=1}^{q} |\hat{y}_r(t)|^{m_4}\Big) \\
& - \bar{\delta}\Big(\sum_{r=1}^{q} |x_r(t)|^{n_1} + \sum_{r=1}^{q} |\hat{x}_r(t)|^{n_2} + \sum_{r=1}^{q} |y_r(t)|^{n_3} + \sum_{r=1}^{q} |\hat{y}_r(t)|^{n_4}\Big) \\
\le & -\rho\Big(\sum_{r=1}^{q} |x_r(t)| + \sum_{r=1}^{q} |\hat{x}_r(t)| + \sum_{r=1}^{q} |y_r(t)| + \sum_{r=1}^{q} |\hat{y}_r(t)|\Big) \\
& - \bar{\lambda}\Big(\big(\sum_{r=1}^{q} |x_r(t)|\big)^{m_1} + \big(\sum_{r=1}^{q} |\hat{x}_r(t)|\big)^{m_2} + \big(\sum_{r=1}^{q} |y_r(t)|\big)^{m_3} + \big(\sum_{r=1}^{q} |\hat{y}_r(t)|\big)^{m_4}\Big) \\
& - \bar{\delta}\Big(q^{1-n_1}\big(\sum_{r=1}^{q} |x_r(t)|\big)^{n_1} + q^{1-n_2}\big(\sum_{r=1}^{q} |\hat{x}_r(t)|\big)^{n_2} + q^{1-n_3}\big(\sum_{r=1}^{q} |y_r(t)|\big)^{n_3} + q^{1-n_4}\big(\sum_{r=1}^{q} |\hat{y}_r(t)|\big)^{n_4}\Big) \\
\le & -\rho\sum_{\upsilon=1}^{4} V_\upsilon(t) - \epsilon\Big(\sum_{\upsilon=1}^{4} V_\upsilon^{m_\upsilon}(t) + \sum_{\upsilon=1}^{4} V_\upsilon^{n_\upsilon}(t)\Big),
\end{aligned} \tag{4.2.16}$$

where $\epsilon = \min_{\upsilon \in S}\{\bar{\lambda}, \bar{\delta}q^{1-n_\upsilon}\}$. Let $m^* = \max_{\upsilon \in S}\{m_\upsilon\} < 1$, $n^* = \max_{\upsilon \in S}\{n_\upsilon\} > 1$, then we can conclude that

$$V_\upsilon^{m_\upsilon}(t) + V_\upsilon^{n_\upsilon}(t) \ge \frac{1}{2}\Big(V_\upsilon^{m^*}(t) + V_\upsilon^{n^*}(t)\Big). \tag{4.2.17}$$

Applying Lemma 3.2.1 and combining inequalities (4.2.16) and (4.2.17) yield

$$\begin{aligned} D^+V(t) &\leq -\rho V(t) - \frac{\epsilon}{2}\Big(\sum_{\upsilon=1}^{4} V_\upsilon^{m^*}(t) + \sum_{\upsilon=1}^{4} V_\upsilon^{n^*}(t)\Big) \\ &\leq -\rho V(t) - \frac{\epsilon}{2}\Big(\sum_{\upsilon=1}^{4} V_\upsilon^{m^*}(t) + \sum_{\upsilon=1}^{4} V_\upsilon^{n^*}(t)\Big) \\ &\leq -\rho V(t) - \frac{\epsilon}{2}\Big(\sum_{\upsilon=1}^{4} V_\upsilon(t)\Big)^{m^*} + \frac{\epsilon 4^{1-n^*}}{2}\Big(\sum_{\upsilon=1}^{4} V_\upsilon(t)\Big)^{n^*} \\ &= -\rho V(t) - \epsilon_1 V^{m^*}(t) - \epsilon_2 V^{n^*}(t), \end{aligned} \tag{4.2.18}$$

where $\epsilon_1 = \frac{\epsilon}{2}$, $\epsilon_2 = \frac{\epsilon 4^{1-n^*}}{2}$.

When $t = t_l$, it follows from Eq. (4.2.11) that

$$\begin{aligned} V(t_l) &= \sum_{\upsilon=1}^{4} V_\upsilon(t_l) \\ &= \sum_{r=1}^{q} |x_r(t_l)| + \sum_{r=1}^{q} |\hat{x}_r(t_l)| + \sum_{r=1}^{q} |y_r(t_l)| + \sum_{r=1}^{q} |\hat{y}_r(t_l)| \\ &= \sum_{r=1}^{q} \mu|x_r(t_l^-)| + \sum_{r=1}^{q} \eta|\hat{x}_r(t_l^-)| + \sum_{r=1}^{q} \mu|y_r(t_l^-)| + \sum_{r=1}^{q} \eta|\hat{y}_r(t_l^-)| \\ &\leq \max\{\mu, \eta\} \sum_{\upsilon=1}^{4} V_\upsilon(t_l^-) = \xi V(t_l^-), \end{aligned} \tag{4.2.19}$$

where $\xi = \max\{\mu, \eta\}$, and we have $0 < \xi \leq 1$.

Further, in order to investigate the fixed-time synchronization of error system (4.2.7) under the controller (4.2.9), the following auxiliary system is constructed for the proof using comparison method:

$$\begin{cases} \dot{U}(t) = \begin{cases} -\epsilon_2 U^{n^*}(t) - \rho U(t), & U(t) \geq 1, t \neq t_l, \\ -\epsilon_1 U^{m^*}(t) - \rho U(t), & 0 < U(t) < 1, t \neq t_l \\ 0, & U(t) = 0, t \neq t_l, \end{cases}, \\ U(t_l) = \xi U(t_l^-),\ t = t_l, \\ U(0) = \sum_{r=1}^{q} |x_r(0)| + \sum_{r=1}^{q} |\hat{x}_r(0)| + \sum_{r=1}^{q} |y_r(0)| + \sum_{r=1}^{q} |\hat{y}_r(0)|. \end{cases} \tag{4.2.20}$$

From Eq. (4.2.18), Eq. (4.2.19) and Eq. (4.2.20), we get the inequality $0 \leq V(t) \leq U(t)$. Thus, if there exists $T_{max} > 0$, for $t \geq T_{max}$ such that $U(t) \equiv 0$ then $V(t) \equiv 0$ for $t \geq T_{max}$. Thus, in order to demonstrate that the error system (4.2.7) can achieve fixed-time synchronization, it is needed to demonstrate that the origin of system (4.2.20) can achieve fixed-time synchronization.

For $U(t) \geq 1$, take $R(t) = U^{1-n^*}(t)$, where $n^* > 1$. From Eq. (4.2.20), we obtain that $R(t)$ tends to 0 as $U(t)$ tends to $+\infty$ and $R(t)$ tends to 1 as $U(t)$ tends to 1. Thus, we have

$$\begin{cases} \dot{R}(t) = \epsilon_2(n^* - 1) + \rho(n^* - 1)R(t), & t \neq t_l, 0 < R(t) \leq 1, \\ R(t_l) = \bar{\xi} R(t_l^-), t = t_l, \\ R(0) = U_0^{1-n^*}, \end{cases} \tag{4.2.21}$$

where $\bar{\xi} = \xi^{1-n^*} \in [1, \infty)$. It is easy to see that $U(t)$ tends to 1 is similar to $R(t)$ tends to 1. Now, take $R(t) = U^{1-m^*}(t)$, where $0 < m^* < 1$ for $0 < U(t) < 1$, so that we get

$$\begin{cases} R(t) = -\epsilon_1(1-m^*) - \rho(1-m^*)R(t), \quad t \neq t_l, 0 \leq R(t) < 1, \\ R(t_l) = \hat{\xi} R(t_l^-), t = t_l, \\ R(0) = 1, \end{cases} \tag{4.2.22}$$

where $\hat{\xi} = \xi^{1-m^*} \in (0,1]$. This implies that $U(t)$ tends to 0 is similar to $R(t)$ tends to 0. The following two cases are considered for the system (4.2.20) to be fixed-time synchronization : (i) the system's state of Eq. (4.2.21) converges 1 in T_1; (ii) the system's state of Eq. (4.2.22) converges 0 from 1 in T_2. Then, we have $U(t) \to 0$ in $T_{max} = T_1 + T_2$ for initial condition $U(0)$. Further, we separate the proof into two situations $0 < \xi < 1$ and $\xi = 1$.

Case 1: When $0 < \xi < 1$, then $\bar{\xi} > 1$ and $0 < \hat{\xi} < 1$. From Eq. (4.2.21) and by using the method of variation of parameters, we have

$$R(t) = e^{\rho(n^*-1)t} \bar{\xi}^{N_\varsigma(t,0)} R(0) + \epsilon_2(n^*-1) \int_0^t e^{\rho(n^*-1)(t-s)} \bar{\xi}^{N_\varsigma(t,s)} ds. \tag{4.2.23}$$

From Definition 2.1.2.4 , Eq. (4.2.23) and $R(0) = \bar{\xi}^{N_\varsigma(0,0)} R(0) < 1$, it is observed that $R(t)$ is increasing and $\lim_{t\to\infty} R(t) = \infty$, when $\bar{\xi} > 1$. There exist T_1 such that $\lim_{t\to T_1} R(t) = 1$ and $0 < R(t) < 1$ for $0 < t < T_1$. In view of Eq. (4.2.23), one can get that

$$e^{\rho(n^*-1)t} \bar{\xi}^{N_\varsigma(t,0)} R(0) + \epsilon_2(n^*-1) \int_0^t e^{\rho(n^*-1)(t-s)} \bar{\xi}^{N_\varsigma(t,s)} ds = 1,$$

which implies that

$$\epsilon_2(n^*-1) \int_0^t e^{\rho(n^*-1)(t-s)} \bar{\xi}^{N_\varsigma(t,s)} ds \leq 1.$$

In view of Definition 2.1.2.4 , it follows that $\bar{\xi}^{\frac{t-s}{\tau_a}-N_0} \leq \bar{\xi}^{N_\varsigma(t,s)} \leq \bar{\xi}^{\frac{t-s}{\tau_a}+N_0}$, such that

$$\epsilon_2(n^*-1)e^{\rho(n^*-1)t}\int\limits_0^t e^{-\rho(n^*-1)s}\bar{\xi}^{\frac{t-s}{\tau_a}-N_0}ds \leq 1.$$

After solving above equation, we obtain that

$$t \leq \frac{\ln\left[1+\frac{\bar{\xi}^{N_0}\left(\rho(1-m^*)+\frac{\ln\bar{\xi}}{\tau_a}\right)}{\epsilon_2(n^*-1)}\right]}{\left[\rho(1-m^*)+\frac{\ln\bar{\xi}}{\tau_a}\right]} = T_1.$$

Substituting $\bar{\xi} = \xi^{1-n^*}$into the above inequalities, we have

$$T_1 = \frac{\ln\left[1+\frac{\xi^{N_0(1-n^*)}\left(\rho-\frac{\ln\xi}{\tau_a}\right)}{\epsilon_2}\right]}{\left(\rho-\frac{\ln\xi}{\tau_a}\right)(n^*-1)}. \tag{4.2.24}$$

Similar analysis as followed above estimate the fixed-time T_2 by which the system (4.2.22) converges to 0 from 1. In view of Eq. (4.2.22), we find that

$$R(t) = e^{-\rho(1-m^*)t}\hat{\xi}^{N_\varsigma(t,0)}R(0) - \epsilon_1(1-m^*)\int\limits_0^t e^{-\rho(1-m^*)(t-s)}\hat{\xi}^{N_\varsigma(t,s)}ds. \tag{4.2.25}$$

Since $0 < \hat{\xi} < 1$, and according to the Definition 2.1.2.4, we have $\hat{\xi}^{\frac{t-s}{\tau_a}+N_0} \leq \hat{\xi}^{N_\xi(t,s)} \leq$

$\hat{\xi}^{\frac{t-s}{\tau_a}-N_0}$ for $0 < \hat{\xi} < 1$ and $R(0) = 1$. From Eq. (4.2.25), we therefore have

$$\begin{aligned} R(t) &= e^{-\rho(1-m^*)t}\hat{\xi}^{N_\zeta(t,0)} - \epsilon_1(1-m^*)\int_0^t e^{-\rho(1-m^*)(t-s)}\hat{\xi}^{N_\zeta(t,s)}ds \\ &\le e^{-\rho(1-m^*)t}\hat{\xi}^{\frac{t}{\tau_a}-N_0} - \epsilon_1(1-m^*)\int_0^t e^{-\rho(1-m^*)(t-s)}\hat{\xi}^{\frac{t-s}{\tau_a}+N_0}ds \\ &= e^{-\rho(1-m^*)t}\hat{\xi}^{\frac{t}{\tau_a}-N_0} - \epsilon_1(1-m^*)\hat{\xi}^{N_0}\left[\frac{1-e^{-\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)t}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)}\right] \\ &= \left[\hat{\xi}^{-N_0} + \frac{\epsilon_1(1-m^*)\hat{\xi}^{N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)}\right]e^{-\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)t} - \frac{\epsilon_1(1-m^*)\hat{\xi}^{N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)} \triangleq Q(t). \end{aligned} \tag{4.2.26}$$

Since $Q(0) = \hat{\xi}^{-N_0} > 0$ and $Q(+\infty) = -\frac{\epsilon_1(1-m^*)\hat{\xi}^{N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)} < 0$, we get $\dot{Q}(t) < 0$ and then there exist unique T_2 satisfies $Q(T_2) = 0$. This implies that $Q(t)$ is decreasing. Since $R(0) = 1$ then $R(t) \to 0$ as $t \to T_2$. From inequality (4.2.26), we therefore have

$$Q(T_2) = \left[\hat{\xi}^{-N_0} + \frac{\epsilon_1(1-m^*)\hat{\xi}^{N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)}\right]e^{-\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)T_2} - \frac{\epsilon_1(1-m^*)\hat{\xi}^{N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)} = 0,$$

which implies that

$$T_2 = \frac{\ln\left[\frac{\epsilon_1(1-m^*)\hat{\xi}^{N_0}}{\hat{\xi}^{-N_0}\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)+\epsilon_1(1-m^*)\hat{\xi}^{N_0}}\right]}{\left(\frac{\ln\hat{\xi}}{\tau_a}-\rho(1-m^*)\right)}.$$

Substituting $\hat{\xi} = \xi^{1-m^*}$ into the above inequality leads to

$$T_2 = \frac{\ln\left[\frac{\epsilon_1\xi^{(1-m^*)N_0}}{\xi^{-(1-m^*)N_0}\left(\rho-\frac{\ln\xi}{\tau_a}\right)+\epsilon_1\xi^{(1-m^*)N_0}}\right]}{\left(\frac{\ln\xi}{\tau_a}-\rho\right)(1-m^*)}. \tag{4.2.27}$$

When $W(t) \ge 1$ and $0 < W(t) < 1$. The condition indicates that $W(t) > 1$ approaches to 1 is definitely smaller than T_1 and $0 < W(t) < 1$ from 1 approaches to 0 is definitely smaller than T_2. We can therefore draw the conclusion that the error system (4.2.7) can ensure fixed-time synchronization and settling-time function is $T_{max} = T_1 + T_2$. In order to $0 < \xi < 1$, the drive-response systems (4.2.3) and (4.2.5) can be ensured fixed-time synchronization and settling-time function is $T_{max} = T_1 + T_2$,where T_{max} is

$$T_{max} = \frac{\ln\left[1 + \frac{\xi^{N_0(1-n^*)}\left(\rho - \frac{\ln \xi}{\tau_a}\right)}{\epsilon_2}\right]}{\left(\rho - \frac{\ln \xi}{\tau_a}\right)(n^* - 1)} + \frac{\ln\left[\frac{\epsilon_1 \xi^{N_0(1-m^*)}}{\xi^{-N_0(1-m^*)}\left(\rho - \frac{\ln \xi}{\tau_a}\right) + \epsilon_1 \xi^{N_0(1-m^*)}}\right]}{\left(\frac{\ln \xi}{\tau_a} - \rho\right)(1 - m^*)}. \tag{4.2.28}$$

Case 2: When $\xi = 1$, it can be obtained that $\bar{\xi} = \hat{\xi} = 1$ and using the same formula as discussed above (4.2.23), one can get that

$$R(t) = e^{\rho(n^*-1)t} R(0) + \epsilon_2(n^* - 1) \int_0^t e^{\rho(n^*-1)(t-s)} ds.$$

Thus, we obtain

$$\epsilon_2(n^* - 1) \int_0^t e^{\rho(n^*-1)(t-s)} ds \leq 1.$$

After solving above inequality, we get

$$t \leq \frac{\ln\left[1 + \frac{\rho}{\epsilon_2}\right]}{\rho(n^* - 1)} = T_1.$$

Since $R(0) = 1$,$R(T_2) = 1$ and $\hat{\xi}_l = 1$, we can get from (4.2.25)

$$R(t) = e^{-\rho(1-m^*)t} - \epsilon_1(1 - m^*) \int_0^t e^{-\rho(1-m^*)(t-s)} ds,$$

$$Q(T_2) = e^{-\rho(1-m^*)t} - \epsilon_1(1 - m^*) \int_0^t e^{-\rho(1-m^*)(t-s)} ds = 0.$$

After solving the above inequality , we find

$$T_2 = \frac{\ln\left[\frac{\epsilon_1}{\epsilon_1+\rho}\right]}{-\rho(1-m^*)}.$$

Hence, when $\xi_l = 1$, the drive-response systems (4.2.3) and (4.2.5) can be ensured fixed-time synchronization and settling-time function T_{max} is given by

$$T_{max} = T_1 + T_2 = \frac{\ln\left[1+\frac{\rho}{\epsilon_2}\right]}{\rho(n^*-1)} - \frac{\ln\left[\frac{\epsilon_1}{\epsilon_1+\rho}\right]}{\rho(1-m^*)}. \tag{4.2.29}$$

Remark 4.2.3.1. In previous articles [92, 154, 156, 157], two controllers were added to a modified first-order system. However, in this subchapter, only one controller has been added to the main system, rather than adding two to the modified one. This configuration is often more suitable for practical applications.

Remark 4.2.3.2. In [153], the impulsive strength is dependent on the parameters of the continuous Lyapunov inequality, i.e., $V(u(t_l^+)) \leq \xi^{\frac{1}{1-\alpha}} V(u(t_l))$, where $0 < \alpha < 1$. It exhibits conservative discrete time behaviour in the Lyapunov inequality. However, in the present subchapter, the impulsive strength does not depend on the parameters of the continuous Lyapunov inequality, i.e., $V(u(t_l^+)) \leq \xi V(u(t_l))$. Therefore, the results obtained in this subchapter are less conservative than those results obtained in the previous studies.

Remark 4.2.3.3. Recently in [153], the authors have introduced fixed-time synchronization of nonlinear impulsive systems and discussed about its application to inertial neural networks. The class of impulsive sequence considered in their work relies on the last impulsive point. However, in this particular study, a general class of impulsive sequence is taken which is based on the average impulsive interval. Further, the design of the Lyapunov function in this study is different from that in [153]. In this study, the Lyapunov function takes the form $\dot{V}(t) \leq -aV(t) - bV(t)^{\alpha} - cV(t)^{\beta}$, where $a, b, c > 0$, $0 < \alpha < 1$ and $\beta > 1$. On the other hand, in [153], the Lyapunov function is of the form $\dot{V}(t) \leq -aV(t)^{\alpha} - bV(t)^{\beta}$, where $a, b > 0$, $0 < \alpha < 1$ and $\beta > 1$.

Remark 4.2.3.4. Numerous CVINNs incorporating impulsive effects have been proposed to achieve exponential synchronization [150, 151, 152, 158, 159] and asymptotic synchronization [160]. Specifically, the use of impulsive control for exponential stabilization of CVINNs has been investigated in [150]. Additionally, Guo et al. [155] recently discussed the fixed-time synchronization for CVINNs without impulses. However, the fixed-time synchronization of CVINNs with impulsive effects has not been explored in the aforementioned literature. The current research is focused on filling this gap and investigating towards this direction.

Remark 4.2.3.5. The obtained results in Theorem 4.2.3.1 show that the CVINNs with impulsive effects can achieve fixed-time synchronization for synchronizing impulses ($\xi < 1$) and inactive impulses ($\xi = 1$). Moreover, the estimated settling-time depends on the system's parameters, impulsive strength and average impulsive interval. In addition, stabilizing impulse ($\xi < 1$) plays the role to stabilize the system within the estimated settling-time.

Theorem 4.2.3.2. Suppose that Assumption 4.2.2.1 holds then the drive-response system (4.2.3) and (4.2.5) achieve fixed-time synchronization under the controller (4.2.9) for desynchronizing impulses if the parameters satisfy Eq. (4.2.10) as given in Theorem 4.2.3.1 and it satisfies the condition:

$$\rho > \frac{\ln \xi}{\tau_a}. \tag{4.2.30}$$

Moreover, the estimated settling-time function is given by $T_{max} = T_1 + T_2 = \frac{\ln\left[1+\frac{\left(\rho-\frac{\ln \xi}{\tau_a}\right)}{\epsilon_2 \xi^{N_0(1-n^*)}}\right]}{\left(\rho-\frac{\ln \xi}{\tau_a}\right)(n^*-1)} + \frac{\ln\left[\frac{\epsilon_1 \xi^{-N_0(1-m^*)}}{\xi^{N_0(1-m^*)}\left(\rho-\frac{\ln \xi}{\tau_a}\right)+\epsilon_1 \xi^{-N_0(1-m^*)}}\right]}{\left(\frac{\ln \xi}{\tau_a}-\rho\right)(1-m^*)}$, for $\xi > 1$, where $\xi = \max\{\mu, \eta\} > 1$, $\rho = \min_{r\in I}\{\rho_r, \varrho_r, \sigma_r, \varsigma_r\}$, $m^* = \max_{\kappa\in S}\{m_\kappa\}$, $n^* = \max_{\kappa\in S}\{n_\kappa\}$ and N_0 is a positive integer.

Proof. The proof is similar to the proof of Theorem 4.2.3.1 for $t \neq t_l$, and therefore, this part is skipped here. Now, when $t = t_l$, we have from Eq. (4.2.10)

$$\begin{aligned}
V(t_l) &= \sum_{\upsilon=1}^{4} V_\upsilon(t_l), \\
&= \sum_{r=1}^{q} |x_r(t_l)| + \sum_{r=1}^{q} |\hat{x}_r(t_l)| + \sum_{r=1}^{q} |y_r(t_l)| + \sum_{r=1}^{q} |\hat{y}_r(t_l)| \\
&= \sum_{r=1}^{q} \mu|x_r(t_l^-)| + \sum_{r=1}^{q} \eta|\hat{x}_r(t_l^-)| + \sum_{r=1}^{q} \mu|y_r(t_l^-)| + \sum_{r=1}^{q} \eta|\hat{y}_r(t_l^-)| \\
&\leq \max\{\mu, \eta\} \sum_{\upsilon=1}^{4} V_\upsilon(t_l^-) = \xi V(t_l^-),
\end{aligned} \tag{4.2.31}$$

where $\xi = \max\{\mu, \eta\}$, and we have $\xi > 1$.

Next, the comparison system is designed to discuss the fixed-time synchronization of error system (4.2.7) under the controller (4.2.9) with desynchronizing impulses.

$$\begin{cases}
\dot{U}(t) = \begin{cases} -\epsilon_2 U^{n^*}(t) - \rho U(t), & U(t) \geq 1, t \neq t_l, \\ -\epsilon_1 U^{m^*}(t) - \rho U(t), & 0 < U(t) < 1, t \neq t_l, \\ 0, & U(t) = 0, t \neq t_l, \end{cases} \\
U(t_l) = \xi U(t_l^-),\ t = t_l, \\
U(0) = \sum_{r=1}^{q} |x_r(0)| + \sum_{r=1}^{q} |\hat{x}_r(0)| + \sum_{r=1}^{q} |y_r(0)| + \sum_{r=1}^{q} |\hat{y}_r(0)|.
\end{cases} \tag{4.2.32}$$

Combining Eq. (4.2.18), Eq. (4.2.31) and Eq. (4.2.32), we get the inequality $0 \leq V(t) \leq U(t)$. Thus, if there exists $T_{max} > 0$, for $t \geq T_{max}$ such that $U(t) \equiv 0$ then $V(t) \equiv 0$ for $t \geq T_{max}$. Thus, to obtain that the error system (4.2.7) can achieve fixed-time synchronization, one has to prove the origin of system (4.2.32) can achieve fixed-time synchronization.

For $U(t) \geq 1$, take $R(t) = U^{1-n^*}(t)$, where $n^* > 1$. From Eq. (4.2.32), we obtain that $R(t)$ tends to 1 as $U(t)$ tends to 1 and $R(t)$ tends to 0 as $U(t)$ tends to $+\infty$. Thus, we get

$$\begin{cases} \dot{R}(t) = \epsilon_2(n^* - 1) + \rho(n^* - 1)R(t), \quad t \neq t_l,\ 0 < R(t) \leq 1, \\ R(t_l) = \bar{\xi} R(t_l^-),\ t = t_l, \\ R(0) = U_0^{1-n^*}, \end{cases} \tag{4.2.33}$$

where $\bar{\xi} = \xi^{1-n^*} \in (0, 1)$. And we can get $U(t)$ tends to 1 is similar to $R(t)$ tends to 1. Now, take $R(t) = U^{1-m^*}(t)$, where $0 < m^* < 1$ for $0 < U(t) < 1$, so that we have

$$\begin{cases} R(t) = -\epsilon_1(1 - m^*) - \rho(1 - m^*)R(t), \quad t \neq t_l,\ 0 \leq R(t) < 1, \\ R(t_l) = \hat{\xi} R(t_l^-),\ t = t_l, \\ R(0) = 1, \end{cases} \tag{4.2.34}$$

where $\hat{\xi} = \xi^{1-m^*} \in (1, \infty)$. This implies that $U(t)$ tends to 0 is similar to $R(t)$ tends to 0. From above analysis, the fixed-time synchronization of Eq. (4.2.32) can be changed into two cases: (i) the system's state of (4.2.33) converges to 1 in fixed-time T_1; (ii) the system's state of (4.2.34) converges 0 from 1 in fixed-time T_2. Thus, $U(t)$ tends to 0 in fixed-time $T_{max} = T_1 + T_2$ for initial condition $U(0)$, and the impulsive strength be

$\xi > 1$. From Eq. (4.2.33) and method of variation of parameters, we have

$$R(t) = e^{\rho(n^*-1)t}\bar{\xi}^{N_\zeta(t,0)}R(0) + \epsilon_2(n^*-1)\int_0^t e^{\rho(n^*-1)(t-s)}\bar{\xi}^{N_\zeta(t,s)}ds. \tag{4.2.35}$$

From Definition 2.1.2.4, inequality (4.2.35) and $R(0) = \bar{\xi}^{N_\zeta(0,0)}R(0) < 1$, we can observe that $R(t)$ is increasing and $\lim_{t\to\infty} R(t) = \infty$, when $0 < \hat{\xi} < 1$. There exist T_1 such that $\lim_{t\to T_1} R(t) = 1$ and $0 < R(t) < 1$ for $0 < t < T_1$ that is,

$$e^{\rho(n^*-1)t}\bar{\xi}^{N_\zeta(t,0)}R(0) + \epsilon_2(n^*-1)\int_0^t e^{\rho(n^*-1)(t-s)}\bar{\xi}^{N_\zeta(t,s)}ds = 1.$$

Furthermore,

$$\epsilon_2(n^*-1)\int_0^t e^{\rho(n^*-1)(t-s)}\bar{\xi}^{N_\zeta(t,s)}ds \leq 1.$$

According to Definition 2.1.2.4, $\bar{\xi}^{\frac{t-s}{\tau_a}+N_0} \leq \bar{\xi}^{N_\zeta(t,s)} \leq \bar{\xi}^{\frac{t-s}{\tau_a}-N_0}$ so that

$$\epsilon_2(n^*-1)e^{\rho(n^*-1)t}\int_0^t e^{-\rho(n^*-1)s}\bar{\xi}^{\frac{t-s}{\tau_a}+N_0}ds \leq 1.$$

After solving above equation, we obtain

$$t \leq \frac{\ln\left[1 + \frac{\left(\rho(n^*-1)+\frac{\ln\bar{\xi}}{\tau_a}\right)}{\epsilon_2(n^*-1)\bar{\xi}^{N_0}}\right]}{\left(\rho(n^*-1) + \frac{\ln\bar{\xi}}{\tau_a}\right)} = T_1.$$

Substituting $\bar{\xi} = \xi^{1-n^*}$, we get

$$T_1 = \frac{\ln\left[1 + \frac{\left(\rho-\frac{\ln\xi}{\tau_a}\right)}{\epsilon_2\xi^{N_0(1-n^*)}}\right]}{\left(\rho - \frac{\ln\xi}{\tau_a}\right)(n^*-1)}. \tag{4.2.36}$$

Similar discussion as given above estimate the fixed-time T_2 such that the system

(4.2.34) converges to 0 from 1. Further, from Eq. (4.2.34), we have

$$R(t) = e^{-\rho(1-m^*)t}\hat{\xi}^{N_\zeta(t,0)}R(0) - \epsilon_1(1-m^*)\int_0^t e^{-\rho(1-m^*)(t-s)}\hat{\xi}^{N_\zeta(t,s)}ds. \tag{4.2.37}$$

Since $1 < \hat{\xi} < \infty$, from Definition 2.1.2.4, we have $\hat{\xi}^{\frac{t-s}{\tau_a}-N_0} \leq \hat{\xi}^{N_\zeta(t,s)} \leq \hat{\xi}^{\frac{t-s}{\tau_a}+N_0}$. Now, considering the fixed-time T_2 from $R(t)$ changes from 1 to 0. In the above equation we get $R(0) = 1$ and $R(T_2) = 0$. From Eq. (4.2.37), we therefore have

$$\begin{aligned}
R(t) &= e^{-\rho(1-m^*)t}\hat{\xi}^{N_\zeta(t,0)} - \epsilon_1(1-m^*)\int_0^t e^{-\rho(1-m^*)(t-s)}\hat{\xi}^{N_\zeta(t,s)}ds \\
&\leq e^{-\rho(1-m^*)t}\hat{\xi}^{\frac{t}{\tau_a}+N_0} - \epsilon_1(1-m^*)\int_0^t e^{-\rho(1-m^*)(t-s)}\hat{\xi}^{\frac{t-s}{\tau_a}-N_0}ds \\
&= e^{-\rho(1-m^*)t}\hat{\xi}^{\frac{t}{\tau_a}+N_0} - \epsilon_1(1-m^*)\hat{\xi}^{-N_0}\left[\frac{1-e^{-\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)t}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)}\right] \\
&= \left[\hat{\xi}^{N_0} + \frac{\epsilon_1(1-m^*)\hat{\xi}^{-N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)}\right]e^{-\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)t} - \frac{\epsilon_1(1-m^*)\hat{\xi}^{-N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)} \triangleq G(t).
\end{aligned} \tag{4.2.38}$$

Since $G(0) = \hat{\xi}^{N_0} > 0$ and $G(+\infty) = -\frac{\epsilon_1(1-m^*)\hat{\xi}^{-N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)} < 0$, we get $\dot{G}(t) < 0$ so that there exist a unique T_2 which satisfies $G(T_2) = 0$. This implies that $G(t)$ is decreasing. Since $R(0) = 1$ then $R(t) \to 0$ as $t \to T_2$. From inequality (4.2.38), we obtain that

$$G(T_2) = \left[\hat{\xi}^{N_0} + \frac{\epsilon_1(1-m^*)\hat{\xi}^{-N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)}\right]e^{-\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)T_2} - \frac{\epsilon_1(1-m^*)\hat{\xi}^{-N_0}}{\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)} = 0,$$

which implies that

$$T_2 = \frac{\ln\left[\frac{\epsilon_1(1-m^*)\hat{\xi}^{-N_0}}{\hat{\xi}^{N_0}\left(\rho(1-m^*)-\frac{\ln\hat{\xi}}{\tau_a}\right)+\epsilon_1(1-m^*)\hat{\xi}^{-N_0}}\right]}{\left(\frac{\ln\hat{\xi}}{\tau_a}-\rho(1-m^*)\right)}.$$

Substituting $\hat{\xi} = \xi^{1-m^*}$, the above inequality leads to

$$T_2 = \frac{\ln\left[\frac{\epsilon_1 \xi^{-N_0(1-m^*)}}{\xi^{N_0(1-m^*)}\left(\rho - \frac{\ln \xi_l}{\tau_a}\right) + \epsilon_1 \xi^{-N_0(1-m^*)}}\right]}{\left(\frac{\ln \xi}{\tau_a} - \rho\right)(1-m^*)}. \tag{4.2.39}$$

When $U(t) \geq 1$ and $0 < U(t) < 1$. The condition $U(t) > 1$ approaches to 1 is obviously smaller than T_1 and $0 < W(t) < 1$ from 1 approaches to 0 is definitely smaller than T_2. Therefore, the error system (4.2.7) can achieved fixed-time synchronization and settling-time function is given by $T_{max} = T_1 + T_2$. For $\xi > 1$, the drive-response systems (4.2.3) and (4.2.5) can achieved fixed-time synchronization and settling-time function is $T_{max} = T_1 + T_2$,where T_{max} is

$$T_{max} = \frac{\ln\left[1 + \frac{\left(\rho - \frac{\ln \xi}{\tau_a}\right)}{\epsilon_2 \xi^{N_0(1-n^*)}}\right]}{\left(\rho - \frac{\ln \xi}{\tau_a}\right)(n^* - 1)} + \frac{\ln\left[\frac{\epsilon_1 \xi^{-N_0(1-m^*)}}{\xi^{N_0(1-m^*)}\left(\rho - \frac{\ln \xi_l}{\tau_a}\right) + \epsilon_1 \xi^{-N_0(1-m^*)}}\right]}{\left(\frac{\ln \xi}{\tau_a} - \rho\right)(1-m^*)}. \tag{4.2.40}$$

Remark 4.2.3.6. The sufficient conditions guaranteeing the fixed-time synchronization of CVINNs in Theorem 4.2.3.2 are established for the case of desynchronizing impulses. The estimated settling-time function depends on the class of impulsive sequence with the condition $\rho > \frac{\ln \xi}{\tau_a}$. In addition, the condition states that the impulsive perturbation to the system must be separated by the lower bound of τ_a for the drive-response systems (4.2.3) and (4.2.5) to be fixed-time synchronization. However, if the condition $\rho > \frac{\ln \xi}{\tau_a}$ does not hold, then the impulsive perturbation destroys the system's stability which is verified numerically in the numerical and simulations section of this subchapter.

Remark 4.2.3.7. Theorem 4.2.3.2 in this study is an extension of previous existing articles [88, 95, 111] in which only stabilizing impulses have been considered. According to Theorem 4.2.3.2, the study identifies that the condition $\rho > \frac{\ln \xi}{\tau_a}$ is sufficient for desynchronizing impulses. This condition plays a crucial role in ensuring fixed-time synchronization of CVINN.

4.2.4 Numerical Simulations and Discussions

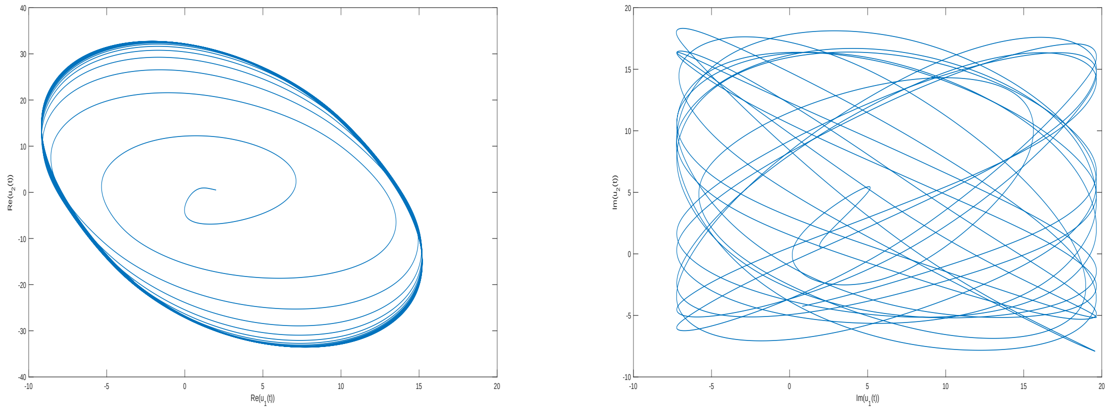
In this subsection, two numerical examples have been provided to verify the key theoretical results as given in previous sections.

Example 4.2.4.1. Consider the following two-dimensional system of model (4.2.1) with the parameters as

$$a_1 = 0.35, a_2 = 0.15, b_1 = 0.45, b_2 = 0.35, c_{11} = 0.45 - i, c_{12} = -1.4 + i, c_{21} = 1 + 1.7i, c_{22} = 0.56 - i,$$
$$d_{11} = 2.5 - i, d_{12} = 1 + 2.5i, d_{21} = -1.5 - i, d_{22} = 2.5 + i.$$

The complex-valued activation function can be selected as $f_s(u_s(t)) = g_s(u_s(t)) = \frac{1-e^{-\mathrm{Re}(u_s(t))}}{1+e^{-\mathrm{Re}(u_s(t))}} + i\frac{1}{1+e^{-\mathrm{Im}(u_s(t))}}$, where $u_s(t) = x_s(t) + iy_s(t)$ $(s = 1, 2)$ with $x_s(t)$, $y_s(t) \in \mathbb{R}$. It is easy to verify that the Assumption 4.2.2.1 hold with the Lipschitz constants $l_s^{RR} = p_s^{RR} = 0.5$, $l_s^{LL} = p_s^{II} = 0.25$ and $l_s^{RI} = l_s^{IR} = p_s^{RI} = p_s^{IR} = 0$. Let us choose $\alpha_1 = 1.3$, $\alpha_2 = 1.6$, $\beta_1 = 0.7$, $\beta_2 = 0.9$ and the time delays are taken as $\tau_s(t) = 1.1|\sin(t)|$. Here, the initial value of CVINNs are selected as $u_1 = 2+i$, $\dot{u}_1 = -0.4+0.4i$, $u_2 = 1.5+0.5i$, $\dot{u}_2 = 0.5 + 0.3i$, $s \in [-1, 0]$. Then, the phase plot and state trajectories of variables $u_1(t)$ and $u_2(t)$ without impulsive effects of CVINNs (4.2.1) is obtained and respectively shown in Fig. 4.2.1.

For numerical verification, the initial value of drive-response systems (4.2.3) and (4.2.5) are respectively taken as $\phi_1(m) = 2+i$, $\phi_2(m) = -0.4+0.4i$, $\psi_1(m) = 1.5+i$, $\psi_2(m) =$



(a) (b)

Figure 4.2.1: The phase plot of complex-valued inertial neural networks (4.2.1) without impulsive effects.

$-0.45+0.6i$ and $\widetilde{\phi}_1(m) = -0.5+0.2i$, $\widetilde{\phi}_2(m) = -0.45+0.25i$, $\widetilde{\psi}_1(m) = 0.4+0.35i$, $\widetilde{\psi}_2(m) = 0.5+0.60i$, where $m \in [-1.1, 0]$. The state trajectories of drive-response systems (4.2.3) and (4.2.5) without impulses and without controller are depicted in Fig. 4.2.2. From figure it is clear that the synchronization can not be achieved without the designed controller (4.2.9).

Based on Theorem 4.2.3.1, by simple calculation the control gains given in Eq. (4.2.10) should satisfy

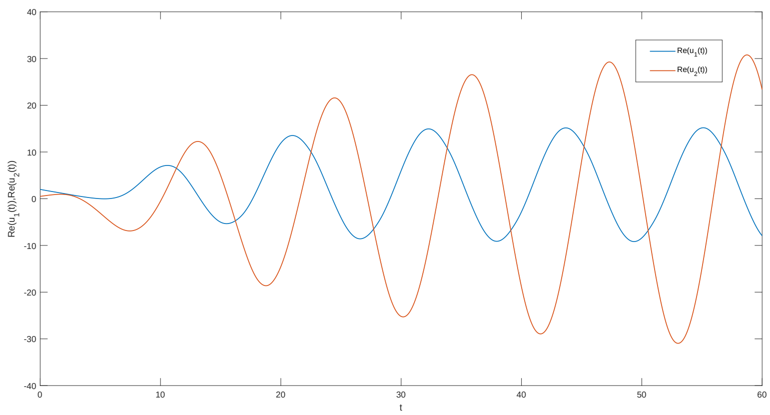
$$\iota_{11} \geq -\frac{\beta_1}{\alpha_1} + |\gamma_1| + \sum_{s=1}^{2} |\alpha_s||c^R_{1s}|l^{RR}_1 + \sum_{q=1}^{2} |\alpha_s||c^I_{1s}|l^{IR}_1 + \sum_{s=1}^{2} |\alpha_s||c^R_{1s}|l^{IR}_1 + \sum_{s=1}^{2} |\alpha_s||c^I_{1s}|l^{RR}_1 = 3.03,$$

$$\iota_{12} \geq -\frac{\beta_2}{\alpha_2} + |\gamma_2| + \sum_{s=1}^{2} |\alpha_s||c^R_{2s}|l^{RR}_2 + \sum_{s=1}^{2} |\alpha_s||c^I_{2s}|l^{IR}_2 + \sum_{s=1}^{2} |\alpha_s||c^R_{2s}|l^{IR}_2 + \sum_{s=1}^{2} |\alpha_s||c^I_{2s}|l^{RR}_2 = 3.26,$$

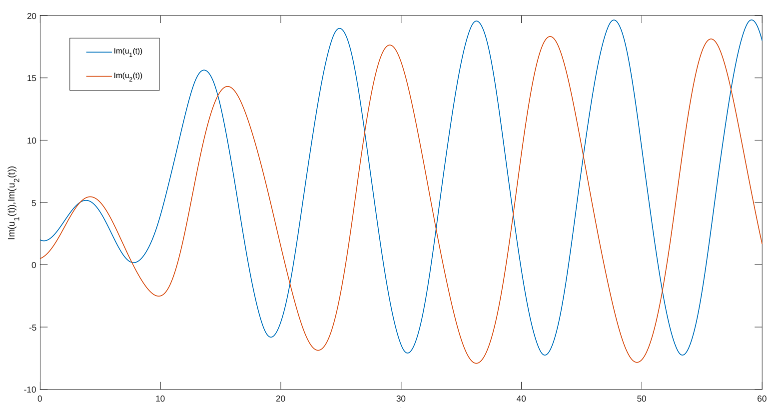
$$\iota_{21} \geq -\frac{\beta_1}{\alpha_1} + |\gamma_1| + \sum_{s=1}^{2} |\alpha_s||c^R_{1s}|l^{RI}_1 + \sum_{s=1}^{2} |\alpha_s||c^I_{1s}|l^{II}_1 + \sum_{s=1}^{2} |\alpha_s||c^R_{1s}|l^{II}_1 + \sum_{s=1}^{2} |\alpha_s||c^I_{1s}|l^{RI}_1 = 1.6,$$

$$\iota_{22} \geq -\frac{\beta_2}{\alpha_2} + |\gamma_2| + \sum_{s=1}^{2} |\alpha_s||c^R_{2s}|l^{RI}_2 + \sum_{s=1}^{2} |\alpha_s||c^I_{2s}|l^{II}_2 + \sum_{s=1}^{2} |\alpha_s||c^R_{2s}|l^{II}_2 + \sum_{s=1}^{2} |\alpha_s||c^I_{2s}|l^{RI}_2 = 2.79.$$

$$\theta_{11} = \theta_{31} \geq \frac{1}{|\alpha_1|} - \pi_1 = 0.95,$$
$$\theta_{12} = \theta_{32} \geq \frac{1}{|\alpha_2|} - \pi_2 = 1.04.$$



(a)



(b)

Figure 4.2.2: State trajectories of complex-valued inertial neural networks (4.2.1) without impulsive effects and without design control (4.2.9).

$$\theta_{21} \geq \sum_{r=1}^{2} |\alpha_1|(|d_{r1}^R|p_r^{RR} + |d_{r1}^I|p_r^{IR} + |d_{r1}^R|p_r^{IR} + |d_{r1}^I|p_r^{RR}) = 4.55,$$
$$\theta_{22} \geq \sum_{r=1}^{2} |\alpha_2|(|d_{r2}^R|p_r^{RR} + |d_{r2}^I|p_r^{IR} + |d_{r2}^R|p_r^{IR} + |d_{r2}^I|p_r^{RR}) = 4.8,$$
$$\theta_{41} \geq \sum_{r=1}^{2} |\alpha_1|(|d_{r1}^R|p_r^{RI} + |d_{r1}^I|p_r^{IR} + |d_{r1}^R|p_r^{II} + |d_{r1}^I|p_r^{RI}) = 1.46,$$
$$\theta_{42} \geq \sum_{r=1}^{2} |\alpha_2|(|d_{r2}^R|p_r^{RI} + |d_{r2}^I|p_r^{IR} + |d_{r2}^R|p_r^{II} + |d_{r2}^I|p_r^{RI}) = 1.6.$$

Then the control gains of control (4.2.9) are chosen as $\theta_{11} = \theta_{31} = 1.2$, $\theta_{12} = \theta_{32} = 1.5$, $\theta_{21} = 4.8$, $\theta_{22} = 5$, $\theta_{41} = 1.6$, $\theta_{42} = 1.8$, $\iota_{11} = 3.2$, $\iota_{12} = 3.5$, $\iota_{21} = 2$, $\iota_{22} = 3$, $\lambda_{11} = 3.4$, $\lambda_{12} = \lambda_{32} = 3.5$, $\lambda_{22} = \lambda_{31} = 3.6$, $\lambda_{41} = \lambda_{42} = 4.4$, $\lambda_{21} = 4.6$, $\delta_{11} = \delta_{21} = \delta_{31} = 3.8$, $\delta_{12} = \delta_{32} = 2.8$, $\delta_{41} = \delta_{42} = 4.4$, $\delta_{22} = 3.8$. Thus, all the conditions (4.2.10) given in Theorem 4.2.3.1 are satisfied. Further, some parameters are selected as $\rho_1 = 1.24$, $\rho_2 = 1.36$, $\varrho_1 = 1.46$, $\varrho_2 = 1.33$, $\sigma_1 = 0.25$, $\sigma_2 = 0.46$, $\varsigma_1 = 3.8$, $\varsigma_2 = 3.97$, $m_1 = 0.4$, $m_2 = 0.5$, $m_3 = 0.7$, $m_4 = 0.8$, $n_1 = 1.3$, $n_2 = 1.5$, $n_3 = 1.7$, $n_4 = 1.9$, then we have $m^* = 0.8$, $n^* = 1.9$, $\epsilon_1 = 1.35$, $\epsilon_2 = 0.39$ and $\rho = 0.25$. The impulsive sequence is taken as $\{t_l, l \in \mathbb{N}\}$ which is chosen from [106], to assume the following constants: average impulsive interval is $\tau_a = 0.8$, positive integer $N_0 = 2$, $\varepsilon = 0.5$. For the case of synchronizing impulses, the impulsive strength becomes $\mu = 0.5$, $\eta = 0.7$ then we have $\xi = 0.7$. Therefore, the drive-response systems (4.2.3) and (4.2.5) yield fixed-time synchronization under the control (4.2.9) for the case of synchronizing impulsive effects and the estimated settling-time function is bounded by $T_{max} = 6.11$. The state trajectories of drive-response systems (4.2.3) and (4.2.5) with impulsive effects are shown in Fig. 4.2.3 and Fig. 4.2.5. Further, the evolutions of error system (4.2.7) with impulsive effects are depicted in Fig. 4.2.4

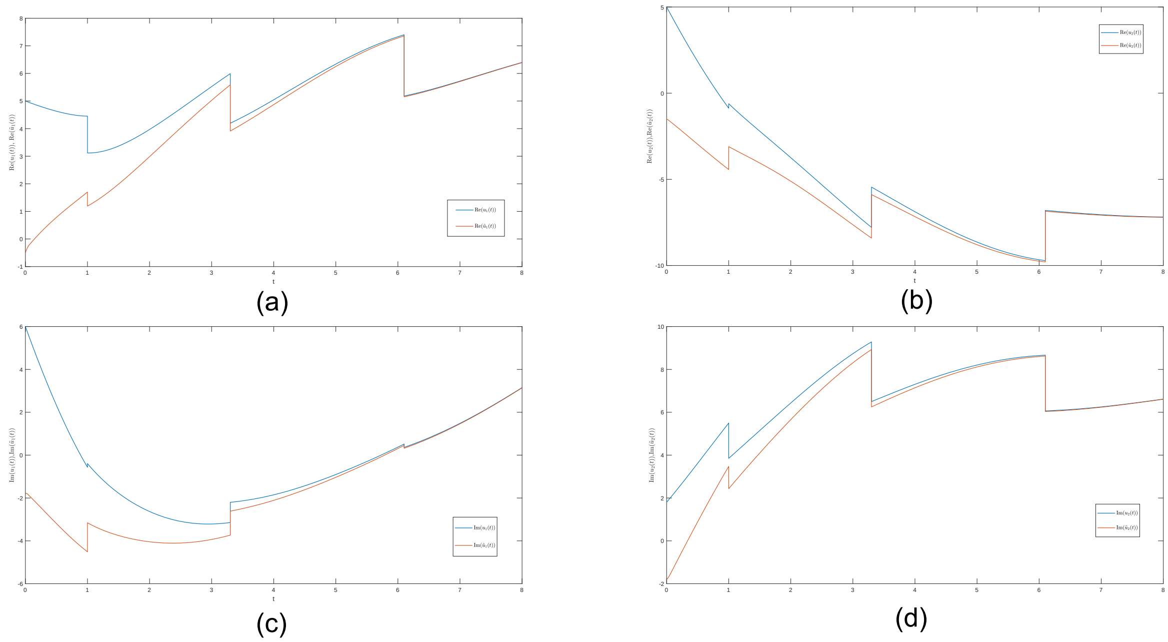


Figure 4.2.3: State trajectories of derive-response system (4.2.3) and (4.2.5) with impulsive effects and design control (4.2.9).

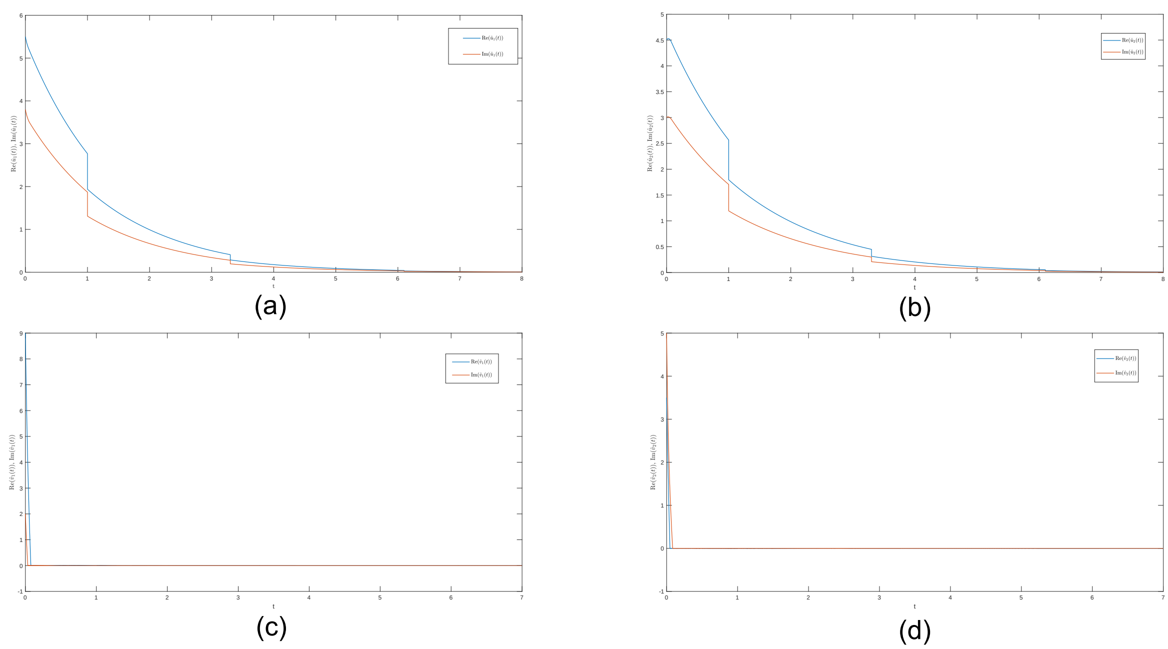


Figure 4.2.4: State trajectories of error system (4.2.7) with stabilizing impulsive effects and design control (4.2.9).

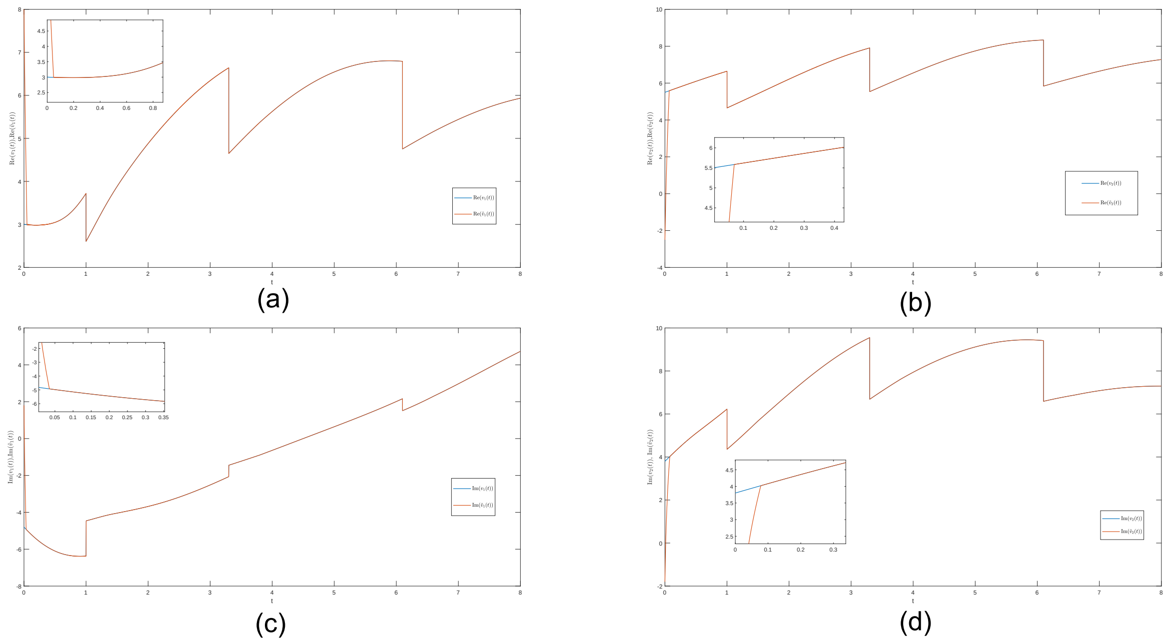


Figure 4.2.5: State trajectories of derive-response system (4.2.3) and (4.2.5) with impulsive effects and design control (4.2.9).

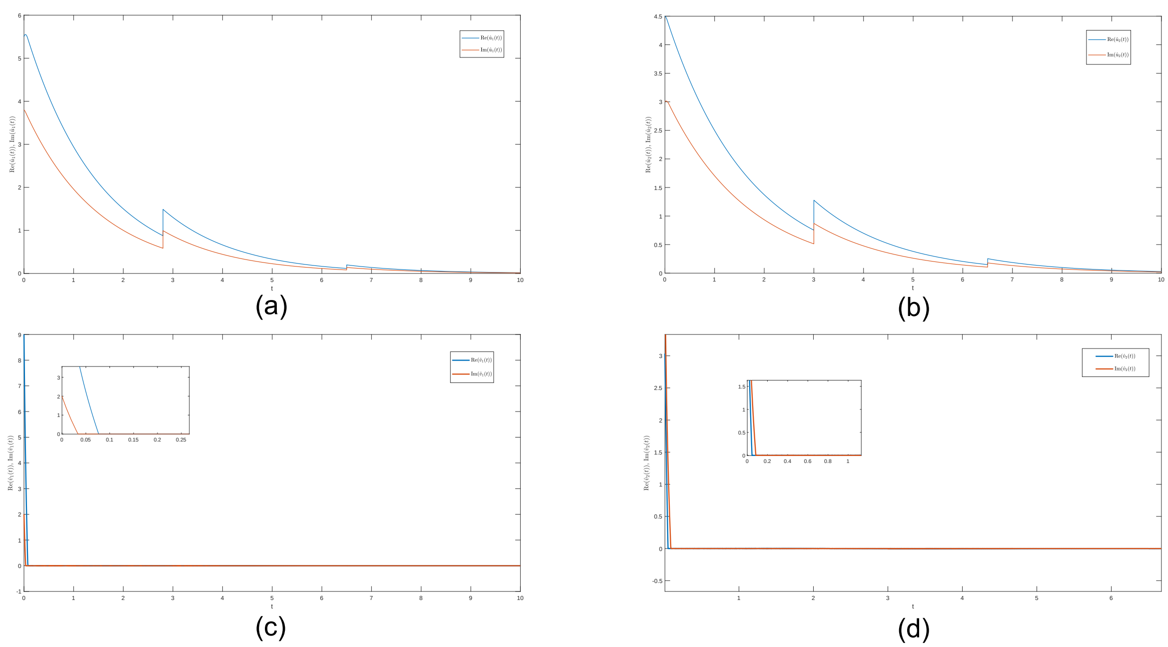
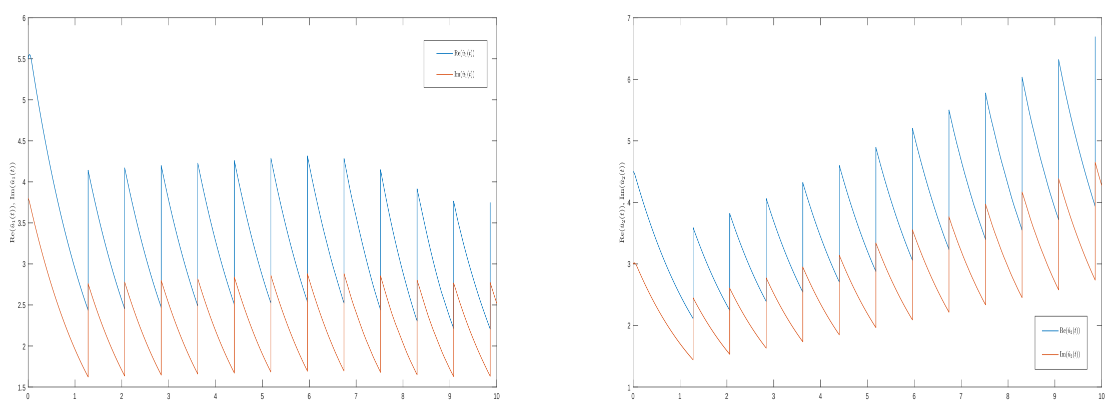


Figure 4.2.6: State trajectories of error system (4.2.7) with design control (4.2.9) and $\tau_a = 2 > 1.88$.

Example 4.2.4.2. Let us consider the same CVINNs systems as shown in Example 4.2.4.1 to verify the results of Theorem 4.2.3.2. Now, we have taken similar type of control gain parameters as defined in Example 4.2.4.1 such that it satisfies all the conditions (4.2.9) of Theorem 4.2.3.2. Let us choose other parameters as $n^* = 0.8$, $m^* = 1.9$, $\epsilon_1 = 1.35$, $\epsilon_2 = 0.39$, $\rho = 0.25$ and the impulsive strength is taken as $\mu = 1.3$, $\eta = 1.6$, then we have $\xi = 1.6$, which means that impulses are desynchronizing impulses. Now, we calculate $\frac{\ln \xi}{\rho} = 1.88$, and choose an average impulsive interval $\tau_a = 2$ and positive number $N_0 = 2$, $\varepsilon = 0.6$ from the impulsive sequence $\{t_l, l \in \mathbb{N}\}$, which is taken from [106]. This implies that the sufficient condition $\rho > \frac{\ln \xi}{\tau_a}$ in Theorem 4.2.3.2 is satisfied and the drive-response systems (4.2.3) and (4.2.5) with the impulsive effects under the design control (4.2.9) is fixed-time synchronization and the estimated settling-time function is bounded by $T_{max} = 8.67$ based on the selected parameters. The evolutions of error system (4.2.7) are shown in Fig. 4.2.6. However, if the sufficient condition $\rho > \frac{\ln \xi}{\tau_a}$ given in Theorem 4.2.3.2 is not satisfied then there exists a $\tau_a = 0.75$ so that the state trajectories of drive-response systems (4.2.3) and (4.2.5) will not be fixed-time synchronization. Let the average impulsive interval is chosen as $\tau_a = 0.75$ then it is clear from the Fig. 4.2.7 that the system's state trajectories are not stable. Hence, the effectiveness of sufficient condition have been verified to achieve the fixed-time synchronization of CVINNs for the case of desynchronizing impulsive effects numerically.



(a) (b)

Figure 4.2.7: State trajectories of error system (4.2.7) with design control (4.2.9) and $\tau_a = 0.75 < 1.88$

4.2.5 Conclusion

This subchapter discusses the problem of fixed-time synchronization for complex-valued impulsive neural networks (CVINNs) with synchronizing and desynchronizing impulsive effects. Here, the second-order CVINN is first reduced to a first-order complex-valued system using variable substitution and then divide it into two equivalent real-valued subsystems. Then, the sufficient conditions have been proposed for achieving fixed-time synchronization of CVINNs by using the comparison principle and the average impulsive interval. For desynchronizing impulses, it is shown that the condition $\rho > \frac{\ln \xi}{\tau_a}$ is required to achieve fixed-time synchronization of CVINNs. The settling-time function obtained in this work depends on the impulsive sequence as well as on the parameters of the continuous-time subsystems. Finally, two numerical examples are considered to demonstrate the effectiveness of the proposed theoretical results. In future work, it will be interesting to investigate fixed-time synchronization of time-delayed impulsive systems with delayed impulses.