Chapter 5

Applications

In this chapter, as an application of the results obtained in the thesis, we will describe the conditions under which the quantum translates of a non-zero Schatten class operator are linearly independent, and we will prove an analog of the Fourier restriction theorem for the Fourier-Wigner transform.

5.1 Quantum translates

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. If $x \in \mathbb{R}^n$, the translate of f by x is the function $T_x f : \mathbb{R}^n \to \mathbb{C}$ given by

$$T_x f(y) = f(x+y), \quad y \in \mathbb{R}^n.$$

In [8, 43], Edgar and Rosenblatt considered the set of translates of a non-zero function in $L^p(\mathbb{R}^n)$ by distinct elements of $\mathbb{R}^n$, and gave conditions under which this set is linearly independent. They proved that any non-zero function in $L^p(\mathbb{R})$,$1 \leq$

$p < \infty$, has linearly independent translates. They also proved that if $n \geq 2$ and $1 \leq p < 2n/(n-1)$, then a non-zero function in $L^p(\mathbb{R}^n)$ has linearly independent translates (see [8, Corollary 2.7]) by proving the following more general result [8, Theorem 2.6].

Theorem 5.1.1. *If $S \subseteq \mathbb{R}^n$ is a closed subset of Hausdorff dimension no larger than $n-1$, and $T \neq 0$ is a tempered distribution with $\operatorname{supp}(T) \subseteq S$ such that $\widehat{T} \in L^p(\mathbb{R}^n)$, then $p \geq 2n/(n-1)$.*

Moreover, they proved that there exists a function in $L^p(\mathbb{R}^n)$, $n \geq 2$, that has linearly dependent translates when $p > 2n/(n-1)$ (see [8, Example 2.4]). Furthermore, in [43], Rosenblatt proved that a non-zero function in $L^p(\mathbb{R}^n)$, $p = 2n/(n-1)$, has linearly independent translates.

In this section we will prove quantum analogs of these results. Recall that there is an action of $\mathbb{R}^{2n}$ on $S^p(\mathcal{H})$ called quantum translation which is defined by

$$(x_1, y_1) \cdot X = \rho(x_1, y_1, 1) X \rho(x_1, y_1, 1)^{-1}.$$

Observe that for any finite measure λ on $\mathbb{R}^{2n}$,

$$(x_1, y_1) \cdot W(\lambda) = W(e((x_1, y_1), \cdot)\lambda), \tag{5.1}$$

where $e : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ is the map

$$e((x', y'), (x, y)) = e^{2\pi i(x' \cdot y - y' \cdot x)}.$$

By the Plancherel theorem for W, this relation continues to hold when λ is replaced by a function $f \in L^2(\mathbb{R}^{2n})$.

Let $(x_1, y_1), \ldots, (x_k, y_k)$ be distinct elements of $\mathbb{R}^{2n}$, and let $c_1, \ldots, c_k$ be nonzero scalars. Define the difference operator D on $S^p(\mathcal{H})$, $1 \le p \le \infty$, by

$$DA = c_1(x_1, y_1) \cdot A + \cdots + c_k(x_k, y_k) \cdot A, \quad A \in S^p(\mathcal{H}).$$

The trigonometric polynomial

$$c_1 e\left((x_1, y_1), (x, y)\right) + \cdots + c_k e\left((x_k, y_k), (x, y)\right)$$

is called the characteristic trigonometric polynomial of the difference equation $DA = 0$.

Let $A \in S^2(\mathcal{H})$ be non-zero, and suppose $DA = 0$. By the Plancherel theorem for W, we may write $A = W(f)$ for some nonzero $f \in L^2(\mathbb{R}^{2n})$. By equation (5.1),

$$0 = DA = W\left((c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot))f\right).$$

Therefore $(c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot))f = 0$. Since f is a non-zero function in $L^2(\mathbb{R}^{2n})$, it follows that the analytic function $c_1 e((x_1, y_1), (x, y)) + \cdots + c_k e((x_k, y_k), (x, y))$ vanishes on a set of positive measure, and thus it is identically zero. Since the set

$$\{e((x_1, y_1), \cdot), \ldots, e((x_k, y_k), \cdot)\}$$

is linearly independent, it follows that $c_1 = \cdots = c_k = 0$. Therefore the quantum translates of A are linearly independent.

Since $S^p(\mathcal{H}) \subseteq S^2(\mathcal{H})$ for $1 \le p \le 2$, it follows that the quantum translates of a non-zero operator in $S^p(\mathcal{H})$, $1 \le p \le 2$, are linearly independent. By Theorem 4.1.2, Theorem 5.1.1, and the fact that the zero set of a trigonometric polynomial on $\mathbb{R}^{2n}$

is of Hausdorff dimension strictly less than $2n$ (see e.g., [66]), we can conclude the following.

Theorem 5.1.2. *Let $A \in S^p(\mathcal{H})$, $1 \leq p < 4n/(2n-1)$, be a non-zero operator and let $(x_1, y_1), \ldots, (x_k, y_k)$ be distinct elements of $\mathbb{R}^{2n}$. Then $\{(x_1, y_1) \cdot A, \ldots, (x_k, y_k) \cdot A\}$ is a linearly independent set.*

We will now find an explicit example of a non-zero operator $A \in S^p(\mathcal{H})$, $p > 4n/(2n-1)$, whose quantum translates are linearly dependent. The argument used here is similar to the one given by Edgar and Rosenblatt in [8, Example 2.4].

Theorem 5.1.3. *There exists a non-zero operator $A \in S^p(\mathcal{H})$, $p > 4n/(2n-1)$ and distinct elements $(x_1, y_1), \ldots, (x_{4n+1}, y_{4n+1}) \in \mathbb{R}^{2n}$ such that*

$$\{(x_1, y_1) \cdot A, \ldots, (x_{4n+1}, y_{4n+1}) \cdot A\}$$

is a linearly dependent set.

Proof. Consider the difference equation given by

$$2(2n-1)A = \sum_{j=1}^{n} \left[(e_j, \mathbf{0}) \cdot A + (-e_j, \mathbf{0}) \cdot A + (\mathbf{0}, e_j) \cdot A + (\mathbf{0}, -e_j) \cdot A\right], \qquad (5.2)$$

where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Then the characteristic trigonometric polynomial

$$p(x_1, \ldots, x_n, y_1, \ldots, y_n) = 2(2n-1) - 2\sum_{j=1}^{n} (\cos(2\pi x_j) + \cos(2\pi y_j))$$

has a zero set which is a disjoint union of compact $(2n-1)$-dimensional surfaces of positive Gaussian curvature. Let S_{2n} be the connected component of the zero set containing the points with all coordinates zero except for one which is $\pm 1/4$. Let σ be the measure on S_{2n} induced by the Lebesgue measure on $\mathbb{R}^{2n}$. Then by Theorem 4.5.1, $W(\sigma) \in S^p(\mathcal{H})$ for $p > 4n/(2n-1)$.

It follows from equation (5.1) that $A = W(\sigma)$ is a non-zero solution of the difference equation (5.2). Hence, for $p > 4n/(2n-1)$, there exists an operator in $S^p(\mathcal{H})$ whose quantum translates are linearly dependent. □

The question of what happens if we consider the quantum translates of a non-zero operator in $S^p(\mathcal{H})$ for $p = 4n/(2n-1)$ is still unanswered.

5.2 Restriction theorem

The restriction problem for the Fourier transform was introduced by Stein in the 1970s. For a given $p \in [1, 2]$ and a subset S of $\mathbb{R}^n$, the question is whether it is possible to meaningfully restrict the Fourier transform of a function in $L^p(\mathbb{R}^n)$ to the set S. If S has a positive Lebesgue measure, then the answer is yes, since $\widehat{f}$ lies in $L^{p'}(\mathbb{R}^n)$ and therefore has a meaningful restriction to S. So, the question is what happens if S has measure zero. If $p = 1$, then $\widehat{f}$ is a continuous function on $\mathbb{R}^n$, and therefore has a meaningful restriction to S.

Suppose $M \subseteq \mathbb{R}^n$, $n \geq 2$, is a smooth submanifold. Let σ denote the measure on M induced by the Lebesgue measure on $\mathbb{R}^n$. Then M is said to have the L^p *restriction property* if there exist $q = q(p)$ and $C > 0$ such that the inequality

$$\left(\int_{M_0} \left| \widehat{f}(\xi) \right|^q d\sigma(\xi) \right)^{1/q} \leq C \left\| f \right\|_p$$

holds for all $f \in \mathcal{S}(\mathbb{R}^n)$, whenever M_0 is an open subset of M with compact closure in M.

A celebrated result by Tomas [58] and Stein [35] is that the unit sphere in $\mathbb{R}^n$ has the L^p restriction property with $q = 2$ if $1 \leq p \leq \frac{2n+2}{n+3}$. Generalizing this, Stein proved the following result (see [49, p352]).

Theorem 5.2.1. *Suppose M is a smooth submanifold of $\mathbb{R}^n$ of type k. Then M has the L^p restriction property with $q = 2$, and $1 \leq p \leq \frac{2nk}{2nk-1}$.*

As an application of Theorem 4.1.2, we prove the following result, which is an analog of Theorem 5.2.1 for the Fourier-Wigner transform.

Theorem 5.2.2. *Suppose M is a smooth submanifold of $\mathbb{R}^{2n}$ of type k. Let σ denote the measure on M induced by the Lebesgue measure on $\mathbb{R}^{2n}$. If $X \in \mathcal{S}(\mathcal{H})$ and $1 \leq p \leq \frac{4nk}{4nk-1}$, then*

$$\left(\int_{M_0} |\alpha(X)(x,y)|^2 \, d\sigma(x,y) \right)^{1/2} \leq C \, \|X\|_{S^p} \, ,$$

where M_0 is an open subset of M with compact closure in M.

Proof. Let η be a compactly supported smooth function on $\mathbb{R}^{2n}$ such that $\eta = 1$ on M. It suffices to show that if $X \in \mathcal{S}(\mathcal{H})$, then

$$\int_M |\alpha(X)(x,y)|^2 \, (\eta(x,y))^2 \, d\sigma(x,y) \leq C \, \|X\|_{S^p}^2 \, .$$

Let $X \in \mathcal{S}(\mathcal{H})$. Observe that

$$\alpha(X)(x,y)\eta(x,y) = \beta(X)(x,y) \left(\eta(x,y) e^{\frac{\pi}{2}(|x|^2+|y|^2)} \right) ,$$

where $\beta(X)(x,y) = \alpha(X)(x,y)e^{-\frac{\pi}{2}(|x|^2+|y|^2)}$. Let $\psi(x,y) = \eta(x,y)e^{\frac{\pi}{2}(|x|^2+|y|^2)}$. By Lemma 4.4.1, we know that if $1 \leq p \leq 2$, then $\breve{\beta}(X) \in L^p(\mathbb{R}^{2n})$, and

$$\left\|\breve{\beta}(X)\right\|_p \leq \|X\|_{S^p}.$$

Therefore $\beta(X)$ is the symplectic Fourier transform of a function in $L^p(\mathbb{R}^{2n})$. Since the symplectic Fourier transform is just a rotation of the Fourier transform, it follows from Theorem 5.2.1 that

$$\int_M |\beta(X)(x,y)|^2 \, d\sigma(x,y) \leq C \left\|\breve{\beta}(X)\right\|_p^2,$$

provided that $1 \leq p \leq \frac{4nk}{4nk-1}$.

Therefore if $1 \leq p \leq \frac{4nk}{4nk-1}$, then

$$\begin{aligned}
\int_M |\alpha(X)(x,y)|^2 (\eta(x,y))^2 \, d\sigma(x,y) &= \int_M |\beta(X)(x,y)|^2 (\psi(x,y))^2 \, d\sigma(x,y) \\
&\leq \|\psi\|_\infty^2 \int_M |\beta(X)(x,y)|^2 \, d\sigma(x,y) \\
&\leq C \left\|\breve{\beta}(X)\right\|_p^2 \\
&\leq C \|X\|_{S^p}^2 .
\end{aligned}$$

□