Chapter 3

Epsilon-subdifferentiability for interval-valued functions and its application in interval optimization problems

3.1 Introduction

In this chapter, the concept of approximate subdifferential for interval-valued functions is defined. The notion of approximate subdifferential or ϵ-subdifferential has been introduced by Rockafellar and Brønsted [33] to deal with the nonsmooth optimization problems. In [30], calculus for ϵ-subdifferential has been presented in which existence, closedness, convexity, etc. of ϵ-subdifferential set have been given. Ioffe [113] related the concept of approximate subdifferential to directional and Clarke derivatives. The notion of ϵ-directional derivative has been given in [151], in which a connection between ϵ-subdifferential and ϵ-directional derivative of a convex function is also established. The geometrical significance of ϵ-directional derivative and optimality conditions using it are provided in [105]. Volle [189] introduced a global approximate minimum for optimization problems and gave optimality conditions using these approximate minima and approximate subdifferential. In [90], an approximate solution using ϵ-subdifferentiability has been obtained for multiobjective optimization problems. A supremum rule has been presented in [101] to find the ϵ-subdifferential set of the pointwise supremum of an arbitrary family of convex functions. Apart from these, many theories and concepts have been proposed using ϵ-subdifferentials, for instance, see [16, 51, 80, 83, 89, 158] and their references.

3.2 Motivation

In Chapter 2, the concepts of gH-subgradient and gH-subdifferentiability have been introduced. In this chapter, the nonemptiness of gH-subdifferential set on $\text{int}(\text{dom}(\mathbf{F}))$ has been given. However, it can be observed that gH-subdifferential set of a convex IVF $\mathbf{F}$ at $\bar{x}$ may be empty even when $\bar{x} \in \text{dom}(\mathbf{F})$ (see Remark 3.3). We thus study a notion of gH_ϵ-subdifferentiability for convex IVFs because gH_ϵ-subdifferential set is always nonempty on $\text{dom}(\mathbf{F})$ (see Remark 3.6). The prime motivation behind the study of gH_ϵ-subdifferential is that there are some IOP whose exact efficient solutions cannot be found; for instance, consider the very simple IOP

$$\min_{x \in \mathbb{R}} e^x \odot \mathbf{T}, \tag{3.1}$$

where $\mathbf{0} = [0, 0] \prec \mathbf{T}$. Clearly, the infimum of the IOP (3.1) is $[0, 0]$. This infimum cannot be attained on $\mathbb{R}$. To our knowledge, there is no optimality condition or theory to handle such IOPs. In such type of situations in conventional optimization, when exact minima can not be found, one attempts to find the approximate minima, and it is well-known that ϵ-subdifferentiability is helpful to give approximate solutions [189] in conventional optimization. Moreover, using ϵ-subdifferentiability, characterizations of global optimality conditions are also given for general convex optimization problems involving convex inequality constraints [122]. Approximate subdifferentials are helpful for solving stochastic programs by nonlinear programming techniques [27]. Besides these, ϵ-subdifferentials are also useful in the study of differential stability of convex optimization problems with possibly empty solution sets, see [5]. It has many applications in different areas, including numerical analysis, stochastic theory, and control theory, etc., for instance, see [30, 105, 151]. Therefore, inspired by these reasons, we introduce the concept of ϵ-subdifferentiability for convex IVFs using gH-difference.

3.3 Contributions

The major contributions in this chapter are as follows:

- The notions of ϵ-directional derivative and ϵ-subdifferentiability for convex IVFs are defined.
- Some necessary properties such as nonemptiness, convexity, closedness, and boundedness of gH_ϵ-subdifferential set are derived.
- The definition of an approximate solution of an IOP is given.
- A few necessary and sufficient optimality conditions are proposed to find an approximate solution for IOPs.
- Theoretical study on finding an approximate solution of an interval minimax optimization problem is provided.

3.4 gH_ϵ-subdifferentiability and its properties

In this section, we propose the concepts of gH_ϵ-directional derivative and gH_ϵ-subdiffer entiability for convex IVFs along with some necessary characterizations of these two notions.

Definition 3.1 (gH_ϵ-directional derivative). *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\epsilon > 0$. The ϵ-directional derivative or gH_ϵ-directional derivative of $\boldsymbol{F}$ at $\bar{x} \in \mathbb{R}^n$ in the direction of $d \in \mathbb{R}^n$ is defined by*

$$\boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) = \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{x} + \lambda d) \ominus_{gH} \boldsymbol{F}(\bar{x}) \oplus \epsilon \right),$$

provided the limit exists.

We consider $\epsilon > 0$ throughout this chapter to avoid writing $\epsilon > 0$ again and again. Next, we have an example of gH_ϵ-directional derivative of an IVF.

Example 3.1 *In this example, we find the gH_ϵ-directional derivative of the indicator function of the unit ball $\mathbb{B}$ of $\mathbb{R}^2$ at $\bar{x} = (0, -1)$. Let $\boldsymbol{F}$ be the indicator IVF of $\mathbb{B}$, i.e.,*

$$\boldsymbol{F}(x) = \delta_{\mathbb{B}}(x) = \begin{cases} \boldsymbol{0}, & \text{if } x \in \mathbb{B} \\ [+\infty, +\infty], & \text{if } z \notin \mathbb{B}, \end{cases}$$

where $\boldsymbol{0} = [0, 0]$. To find the gH_ϵ-directional derivative of $\boldsymbol{F}$ at $\bar{x} = (0, -1)$ in the direction $d = (d_1, d_2) \neq (0, 0) \in \mathbb{R}^2$, we calculate

$$\begin{aligned} \boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) &= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{x} + \lambda d) \ominus_{gH} \boldsymbol{F}(\bar{x}) \oplus \epsilon \right), \text{ provided the limit exists} \\ &= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\boldsymbol{F}((0, -1) + \lambda(d_1, d_2)) \ominus_{gH} \boldsymbol{F}(0, -1) \oplus \epsilon \right) \\ &= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\boldsymbol{F}(\lambda d_1, \lambda d_2 - 1) \ominus_{gH} \boldsymbol{F}(0, -1) \oplus \epsilon \right). \end{aligned} \tag{3.2}$$

In (3.2), the value of $F(\lambda d_1, \lambda d_2 - 1)$ will depend on whether $(\lambda d_1, \lambda d_2 - 1)$ belongs to $\mathbb{B}$ or not. Therefore, we have two following cases.

- *Case 1. Let $(\lambda d_1, \lambda d_2 - 1) \in \mathbb{B}$. In this case, we claim that $d_2 \neq 0$. The reason is as follows. If $d_2 = 0$ and $(\lambda d_1, \lambda d_2 - 1) = (\lambda d_1, -1) \in \mathbb{B}$, then $(\lambda d_1)^2 = 0$ which is not possible as $d = (d_1, d_2) \neq (0, 0)$. Therefore, for $(\lambda d_1, \lambda d_2 - 1) \in \mathbb{B}$ and*

$d_2 \neq 0$, we have

$$\begin{aligned}
&(\lambda d_1)^2 + (\lambda d_2 - 1)^2 = 1\\
or,\ &\lambda^2 d_1^2 + \lambda^2 d_2^2 - 2\lambda d_2 + 1 = 1\\
or,\ &\lambda^2 d_1^2 + \lambda^2 d_2^2 = 2\lambda d_2\\
or,\ &\tfrac{d_1^2+d_2^2}{2d_2} = \tfrac{1}{\lambda}\\
or,\ &\tfrac{(d_1^2+d_2^2)\epsilon}{2d_2} = \tfrac{\epsilon}{\lambda}\ because\ \epsilon > 0.
\end{aligned} \tag{3.3}$$

From (3.2), we then have $\boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) = \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\boldsymbol{0} \ominus_{gH} \boldsymbol{0} \oplus \epsilon) = \lim_{\lambda\to 0+} \frac{\epsilon}{\lambda} = \frac{(d_1^2+d_2^2)\epsilon}{2d_2}$ *using (3.3).*

- *Case 2. If $(\lambda d_1, \lambda d_2 - 1) \notin \mathbb{B}$, then from (3.2), we have*

$$\begin{aligned}
\boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) &= \lim_{\lambda\to 0+} \tfrac{1}{\lambda} \odot \left([+\infty, +\infty] \ominus_{gH} \boldsymbol{0} \oplus \epsilon\right)\\
&= [+\infty, +\infty].
\end{aligned}$$

Hence, from Case 1 and Case 2, we have

$$\boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) = \begin{cases} \frac{\left(d_1^2+d_2^2\right)\epsilon}{2d_2}, & if\ (\lambda d_1, \lambda d_2 - 1) \in \mathbb{B}\\ [+\infty, +\infty], & if\ (\lambda d_1, \lambda d_2 - 1) \notin \mathbb{B}. \end{cases}$$

Theorem 3.1 *Let $\boldsymbol{F}$ be a convex IVF on $\mathbb{R}^n$. Then, at $\bar{x} \in \mathbb{R}^n$ and for $d \in \mathbb{R}^n$, the function $d \mapsto \boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d)$ is sublinear.*

Proof: We prove positive homogeneity first. For this, let $\beta > 0$, then

$$\begin{aligned}
\mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(\beta d) =& \lim_{\lambda\to 0+} \tfrac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x} + \beta\lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right)\\
=& \lim_{\lambda\to 0+} \tfrac{\beta}{\beta\lambda} \odot \left(\mathbf{F}(\bar{x} + \beta\lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right)
\end{aligned}$$

$$= \beta \odot \lim_{\lambda \to 0+} \tfrac{1}{\beta\lambda} \odot \left(\mathbf{F}(\bar{x} + \beta\lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right)$$
$$= \beta \odot \lim_{\lambda' \to 0+} \tfrac{1}{\lambda'} \odot \left(\mathbf{F}(\bar{x} + \lambda' d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right), \text{ where } \lambda' = \beta\lambda$$
$$= \beta \odot \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d).$$

Hence, positive homogeneity is proved.

Next, we prove the subadditivity. For this, let d^1 and d^2 be in $\mathbb{R}^n$. Thus, $d^1 + d^2$ is also in $\mathbb{R}^n$. The gH_ϵ-directional derivative at $\bar{x} \in \mathbb{R}^n$ in the direction of $d^1 + d^2$ is given by $\mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1 + d^2)$

$$= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x} + \lambda(d^1 + d^2)) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right), \text{ provided the limit exists}$$
$$= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x} + \lambda(d^1 + d^2)) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right)$$
$$= \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\mathbf{F}\left(\tfrac{\bar{x}+2\lambda d^1}{2} + \tfrac{\bar{x}+2\lambda d^2}{2}\right) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right)$$
$$\preceq \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\tfrac{1}{2} \odot \mathbf{F}(\bar{x} + 2\lambda d^1) \oplus \tfrac{1}{2} \odot \mathbf{F}(\bar{x} + 2\lambda d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right) \text{ because } \mathbf{F} \text{ is convex}$$
$$\nsucc \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \left(\tfrac{1}{2} \odot \left(\mathbf{F}(\bar{x} + 2\lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right) \oplus \tfrac{1}{2} \odot \left(\mathbf{F}(\bar{x} + 2\lambda d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right)\right)$$
$$\text{by Lemma 1.6}$$
$$= \lim_{\lambda \to 0+} \tfrac{1}{2\lambda} \odot \left(\mathbf{F}(\bar{x} + 2\lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right) \oplus \lim_{\lambda \to 0+} \tfrac{1}{2\lambda} \odot \left(\mathbf{F}(\bar{x} + 2\lambda d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right)$$
$$= \lim_{\lambda' \to 0+} \tfrac{1}{\lambda'} \odot \left(\mathbf{F}(\bar{x} + \lambda' d^1) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right) \oplus \lim_{\lambda' \to 0+} \tfrac{1}{\lambda'} \odot \left(\mathbf{F}(\bar{x} + \lambda' d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\right),$$
$$\text{where } \lambda' = 2\lambda$$
$$= \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1) \oplus \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^2).$$

Thus, $\mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1 + d^2) \nsucc \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1) \oplus \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^2)$ for any $d^1, d^2 \in \mathbb{R}^n$, which completes the proof. □

Theorem 3.2 *Let $\boldsymbol{F}$ be a gH-Lipschitz continuous IVF on $\mathbb{R}^n$ with Lipschitz constant M. Then, $d \mapsto \boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d)$ is also a gH-Lipschitz continuous with the constant M.*

Proof: Let $\bar{x} \in \mathbb{R}^n$ and $d^1, d^2 \in \mathbb{R}^n$. Since $\mathbf{F}$ is gH-Lipschitz continuous with constant M, using Definition 1.16 and $\lambda > 0$, we have

$$\begin{aligned}
&\mathbf{F}(\bar{x} + \lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x} + \lambda d^2) \preceq M\|(\bar{x} + \lambda d^1) - (\bar{x} + \lambda d^2)\| \\
\Longrightarrow\ &\mathbf{F}(\bar{x} + \lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x} + \lambda d^2) \preceq M\lambda\|d^1 - d^2\| \\
\Longrightarrow\ &\mathbf{F}(\bar{x} + \lambda d^1) \preceq \mathbf{F}(\bar{x} + \lambda d^2) \oplus M\lambda\|d^1 - d^2\| \\
\Longrightarrow\ &\mathbf{F}(\bar{x} + \lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}(\bar{x} + \lambda d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus M\lambda\|d^1 - d^2\| \\
\Longrightarrow\ &\mathbf{F}(\bar{x} + \lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \preceq \mathbf{F}(\bar{x} + \lambda d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \oplus M\lambda\|d^1 - d^2\| \\
\Longrightarrow\ &\tfrac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\big) \preceq \tfrac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\big) \oplus M\|d^1 - d^2\| \\
\Longrightarrow\ &\lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda d^1) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\big) \qquad (3.4) \\
&\preceq \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda d^2) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\big) \oplus M\|d^1 - d^2\| \\
\Longrightarrow\ &\mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1) \preceq \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^2) \oplus M\|d^1 - d^2\| \\
\Longrightarrow\ &\mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1) \ominus_{gH} \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^2) \preceq M\|d^1 - d^2\|. \qquad (3.5)
\end{aligned}$$

Similarly, on reversing the order of d^1 and d^2, we get

$$\mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^2) \ominus_{gH} \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1) \preceq M\|d^1 - d^2\|. \qquad (3.6)$$

Thus, in view of (3.4), (3.6), and the definition of norm of an interval, we get the desired result:

$$\|\mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^1) \ominus_{gH} \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d^2)\|_{I(\mathbb{R})} \leq M\|d^1 - d^2\|.$$

□

Next, we give the definition of gH_ϵ-subdifferentiability of convex IVF.

Definition 3.2 *(*gH_ϵ-subdifferentiability*). Let* $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ *be a proper convex IVF and* $\bar{x} \in \mathrm{dom}(\boldsymbol{F})$*. Then,* gH_ϵ*-subdifferential set of* $\boldsymbol{F}$ *at* $\bar{x}$*, denoted by* $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$*, is defined* *by*

$$\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}) = \left\{ \widehat{\boldsymbol{L}} \in I(\mathbb{R})^n : (x - \bar{x})^\top \odot \widehat{\boldsymbol{L}} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(\bar{x}) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n \right\}. \quad (3.7)$$

The elements of (2.9) *are called* gH_ϵ*-subgradients of* $\boldsymbol{F}$ *at* $\bar{x}$*. If* $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}) \neq \emptyset$*, we say that* $\boldsymbol{F}$ *is* gH_ϵ*-subdifferentiable at* $\bar{x}$*.*

Below we give some examples and remarks for gH_ϵ-subdifferential set of IVFs.

Example 3.2 *Consider a convex IVF* $\boldsymbol{F} : \mathbb{R} \to \overline{I(\mathbb{R})}$ *defined by*

$$\boldsymbol{F}(x) = \begin{cases} [-2, -1] \odot \sqrt{x}, & \text{if } x \geq 0 \\ [+\infty, +\infty], & \text{otherwise.} \end{cases} \quad (3.8)$$

Let us find the gH_ϵ*-subdifferential set of* $\boldsymbol{F}$ *at* 0*. If* $\boldsymbol{L} = [a, b] \in \boldsymbol{\partial}_\epsilon \boldsymbol{F}(0)$*, then by Definition* 3.2*,*

$$(x - 0) \odot \boldsymbol{L} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(0) \oplus \epsilon \text{ for all } x \in \mathbb{R}. \quad (3.9)$$

Note that (3.9) *is trivial when* $x < 0$*. Therefore, we take* $x \geq 0$ *in* (3.9)*, i.e.,*

$$\begin{aligned} & (x - 0) \odot \boldsymbol{L} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(0) \oplus \epsilon \text{ for all } x \geq 0 \\ \Longrightarrow\ & x \odot [a, b] \preceq [-2, -1] \odot \sqrt{x} \ominus_{gH} [0, 0] \oplus \epsilon \\ \Longrightarrow\ & ax \leq -2\sqrt{x} + \epsilon \text{ and } bx \leq -\sqrt{x} + \epsilon \\ \Longrightarrow\ & ax + 2\sqrt{x} - \epsilon \leq 0 \text{ and } bx + \sqrt{x} + \epsilon \leq 0 \\ \Longrightarrow\ & a \leq -\tfrac{1}{\epsilon} \text{ and } b \leq -\tfrac{1}{4\epsilon} \text{ since } x \geq 0. \end{aligned}$$

Hence,

$$\boldsymbol{\partial}_\epsilon \boldsymbol{F}(0) = \left\{ [a, b] \in I(\mathbb{R}) : a \leq -\tfrac{1}{\epsilon} \text{ and } b \leq -\tfrac{1}{4\epsilon} \right\}.$$

Remark 3.1 *The importance of adding* ϵ *in Definition* 3.2 *cannot be avoided. For instance, take the IVF* $\boldsymbol{F}$ *of Example* 3.2*, and its* gH_ϵ*-subdifferential set at* 0*, i.e.,*

$\boldsymbol{\partial}_\epsilon \boldsymbol{F}(0)$. *We take* $\epsilon = 0.25$ *and depict the graphs of* $\boldsymbol{F} \oplus \epsilon$ *and* $x \odot \boldsymbol{L}$ *by light purple-shaded portion and light cyan-shaded portion, respectively, in (a) of Figure 3.1. It can be seen clearly from (b) of Figure3.1 that the graph of* $x \odot \boldsymbol{L}$ *is always below the graph of* $\boldsymbol{F}(x) \oplus \epsilon$*. However, if we consider* $\boldsymbol{F}(x)$ *instead of* $\boldsymbol{F}(x) \oplus \epsilon$*, then the graph of* $x \odot \boldsymbol{L}$ *is not below always the graph of* $\boldsymbol{F}(x)$*, which can be seen in (d) of Figure 3.1.*

Example 3.3 *Let us take an IVF* $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ *defined by* $\boldsymbol{F}(x) = \boldsymbol{\delta}_S(x)$*, where* S *is a nonempty convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{\delta}_S(x)$ *is the indicator IVF of* S*. If* $\widehat{\boldsymbol{L}}$ *is a* gH_ϵ*-subgradient of* $\boldsymbol{F}$ *at any* $\bar{x} \in S$*, then for all* $x \in \mathbb{R}^n$*, we have*

$$(x - \bar{x})^\top \odot \widehat{\boldsymbol{L}} \preceq \boldsymbol{\delta}_S(x) \ominus_{gH} \boldsymbol{\delta}_S(\bar{x}) \oplus \epsilon$$
$$\iff (x - \bar{x})^\top \odot \widehat{\boldsymbol{L}} \preceq \epsilon \text{ for all } x \in S.$$

Hence, the gH_ϵ*-subdifferential set of* $\boldsymbol{\delta}_S(x)$ *at* $\bar{x}$ *is*

$$\{\widehat{\boldsymbol{L}} \in I(\mathbb{R})^n : (x - \bar{x})^\top \odot \widehat{\boldsymbol{L}} \preceq \epsilon \text{ for all } x \in S\}. \tag{3.10}$$

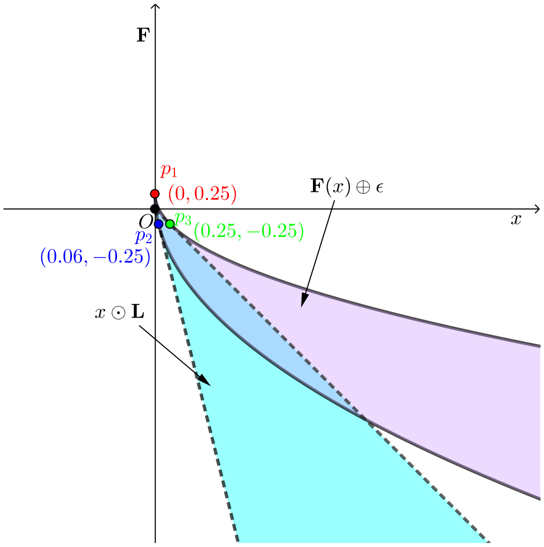
Remark 3.2 *It is noteworthy to see that if* $\boldsymbol{F}(x) = \boldsymbol{\delta}_S(x)$ *in Example 3.3 is real-valued, then* $\widehat{\boldsymbol{L}} \in \mathbb{R}^n$*. In this case, the* gH_ϵ*-subdifferential set of an indicator function* $\boldsymbol{\delta}_S(x)$ *at* $\bar{x} \in S$ *is the* ϵ*-normal cone of* S *at* $\bar{x}$*.*

Example 3.4 *Let* $\boldsymbol{F}$ *be an IVF defined by*

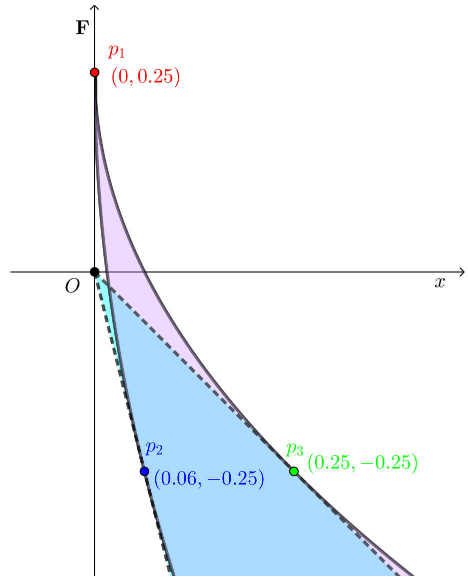
$$\boldsymbol{F}(x) = \boldsymbol{\Phi}(Ax) \text{ for all } x \in \mathbb{R}^n,$$

where $\boldsymbol{\Phi} : Z \to I(\mathbb{R})$ *with* $Z = \{Ax \ : \ x \in \mathbb{R}^n\}$ *and* A *is an* $m \times n$ *real matrix. We calculate* gH_ϵ*-subdifferential set of* $\boldsymbol{F}$ *at* $\bar{x}$*.*

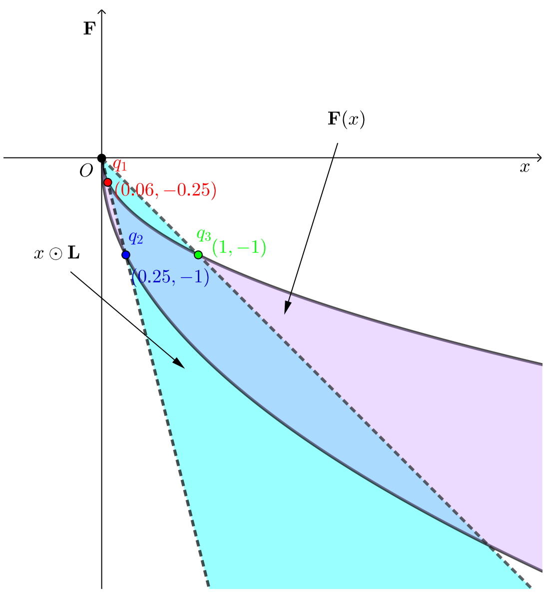
By Definition 3.2 of gH_ϵ*-subdifferential set of* $\boldsymbol{\Phi}$ *at* $A\bar{x}$*, for any* $\bar{x} \in \mathbb{R}^n$*, we have an*



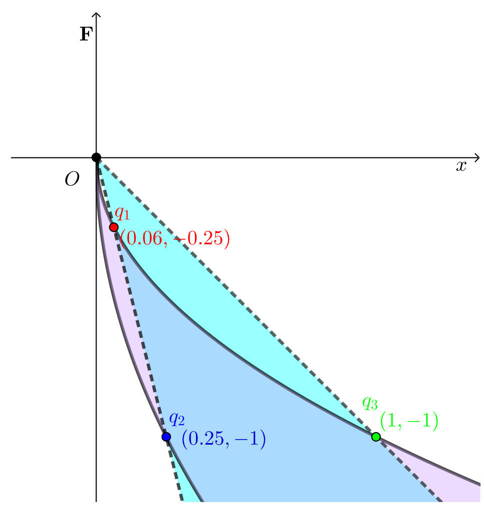
(a) Graphs of $\mathbf{F}(x) \oplus \epsilon$ and $x \odot \mathbf{L}$ of Example 3.2



(b) Zoomed version of (a) near the origin



(c) Graphs of $\mathbf{F}(x)$ and $x \odot \mathbf{L}$ of Example 3.2



(d) Zoomed version of (c) near the origin

Figure 3.1: Geometrical representation of gH_ϵ-subgradient of IVF $\mathbf{F}$ of Example 3.2

$\widehat{\boldsymbol{L}} \in I(\mathbb{R})^m$ such that

$$\begin{aligned}
&(Ax - A\bar{x})^\top \odot \widehat{\boldsymbol{L}} \preceq \boldsymbol{\Phi}(Ax) \ominus_{gH} \boldsymbol{\Phi}(A\bar{x}) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n \\
\implies &(A(x - \bar{x}))^\top \odot \widehat{\boldsymbol{L}} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(\bar{x}) \oplus \epsilon \\
\implies &(x - \bar{x})^\top \odot (A^\top \odot \widehat{\boldsymbol{L}}) \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(\bar{x}) \oplus \epsilon \\
\implies &A^\top \odot \widehat{\boldsymbol{L}} \in \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}) \text{ using Definition 3.2.}
\end{aligned}$$

Thus,

$$\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}) = \{A^\top \odot \widehat{\boldsymbol{L}} \ : \ \widehat{\boldsymbol{L}} \in \boldsymbol{\partial}_\epsilon \boldsymbol{\Phi}(A\bar{x}), \widehat{\boldsymbol{L}} \in I(\mathbb{R})^m \text{ and } \bar{x} \in \mathbb{R}^n\}.$$

Remark 3.3 *Suppose $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ is a proper convex IVF and $\bar{x} \in \mathrm{dom}(\boldsymbol{F})$. Then, by using Definition 2.2 and Definition 3.2, we have*

$$\boldsymbol{\partial} \boldsymbol{F}(\bar{x}) \subseteq \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}). \tag{3.11}$$

Note that the converse of (3.11) is not true in general. For instance, consider the IVF $\boldsymbol{F} : \mathbb{R} \to \overline{I(\mathbb{R})}$ as defined in Example 3.2, i.e.,

$$\boldsymbol{F}(x) = \begin{cases} [-2, -1] \odot \sqrt{x}, & \text{if } x \geq 0 \\ [+\infty, +\infty], & \text{otherwise.} \end{cases} \tag{3.12}$$

Let us try to find gH-subdifferential set of $\boldsymbol{F}$ at 0. Let $\boldsymbol{L} = [m, n] \in \boldsymbol{\partial} \boldsymbol{F}(0)$. Then, by Definition 2.2,

$$\begin{aligned}
&(x - 0) \odot \boldsymbol{L} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(0) \text{ for all } x \geq 0 \\
\implies &x \odot [m, n] \preceq [-2\sqrt{x}, -\sqrt{x}] \\
\implies &mx \leq -2\sqrt{x} \text{ and } nx \leq -\sqrt{x} \text{ because } x \geq 0.
\end{aligned} \tag{3.13}$$

Take $z = \frac{1}{2n^2}$ *in* (3.13). *Then,*

$$\begin{aligned}
& \tfrac{m}{2n^2} \leq -2\sqrt{\tfrac{1}{2n^2}} \text{ and } \tfrac{n}{2n^2} \leq -\sqrt{\tfrac{1}{2n^2}} \\
\Longrightarrow\ & \tfrac{m}{2n^2} \leq -\tfrac{\sqrt{2}}{n} \text{ and } \tfrac{1}{2n} \leq -\sqrt{\tfrac{1}{2n^2}} \\
\Longrightarrow\ & \tfrac{2}{n^2} \leq \tfrac{m^2}{4n^4} \text{ and } \tfrac{1}{2n^2} \leq \tfrac{1}{4n^2} \text{ on squaring on both sides} \\
\Longrightarrow\ & 8n^2 \leq m^2 \text{ and } 2 \leq 1,
\end{aligned}$$

which is impossible. Thus, we get a contradiction, and our assumption is wrong. Therefore, $\boldsymbol{\partial F}(0) = \emptyset$. *However,* $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(0)$ *is nonempty (see Example* 3.2*). Hence, the converse of* (3.11) *is not true.*

Remark 3.4 *Since* $\epsilon > 0$ *in* (3.11), *one may think if* gH*-subdifferential set and* gH_ϵ*-subdifferential set of a convex IVF at a point can be equal. Yes, they can be equal. To see this, consider the convex IVF* $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ *given by* $\boldsymbol{F}(x) = |x| \odot \boldsymbol{T}$, *where* $\boldsymbol{0} \preceq \boldsymbol{T}$.

For this $\boldsymbol{F}$, *the* gH*-subdifferential set at* $\bar{x} = 0$ *is given by (see Example 2.1 in Chapter 2)* $\boldsymbol{\partial F}(0) = \{\boldsymbol{L} \in I(\mathbb{R}) : (-1) \odot \boldsymbol{T} \preceq \boldsymbol{L} \preceq \boldsymbol{T}\}$. *Next, we find the* gH_ϵ*-subdifferential set of* $\boldsymbol{F}$ *at* 0. *Let* $\boldsymbol{L} \in \boldsymbol{\partial}_\epsilon \boldsymbol{F}(0)$. *Then, by Definition* 3.2, *for all* $x \in \mathbb{R}$

$$\begin{aligned}
& (x - 0) \odot \boldsymbol{L} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(0) \oplus \epsilon \\
\Longrightarrow\ & x \odot \boldsymbol{L} \preceq |x| \odot \boldsymbol{T} \ominus_{gH} [0, 0] \oplus \epsilon \\
\Longrightarrow\ & x \odot \boldsymbol{L} \preceq |x| \odot \boldsymbol{T} \oplus \epsilon. \qquad (3.14)
\end{aligned}$$

Here, three cases arise.

- *Case 1. If* $x < 0$, *then* (3.14) *becomes*

$$\begin{aligned}
& x \odot \boldsymbol{L} \preceq (-x) \odot \boldsymbol{T} \oplus \epsilon \\
\Longrightarrow\ & (-1) \odot \boldsymbol{T} \oplus \tfrac{\epsilon}{x} \preceq \boldsymbol{L}. \qquad (3.15)
\end{aligned}$$

Since $x < 0$ and $\epsilon > 0$, therefore (3.15) gives

$$(-1) \odot \boldsymbol{T} \preceq \boldsymbol{L}.$$

- *Case 2. If $x > 0$, then (3.14) becomes*

$$\begin{aligned} & x \odot \boldsymbol{L} \preceq x \odot \boldsymbol{T} \oplus \epsilon \\ \Longrightarrow \ & \boldsymbol{L} \preceq \boldsymbol{T} \oplus \tfrac{\epsilon}{x}. \end{aligned} \tag{3.16}$$

Since $x > 0$ and $\epsilon > 0$, therefore (3.16) gives

$$\boldsymbol{L} \preceq \boldsymbol{T}.$$

- *Case 3. If $x = 0$. This case is trivial.*

Thus, from Case 1, Case 2, and Case 3, we have $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(0) = \{\boldsymbol{L} \in I(\mathbb{R}) : (-1) \odot \boldsymbol{T} \preceq \boldsymbol{L} \preceq \boldsymbol{T}\}$, which is identical to $\boldsymbol{\partial F}(0)$.

Remark 3.5 *If an IVF $\boldsymbol{F} : Y \to I(\mathbb{R})$ is gH-differentiable on Y, where Y is an open subset of $\mathbb{R}^n$, then*

$$\{\boldsymbol{\nabla F}(\bar{x})\} \subseteq \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}), \tag{3.17}$$

The reason is as follows. For a gH-differentiable IVF $\boldsymbol{F}$, we have $\boldsymbol{\partial F}(\bar{x}) = \{\boldsymbol{\nabla F}(\bar{x})\}$ from Theorem 2.3. Then, using (3.11), we get (3.17).

Next, we show the nonemptiness of gH_ϵ-subdifferential set of a convex IVF (Remark 3.6). To prove so, we present a lemma below. This lemma establishes a correspondence between gH_ϵ-subgradient of an IVF and ϵ-subgradients of its lower and upper functions. Basically, we show how to obtain gH_ϵ-subgradient of $\mathbf{F}$ at $\bar{x} \in \text{dom}(\mathbf{F})$ from the ϵ-subgradients of $\underline{F}$ and $\overline{F}$ at $\bar{x}$ and vice-versa (Lemma 3.1).

Lemma 3.1 *Consider a proper convex IVF* $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ *such that* $\underline{F}$ *and* $\overline{F}$ *are its lower and upper functions. Let* $\bar{x} \in \mathrm{dom}(\boldsymbol{F})$. *Then,* $\underline{l} = (\underline{l}_1, \underline{l}_2, \ldots, \underline{l}_n)$ *and* $\bar{l} = (\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_n)$ *be* ϵ*-subgradients of* $\underline{F}$ *and* $\overline{F}$ *at* $\bar{x}$*, respectively if and only if* $\widehat{\boldsymbol{L}} \in \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$*, where* $\widehat{\boldsymbol{L}} = (\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_n)$ *with* $\boldsymbol{L}_j = \left[\min\left\{\underline{l}_j, \bar{l}_j\right\}, \max\left\{\underline{l}_j, \bar{l}_j\right\}\right]$ *for all* $j = 1, 2, \ldots, n$.

Proof: As $\mathbf{F}$ is proper and convex, therefore, by Lemma 1.3, $\underline{F}$ and $\overline{F}$ are also proper and convex. We know that ϵ-subdifferential set of a proper convex real-valued function is nonempty at any point of its effective domain (see [106]). Let $\underline{l} = (\underline{l}_1, \underline{l}_2, \ldots, \underline{l}_n)$ and $\bar{l} = (\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_n)$ be two ϵ-subgradients of $\underline{F}$ and $\overline{F}$ at $\bar{x} \in \mathrm{dom}(\mathbf{F})$, respectively. Then, by Definition 3.2 of gH_ϵ-subdifferentiability, for any $d \in \mathbb{R}^n$, we have

$$d^\top \odot \underline{l} \leq \underline{F}(\bar{x} + d) - \underline{F}(\bar{x}) + \epsilon \quad \text{and} \quad d^\top \odot \bar{l} \leq \overline{F}(\bar{x} + d) - \overline{F}(\bar{x}) + \epsilon. \tag{3.18}$$

Note that for all $d \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\Big[\min\left\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+d) - \overline{F}(\bar{x})\right\},\\
&\qquad \max\left\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+d) - \overline{F}(\bar{x})\right\}\Big] \oplus \epsilon = \mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\\
&\Longrightarrow \left[\min\left\{d^\top \odot \underline{l}, d^\top \odot \bar{l}\right\}, \max\left\{d^\top \odot \underline{l}, d^\top \odot \bar{l}\right\}\right] \preceq \mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \text{ by (3.18)}\\
&\Longrightarrow d^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon, \text{ where } \widehat{\mathbf{L}} = (\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n) \text{ with } \mathbf{L}_j = [\underline{l}_j, \bar{l}_j]\\
&\Longrightarrow \widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x}).
\end{aligned}$$

Thus, for any $\underline{l} \in \partial_\epsilon \underline{F}(\bar{x})$ and $\bar{l} \in \partial_\epsilon \overline{F}(\bar{x})$, we have the corresponding $\widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$.
To prove the converse part, for any $\bar{x} \in \mathrm{dom}(\mathbf{F})$, take $\widehat{\mathbf{L}} = (\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n) \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ with $\mathbf{L}_j = [\underline{l}_j, \bar{l}_j]$. Then, by Definition 3.2 of gH_ϵ-subdifferentiability, we have

$$d^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \text{ for all } d \in \mathbb{R}^n$$

$$\implies \left[\min\left\{\sum_{j=1}^{n} d_j \underline{l}_j, \sum_{j=1}^{n} d_j \bar{l}_j\right\}, \max\left\{\sum_{j=1}^{n} d_j \underline{l}_j, \sum_{j=1}^{n} d_j \bar{l}_j\right\}\right] \preceq \mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon.$$

Thus,

$$\min\left\{\sum_{j=1}^{n} d_j \underline{l}_j, \sum_{j=1}^{n} d_j \bar{l}_j\right\} \leq \min\left\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon, \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon\right\} \tag{3.19}$$

and

$$\max\left\{\sum_{j=1}^{n} d_j \underline{l}_j, \sum_{j=1}^{n} d_j \bar{l}_j\right\} \leq \max\left\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon, \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon\right\}. \tag{3.20}$$

We now consider the following two possible cases.

- Case 1. Let $\min\left\{\sum\limits_{j=1}^{n} d_j \underline{l}_j, \sum\limits_{j=1}^{n} d_j \bar{l}_j\right\} = \sum\limits_{j=1}^{n} d_j \underline{l}_j$ and $\min\Big\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon,$ $\overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon\Big\} = \underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon.$

  In this case, by (3.19) and (3.20), we have

$$\sum_{j=1}^{n} d_j \underline{l}_j \leq \underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon \text{ and } \sum_{j=1}^{n} d_j \bar{l}_j \leq \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon$$

$$\implies d^\top \odot \underline{l} \leq \underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon \text{ and } d^\top \odot \bar{l} \leq \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon,$$

$$\text{where } \underline{l} = (\underline{l}_1, \underline{l}_2, \ldots, \underline{l}_n) \in \mathbb{R}^n \text{ and } \bar{l} = (\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_n) \in \mathbb{R}^n.$$

  Thus, we get $\underline{l} \in \partial_\epsilon \underline{F}(\bar{x})$ and $\bar{l} \in \partial_\epsilon \overline{F}(\bar{x})$, which are as required.

- Case 2. Let $\min\left\{\sum\limits_{j=1}^{n} d_j \underline{l}_j, \sum\limits_{j=1}^{n} d_j \bar{l}_j\right\} = \sum\limits_{j=1}^{n} d_j \bar{l}_j$ and $\min\Big\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon,$ $\overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon\Big\} = \underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon.$

  By (3.19) and (3.20), we get

$$\sum_{j=1}^{n} d_j \bar{l}_j \leq \underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon \text{ and } \sum_{j=1}^{n} d_j \underline{l}_j \leq \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon$$

$$\implies d^\top \odot \bar{l} \leq \underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon \text{ and } d^\top \odot \underline{l} \leq \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon,$$

$$\text{where } \bar{l} = (\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_n) \in \mathbb{R}^n \text{ and } \underline{l} = (\underline{l}_1, \underline{l}_2, \ldots, \underline{l}_n) \in \mathbb{R}^n.$$

Thus, in this case, we get $\bar{l} \in \partial_\epsilon \underline{F}(\bar{x})$ and $\underline{l} \in \partial_\epsilon \overline{F}(\bar{x})$.

Thus, from Case 1 and Case 2, the converse part is proved. □

Remark 3.6 *As ϵ-subdifferential set of a proper convex real-valued function is nonempty at any point of its effective domain, by Lemma 3.1, it can be noted that for any proper convex IVF $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$ for all $x \in \mathbb{R}^n$ and $\bar{x} \in \mathrm{dom}(\boldsymbol{F})$, $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$ is nonempty.*

In order to prove the next result (Theorem 3.3) on the convexity of gH_ϵ-subdifferential set of $\mathbf{F}$, the following Lemma 3.2 is useful.

Lemma 3.2 *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF. Then, for any $\bar{x} \in \mathrm{dom}(\boldsymbol{F})$, the gH_ϵ-subdifferential of $\boldsymbol{F}$ is*

$$\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}) = \left\{ \widehat{\boldsymbol{L}} \in I(\mathbb{R})^n : d^\top \odot \widehat{\boldsymbol{L}} \preceq \boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) \textit{ for any } d \in \mathbb{R}^n \right\}.$$

Proof: Suppose $\widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$. Then, by Definition 3.2, we have

$$(x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n. \tag{3.21}$$

Let $d \in \mathbb{R}^n$. By taking $x = \bar{x} + \lambda d$ with sufficiently small $\lambda > 0$ in (3.21), we get

$$\begin{aligned}
& d^\top \odot \widehat{\mathbf{L}} \preceq \tfrac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon) \\
\iff & d^\top \odot \widehat{\mathbf{L}} \preceq \lim_{\lambda \to 0^+} \left(\tfrac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon) \right) \\
\iff & d^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d).
\end{aligned}$$

Arbitrariness of $d \in \mathbb{R}^n$ proves the lemma. □

Theorem 3.3 *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \mathrm{dom}(\boldsymbol{F})$. Then, the set $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$ is convex.*

Proof: Consider two arbitrary elements $\widehat{\mathbf{T}} = (\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n)$ and $\widehat{\mathbf{U}} = (\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n)$ of $\boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ with $\mathbf{T}_j = [\underline{t}_j, \overline{t}_j]$ and $\mathbf{U}_j = [\underline{u}_j, \overline{u}_j]$ for all $j = 1, 2, \ldots, n$. Then, for any $\beta_1, \beta_2 \geq 0$, with $\beta_1 + \beta_2 = 1$ and for any $d \in \mathbb{R}^n$, we have

$$\begin{aligned} d^\top \odot \left(\beta_1 \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot \widehat{\mathbf{U}}\right) = \Bigg[&\min \left\{ \sum_{j=1}^n d_j(\beta_1 \underline{t}_j + \beta_2 \underline{u}_j), \sum_{j=1}^n d_j(\beta_1 \overline{t}_j + \beta_2 \overline{u}_j) \right\}, \\ &\max \left\{ \sum_{j=1}^n d_j(\beta_1 \underline{t}_j + \beta_2 \underline{u}_j), \sum_{j=1}^n d_j(\beta_1 \overline{t}_j + \beta_2 \overline{u}_j) \right\} \Bigg]. \end{aligned}$$

- Case 1. Let $\min \left\{ \sum_{j=1}^n d_j(\beta_1 \underline{t}_j + \beta_2 \underline{u}_j), \sum_{j=1}^n d_j(\beta_1 \overline{t}_j + \beta_2 \overline{u}_j) \right\} = \sum_{j=1}^n d_j(\beta_1 \underline{t}_j + \beta_2 \underline{u}_j)$. Then,

$$\begin{aligned} & d^\top \odot \left(\beta_1 \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot \widehat{\mathbf{U}}\right) \\ =& \left[\sum_{j=1}^n d_j(\beta_1 \underline{t}_j + \beta_2 \underline{u}_j), \sum_{j=1}^n d_j(\beta_1 \overline{t}_j + \beta_2 \overline{u}_j) \right] \\ =& \left[\sum_{j=1}^n \beta_1 d_j \underline{t}_j, \sum_{j=1}^n \beta_1 d_j \overline{t}_j \right] \oplus \left[\sum_{j=1}^n \beta_2 d_j \underline{u}_j, \sum_{j=1}^n \beta_2 d_j \overline{u}_j \right] \\ =& \beta_1 \odot d^\top \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot d^\top \odot \widehat{\mathbf{U}} \\ \preceq& \beta_1 \odot \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) \oplus \beta_2 \odot \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) \text{ by Lemma 3.2 and } \beta_1, \beta_2 \geq 0 \\ =& \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d). \end{aligned}$$

  Hence, for any $d \in \mathbb{R}^n$, $d^\top \odot \left(\beta_1 \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot \widehat{\mathbf{U}}\right) \preceq \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d)$. Thus, by Lemma 3.2, $\left(\beta_1 \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot \widehat{\mathbf{U}}\right) \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$.

- Case 2. Let $\min\left\{\sum_{j=1}^{n} d_j(\beta_1\underline{t}_j + \beta_2\underline{u}_j), \sum_{j=1}^{n} d_j(\beta_1\bar{t}_j + \beta_2\bar{u}_j)\right\} = \sum_{j=1}^{n} d_j(\beta_1\bar{t}_j+\beta_2\bar{u}_j)$. Then,

$$\begin{aligned}
& d^\top \odot \left(\beta_1 \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot \widehat{\mathbf{U}}\right) \\
=& \left[\sum_{j=1}^{n} d_j(\beta_1\bar{t}_j + \beta_2\bar{u}_j), \sum_{j=1}^{n} d_j(\beta_1\underline{t}_j + \beta_2\underline{u}_j)\right] \\
=& \left[\sum_{j=1}^{n} \beta_1 d_j\bar{t}_j, \sum_{j=1}^{n} \beta_1 d_j\underline{t}_j\right] \oplus \left[\sum_{j=1}^{n} \beta_2 d_j\bar{u}_j, \sum_{j=1}^{n} \beta_2 d_j\underline{u}_j\right] \\
=& \beta_1 \odot d^\top \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot d^\top \odot \widehat{\mathbf{U}} \\
\preceq& \beta_1 \odot \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) \oplus \beta_2 \odot \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d) \text{ by Lemma 3.2 and } \beta_1, \beta_2 \geq 0 \\
=& \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d),
\end{aligned}$$

which implies that $d^\top \odot \left(\beta_1 \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot \widehat{\mathbf{U}}\right) \preceq \mathbf{F}_{\mathscr{D}_\epsilon}(\bar{x})(d)$ for any $d \in \mathbb{R}^n$. Thus, by Lemma 3.2, $\left(\beta_1 \odot \widehat{\mathbf{T}} \oplus \beta_2 \odot \widehat{\mathbf{U}}\right) \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$.

Hence, from Case 1 and Case 2, for any $\bar{x} \in \text{dom}(\mathbf{F})$, $\boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ is convex. $\square$

Theorem 3.4 *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{dom}(\boldsymbol{F})$. Then, the set $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$ is closed.*

Proof: Consider an arbitrary sequence $\left\{\widehat{\mathbf{L}}_k\right\}$ in $\boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$, where $\widehat{\mathbf{L}}_k = (\mathbf{L}_{k1}, \mathbf{L}_{k2}, \ldots, \mathbf{L}_{kn})$ with $\mathbf{L}_{kj} = [\underline{l}_{kj}, \bar{l}_{kj}]$, which converges to $\widehat{\mathbf{L}} = (\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n) \in I(\mathbb{R})^n$ with $\mathbf{L}_j = [\underline{l}_j, \bar{l}_j]$ for all $j = 1, 2, \ldots, n$. Since $\widehat{\mathbf{L}}_k \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ for all $k \in \mathbb{N}$, thus for all $d \in \mathbb{R}^n$, we have

$$\begin{aligned}
& d^\top \odot \widehat{\mathbf{L}}_k \preceq \mathbf{F}(\bar{x} + d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \\
\implies & \min\left\{\sum_{j=1}^{n} d_j\underline{l}_{kj}, \sum_{j=1}^{n} d_j\bar{l}_{kj}\right\} \leq \min\left\{\underline{F}(\bar{x} + d) - \underline{F}(\bar{x}) + \epsilon, \overline{F}(\bar{x} + d) - \overline{F}(\bar{x}) + \epsilon\right\} \\
\text{and } & \max\left\{\sum_{j=1}^{n} d_j\underline{l}_{kj}, \sum_{j=1}^{n} d_j\bar{l}_{kj}\right\} \leq \min\left\{\underline{F}(\bar{x} + d) - \underline{F}(\bar{x}) + \epsilon, \overline{F}(\bar{x} + d) - \overline{F}(\bar{x}) + \epsilon\right\}.
\end{aligned} \tag{3.22}$$

As $k \to \infty$, we have

$$\begin{aligned}
&\left\{\widehat{\mathbf{L}}_k\right\} \to \widehat{\mathbf{L}}\\
\implies &\{\mathbf{L}_{kj}\} \to \mathbf{L}_j \text{ for all } j = 1, 2, \ldots, n \text{ using Remark 1.6}\\
\implies &\{\underline{l}_{kj}\} \to \underline{l}_j \text{ and } \{\overline{l}_{kj}\} \to \overline{l}_j \text{ for all } j = 1, 2, \ldots, n \text{ as } \mathbf{L}_{kj} = [\underline{l}_{kj}, \overline{l}_{kj}] \text{ and } \mathbf{L}_j = [\underline{l}_j, \overline{l}_j]\\
\implies &\sum_{j=1}^{n} d_j \underline{l}_{kj} \to \sum_{j=1}^{n} d_j \underline{l}_j \text{ and } \sum_{j=1}^{n} d_j \overline{l}_{kj} \to \sum_{j=1}^{n} d_j \overline{l}_j. \qquad (3.23)
\end{aligned}$$

Therefore, for $k \to \infty$, (3.22) and (3.23) imply

$$\begin{aligned}
&\left(\min\left\{\sum_{j=1}^{n} d_j \underline{l}_{kj}, \sum_{j=1}^{n} d_j \overline{l}_{kj}\right\}\right) \to \left(\min\left\{\sum_{j=1}^{n} d_j \underline{l}_{j}, \sum_{j=1}^{n} d_j \overline{l}_{j}\right\}\right)\\
&\qquad\qquad \leq \min\left\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon, \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon\right\}\\
\text{and } &\left(\max\left\{\sum_{j=1}^{n} d_j \underline{l}_{kj}, \sum_{j=1}^{n} d_j \overline{l}_{kj}\right\}\right) \to \left(\max\left\{\sum_{j=1}^{n} d_j \underline{l}_{j}, \sum_{j=1}^{n} d_j \overline{l}_{j}\right\}\right)\\
&\qquad\qquad \leq \max\left\{\underline{F}(\bar{x}+d) - \underline{F}(\bar{x}) + \epsilon, \overline{F}(\bar{x}+d) - \overline{F}(\bar{x}) + \epsilon\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left[\min\left\{\sum_{j=1}^{n} d_j \underline{l}_{j}, \sum_{j=1}^{n} d_j \overline{l}_{j}\right\}, \max\left\{\sum_{j=1}^{n} d_j \underline{l}_{j}, \sum_{j=1}^{n} d_j \overline{l}_{j}\right\}\right] \preceq \mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon\\
\implies &d^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \text{ for all } d \in \mathbb{R}^n.
\end{aligned}$$

Therefore, $\widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$, and hence $\boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ is closed. □

Next, to show the boundedness of gH_ϵ-subdifferential set of a convex IVF, we consider a mapping $\mathcal{W} : I(\mathbb{R})^n \to \mathbb{R}^n$ given by (see [75])

$$\mathcal{W}(\widehat{\mathbf{T}}) = \mathcal{W}(\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n) = (w\underline{t}_1 + w'\overline{t}_1, w\underline{t}_2 + w'\overline{t}_2, \ldots, w\underline{t}_n + w'\overline{t}_n)^\top, \qquad (3.24)$$

where $w, w' \in [0,1]$ with $w + w' = 1$. Now, we present two lemmas that are useful to prove boundedness of gH_ϵ-subdifferential sets (Theorem 3.5).

Lemma 3.3 *For any $\widehat{\boldsymbol{T}} \in I(\mathbb{R})^n$ and $x \in \mathbb{R}^n$,*

$$x^\top \odot \widehat{\boldsymbol{T}} \preceq [r, r] \implies x^\top \mathcal{W}(\widehat{\boldsymbol{T}}) \leq 2r,$$

where r is any real number and $\mathcal{W}$ is a mapping as defined in (3.24).

Proof: We have

$$\begin{aligned}
& x^\top \odot \widehat{\mathbf{T}} \preceq [r, r] \\
\implies & \left[\min\left\{\sum_{j=1}^n x_j \underline{t}_j, \sum_{j=1}^n x_j \bar{t}\right\}, \max\left\{\sum_{j=1}^n x_j \underline{t}_j, \sum_{j=1}^n x_j \bar{t}\right\}\right] \preceq [r, r] \\
\implies & \sum_{j=1}^n x_j \underline{t}_j \leq r \text{ and } \sum_{j=1}^n x_j \bar{t} \leq r \\
\implies & w \sum_{j=1}^n x_j \underline{t}_j \leq wr \text{ and } w' \sum_{j=1}^n x_j \bar{t} \leq w' r, \text{ as } w, w' \in [0,1] \\
\implies & \sum_{j=1}^n x_j (w \underline{t}_j + w' \bar{t}_j) \leq 2r \\
\implies & x^\top \mathcal{W}(\widehat{\mathbf{T}}) \leq 2r.
\end{aligned}$$

□

Lemma 3.4 (See [75]). *For any $\widehat{\boldsymbol{T}} \in I(\mathbb{R})^n$,*

$$\|\mathcal{W}(\widehat{\boldsymbol{T}})\| \text{ is finite} \implies \|\widehat{\boldsymbol{T}}\|_{I(\mathbb{R})^n} \text{ is finite.}$$

Theorem 3.5 *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF. Then, the set $\bigcup_{\bar{x} \in \mathrm{int}(\mathrm{dom}(\boldsymbol{F}))} \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$ is bounded.*

Proof: We prove this theorem by contradiction. Let us assume that there exists a sequence $\{x_k\}$ in $\text{int}(\text{dom}(\mathbf{F}))$ and an unbounded sequence $\{\widehat{\mathbf{L}}_k\}$, where $\widehat{\mathbf{L}}_k \in \boldsymbol{\partial}_\epsilon \mathbf{F}(x_k)$ is such that

$$0 < \|\widehat{\mathbf{L}}_k\| < \|\widehat{\mathbf{L}}_{k+1}\|, \ k \in \mathbb{N}.$$

By Definition 3.2, for any $x_k \in \text{int}(\text{dom}(\mathbf{F}))$ and $d \in \mathbb{R}^n$,

$$\begin{aligned} & d_k^\top \odot \widehat{\mathbf{L}}_k \preceq \mathbf{F}(x_k + d_k) \ominus_{gH} \mathbf{F}(x_k) \oplus \epsilon \\ \implies & d_k^\top \odot \widehat{\mathbf{L}}_k \preceq \Big[\min\{\underline{F}(x_k + d_k) - \underline{F}(x_k), \overline{F}(x_k + d_k) - \overline{F}(x_k)\}, \\ & \qquad \max\{\underline{F}(x_k + d_k) - \underline{F}(x_k), \overline{F}(x_k + d_k) - \overline{F}(x_k)\}\Big] \oplus \epsilon. \end{aligned} \tag{3.25}$$

Since $\mathbf{F}$ is convex on $\mathbb{R}^n$, therefore using Theorem 1.1 and Lemma 1.1, real-valued functions $\underline{F}$ and $\overline{F}$ are continuous on $\mathbb{R}^n$. We can take a $c \in \mathbb{R}$ such that $\max\{\underline{F}(x_k + d_k) - \underline{F}(x_k), \overline{F}(x_k + d_k) - \overline{F}(x_k)\} \leq c$ using the property of convex functions (see [22]). Thus, in view of (3.25), we have

$$\begin{aligned} & d_k^\top \odot \widehat{\mathbf{L}}_k \preceq [c + \epsilon, c + \epsilon] \\ \implies & d_k^\top \mathcal{W}(\widehat{\mathbf{L}}_k) \leq 2(c + \epsilon) \text{ using Lemma 3.3} \\ \implies & \|\mathcal{W}(\widehat{\mathbf{L}}_k)\| \leq 2(c + \epsilon) \text{ taking } d_k = \tfrac{\mathcal{W}(\widehat{\mathbf{L}}_k)}{\|\mathcal{W}(\widehat{\mathbf{L}}_k)\|}. \end{aligned}$$

Thus, $\|\mathcal{W}(\widehat{\mathbf{L}}_k)\|$ is finite and so $\|\widehat{\mathbf{L}}_k\|$ is finite using Lemma 3.4. Therefore, the sequence $\|\widehat{\mathbf{L}}_k\|$ is bounded, which is a contradiction. Hence, the set $\bigcup\limits_{\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))} \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ is bounded. □

Remark 3.7 *In Theorem 3.5, we cannot consider $\bar{x}$ from $\text{dom}(\boldsymbol{F})$ instead of interior of $\text{dom}(\boldsymbol{F})$ in general as gH_ϵ-subdifferential set of $\boldsymbol{F}$ can be unbounded on the boundary point. To check this, see Example 3.2. In this example, 0 is not an interior point of $\text{dom}(\boldsymbol{F})$ and $\boldsymbol{\partial}_\epsilon \boldsymbol{F}(0)$ is unbounded.*

Theorem 3.6 *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{dom}(\boldsymbol{F})$. Then,*

$$\boldsymbol{\partial F}(\bar{x}) = \bigcap_{\epsilon > 0} \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x}), \text{ provided } \boldsymbol{\partial F}(\bar{x}) \text{ is nonempty.}$$

Proof: Let $\widehat{\mathbf{L}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$. Then, by Definition 2.2 of gH-subdifferentiability, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
&(x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \\
\implies &(x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \text{ for every } \epsilon > 0 \\
\implies &\widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x}).
\end{aligned}$$

Since $\widehat{\mathbf{L}}$ is arbitrary,

$$\boldsymbol{\partial}\mathbf{F}(\bar{x}) \subseteq \bigcap_{\epsilon > 0} \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x}).$$

Now consider $\widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ for every $\epsilon > 0$. Thus, by Definition 3.2, for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
&(x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \\
\implies &(x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ taking the limit as } \epsilon \to 0 \\
\implies &\widehat{\mathbf{L}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x}) \\
\implies &\bigcap_{\epsilon > 0} \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x}) \subseteq \boldsymbol{\partial}\mathbf{F}(\bar{x}),
\end{aligned}$$

which proves the theorem. □

Theorem 3.7 *Let $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF whose gH_ϵ-subdifferential set is nonempty on $\mathbb{R}^n$. Then, $\boldsymbol{F}$ is gH-Lipschitz continuous on $\mathbb{R}^n$ with the Lipschitz constant*

$$M = \sup_{\widehat{\boldsymbol{L}} \in \bigcup\limits_{\bar{x} \in \mathbb{R}^n} \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})} \|\widehat{\boldsymbol{L}}\|_{I(\mathbb{R})^n} + \epsilon.$$

Proof: It is given that $\boldsymbol{\partial}_\epsilon \mathbf{F}(x) \neq \emptyset$ for each $x \in \mathbb{R}^n$. Therefore, there exists an $\widehat{\mathbf{L}}$ such that for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
& (x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \\
\implies & \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(x) \preceq (\bar{x} - x)^\top \odot \widehat{\mathbf{L}} \oplus \epsilon \text{ by Lemma 1.7} \\
\implies & \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(x) \preceq \|\bar{x} - x\| \odot \left[\|\widehat{\mathbf{L}}\|_{I(\mathbb{R})^n} + \epsilon, \|\widehat{\mathbf{L}}\|_{I(\mathbb{R})^n} + \epsilon\right] \text{ by Lemma 1.8} \\
\implies & \|\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(x)\|_{I(\mathbb{R})} \leq (\|\widehat{\mathbf{L}}\|_{I(\mathbb{R})^n} + \epsilon)\|\bar{x} - x\| \text{ by Lemma 1.5} \\
\implies & \|\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(x)\|_{I(\mathbb{R})} \leq M\|\bar{x} - x\|, \text{ where } M = \sup_{\widehat{\mathbf{L}} \in \bigcup_{\bar{x} \in \mathbb{R}^n} \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})} \|\widehat{\mathbf{L}}\|_{I(\mathbb{R})^n} + \epsilon,
\end{aligned}
$$

which proves the theorem. $\square$

Theorem 3.8 *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \mathrm{dom}(\boldsymbol{F})$. Then, for any $\alpha > 0$,*

$$\boldsymbol{\partial}_\epsilon(\alpha \odot \boldsymbol{F})(\bar{x}) = \alpha \odot \boldsymbol{\partial}_{\epsilon_1} \boldsymbol{F}(\bar{x}), \text{ where } \epsilon_1 = \tfrac{\epsilon}{\alpha}.$$

Proof: Let $\widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon(\alpha \odot \mathbf{F})(\bar{x})$. Then, by Definition 3.2, for all $x \in \mathbb{R}^n$, we have

$$
\begin{aligned}
& (x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq (\alpha \odot \mathbf{F})(x) \ominus_{gH} (\alpha \odot \mathbf{F})(\bar{x}) \oplus \epsilon \\
\implies & (x - \bar{x})^\top \odot \tfrac{\widehat{\mathbf{L}}}{\alpha} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \tfrac{\epsilon}{\alpha} \\
\implies & \tfrac{\widehat{\mathbf{L}}}{\alpha} \in \boldsymbol{\partial}_{\epsilon_1} \mathbf{F}(\bar{x}), \text{ where } \epsilon_1 = \tfrac{\epsilon}{\alpha} \\
\implies & \widehat{\mathbf{L}} \in \alpha \odot \boldsymbol{\partial}_{\epsilon_1} \mathbf{F}(\bar{x}) \\
\implies & \boldsymbol{\partial}_\epsilon(\alpha \odot \mathbf{F})(\bar{x}) \subseteq \alpha \odot \boldsymbol{\partial}_{\epsilon_1} \mathbf{F}(\bar{x}).
\end{aligned}
$$

Conversely, suppose that $\widehat{\mathbf{L}} \in \alpha \odot \boldsymbol{\partial}_{\epsilon_1} \mathbf{F}(\bar{x})$. Thus, by Definition 3.2, we have

$$
\begin{aligned}
& (x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \alpha \odot \{\mathbf{F}(x) \ominus_{gH} \odot \mathbf{F}(\bar{x}) \oplus \epsilon_1\} \\
\implies & (x - \bar{x})^\top \odot \widehat{\mathbf{L}} \preceq \alpha \odot \mathbf{F}(x) \ominus_{gH} \alpha \odot \mathbf{F}(\bar{x}) \oplus \epsilon \text{ using } \alpha > 0 \text{ and } \epsilon_1 = \tfrac{\epsilon}{\alpha} \\
\implies & \widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon(\alpha \odot \mathbf{F})(\bar{x})
\end{aligned}
$$

$$\implies \alpha \odot \boldsymbol{\partial}_{\epsilon_1}\mathbf{F}(\bar{x}) \subseteq \boldsymbol{\partial}_{\epsilon}(\alpha \odot \mathbf{F})(\bar{x}).$$

Hence, we get the desired result. □

3.5 Application to nonsmooth interval optimization

In this section, we define an approximate solution called ϵ-solution of an IOP. Using this definition and the theory derived in Section 3.4, we provide two optimality conditions. After that we attempt to solve an interval minimax optimization problem using gH_ϵ-subdifferentiability of its objective function.

Definition 3.3 (ϵ-solution of an IOP). *Let $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be a convex IVF. A point $\bar{x} \in \mathbb{R}^n$ is called an ϵ-solution or an approximate solution up to ϵ of the IOP*

$$\min_{x \in \mathbb{R}^n} \boldsymbol{F}(x), \tag{IOP}$$

if $\boldsymbol{F}(\bar{x}) \preceq \boldsymbol{F}(x) \oplus \epsilon$ for all $x \in \mathbb{R}^n$.

Example 3.5 *Consider an IOP*

$$\min_{x \in \mathbb{R}} e^x \odot \boldsymbol{T}, \tag{3.26}$$

where $\boldsymbol{0} \prec \boldsymbol{T} = [\underline{t}, \overline{t}]$. As $\lim\limits_{z \to -\infty} e^x \odot \boldsymbol{T} = 0$, we choose an $\bar{x} \in \mathbb{R}$ for a given $\epsilon > 0$ such that

$$e^{\bar{x}} \odot \boldsymbol{T} \prec \epsilon \tag{3.27}$$

$$\implies e^{\bar{x}}\underline{t} < \epsilon \text{ and } e^{\bar{x}}\overline{t} < \epsilon$$

$$\implies \bar{x} < \ln\left(\frac{\epsilon}{\underline{t}}\right) \text{ and } \bar{x} < \ln\left(\frac{\epsilon}{\overline{t}}\right).$$

Therefore, if $\bar{x} < \min\left\{\ln\left(\frac{\epsilon}{\underline{t}}\right), \ln\left(\frac{\epsilon}{\overline{t}}\right)\right\}$, *then from* (3.27) *and* $\boldsymbol{0} \prec e^x \odot \boldsymbol{T}$, *we get*

$$e^{\bar{x}} \odot \boldsymbol{T} \preceq e^x \odot \boldsymbol{T} \oplus \epsilon \textit{ for all } x \in \mathbb{R},$$

Thus, every $\bar{x}$ *such that* $\bar{x} < \min\left\{\ln\left(\frac{\epsilon}{\underline{t}}\right), \ln\left(\frac{\epsilon}{\overline{t}}\right)\right\}$ *is an* ϵ*-solution of the IOP* (3.26).

Theorem 3.9 *Let* $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ *be a convex IVF. Then,* $\bar{x} \in \mathbb{R}^n$ *is an* ϵ*-solution of* (IOP) *if and only if* $\bar{\boldsymbol{0}} \in \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$, *where* $\bar{\boldsymbol{0}} = (\boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{0}) \in I(\mathbb{R})^n$.

Proof: Let $\bar{x} \in \mathbb{R}^n$ be an ϵ-solution of (IOP). Then, we have

$$\begin{aligned}
& \mathbf{F}(\bar{x}) \preceq \mathbf{F}(x) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n \\
\Longrightarrow\ & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \\
\Longrightarrow\ & (x - \bar{x})^\top \odot \bar{\mathbf{0}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon, \text{ where } \bar{\mathbf{0}} = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) \in I(\mathbb{R})^n \\
\Longrightarrow\ & \bar{\mathbf{0}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x}) \text{ using Definition 3.2.}
\end{aligned}$$

Conversely, let $\bar{\mathbf{0}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$. Thus, by Definition 3.2, we have

$$\begin{aligned}
& (x - \bar{x})^\top \odot \bar{\mathbf{0}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n \\
\Longrightarrow\ & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \\
\Longrightarrow\ & \mathbf{F}(\bar{x}) \preceq \mathbf{F}(x) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n.
\end{aligned}$$

Hence, $\bar{x}$ is an ϵ-solution of (IOP) if and only if $\bar{\mathbf{0}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$. □

Theorem 3.10 *Let* $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ *be a convex IVF. A point* $\bar{x} \in \mathbb{R}^n$ *is an* ϵ*-solution of* (IOP) *if and only if there exists an* $\widehat{\boldsymbol{L}} \in \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$ *such that*

$$\boldsymbol{0} \preceq (x - \bar{x})^\top \odot \widehat{\boldsymbol{L}} \textit{ for all } x \in \mathbb{R}^n. \tag{3.28}$$

Proof: Let $\bar{x} \in \mathbb{R}^n$ be an ϵ-solution of (IOP). Then, by Definition 3.3, we have

$$\begin{aligned} & \mathbf{F}(\bar{x}) \preceq \mathbf{F}(x) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n \\ \implies \quad & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon. \end{aligned} \tag{3.29}$$

Suppose that there does not exist $x \in \mathbb{R}^n$ such that

$$\mathbf{0} \preceq (x - \bar{x})^\top \odot \widehat{\mathbf{L}}.$$

Then, using Definition 3.2, there does not exist $x \in \mathbb{R}^n$ such that

$$\mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon,$$

which is a contradiction. Therefore,

$$\mathbf{0} \preceq (x - \bar{x})^\top \odot \widehat{\mathbf{L}} \text{ for all } x \in \mathbb{R}^n.$$

Conversely, let $\widehat{\mathbf{L}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x})$ for some $\bar{x} \in \mathbb{R}^n$, which satisfies (3.28), i.e.,

$$\begin{aligned} & \mathbf{0} \preceq (x - \bar{x})^\top \odot \widehat{\mathbf{L}} \text{ for all } x \in \mathbb{R}^n \\ \implies & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \\ \implies & \mathbf{F}(\bar{x}) \preceq \mathbf{F}(x) \oplus \epsilon \text{ for all } x \in \mathbb{R}^n. \end{aligned}$$

Hence, $\bar{x}$ is an ϵ-solution of (IOP). □

Remark 3.8 *An optimality condition in the form of gH_ϵ-directional derivative can also be given by using Lemma 3.2 and Theorem 3.10: for a convex IVF $\boldsymbol{F}$ on $\mathbb{R}^n$, a point $\bar{x} \in \mathbb{R}^n$ is an ϵ-solution of (IOP) if and only if $\boldsymbol{0} \preceq \boldsymbol{F}_{\mathscr{D}_\epsilon}(\bar{x})(d)$ for any $d \in \mathbb{R}^n$.*

Next, we consider an interval minimax optimization problem of the form

$$\inf_{x\in\mathbb{R}^n} \mathbf{F}(x), \tag{3.30}$$

where $\mathbf{F} : \mathbb{R}^n \to I(\mathbb{R})$ is the IVF $\mathbf{F}(x) = \sup\limits_{z\in\mathbb{R}^m} \mathbf{\Phi}(x,z)$ and $\mathbf{\Phi} : \mathbb{R}^n \times \mathbb{R}^m \to I(\mathbb{R})$ being an IVF. Note that it is not an easy task to find the gH_ϵ-subdifferential set of the IVF $\mathbf{F}$ at an arbitrary $x \in \mathbb{R}^n$ directly due to supremum occurring in the definition of $\mathbf{F}$. However, if we are interested in obtaining a single gH_ϵ-subgradient at a point $\bar{x} \in \mathbb{R}^n$, the calculation is often much easier. In this context, we give the following theorem.

Theorem 3.11 *Let $\boldsymbol{F}$ and $\mathbf{\Phi}$ be the same IVFs as in (3.30). Assume $\mathbf{\Phi}(\cdot, z)$ is convex for each fixed $z \in \mathbb{R}^m$. Also, for a fixed $\bar{x} \in \mathbb{R}^n$, let us assume that there exists a $z_{\bar{x}} \in \mathbb{R}^m$ such that $\mathbf{\Phi}(\bar{x}, z)$ attains the supremum, i.e., $\sup\limits_{z\in\mathbb{R}^m} \mathbf{\Phi}(\bar{x}, z) = \mathbf{\Phi}(\bar{x}, z_{\bar{x}})$ and $\widehat{\boldsymbol{L}}_{\bar{x}}$ is a gH_ϵ-subgradient of $\mathbf{\Phi}(\cdot, z_{\bar{x}})$ at $\bar{x} \in \mathbb{R}^n$. Then, $\widehat{\boldsymbol{L}}_{\bar{x}} \in \boldsymbol{\partial}_\epsilon \boldsymbol{F}(\bar{x})$.*

Proof: Note that if $\mathbf{\Phi}(\cdot, z)$ is convex for each fixed $z \in \mathbb{R}^m$, then $\mathbf{F}(\cdot)$ is also convex. It is given that $\widehat{\mathbf{L}}_{\bar{x}}$ is a gH_ϵ-subgradient of $\mathbf{\Phi}(\cdot, z_{\bar{x}})$ at $\bar{x} \in \mathbb{R}^n$. Then, by using Definition 3.2, for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
& (x-\bar{x})^\top \odot \widehat{\mathbf{L}}_{\bar{x}} \preceq \mathbf{\Phi}(x, z_{\bar{x}}) \ominus_{gH} \mathbf{\Phi}(\bar{x}, z_{\bar{x}}) \oplus \epsilon \\
\implies \quad & (x-\bar{x})^\top \odot \widehat{\mathbf{L}}_{\bar{x}} \preceq \sup_{z\in\mathbb{R}^m} \mathbf{\Phi}(x, z) \ominus_{gH} \mathbf{\Phi}(\bar{x}, z_{\bar{x}}) \oplus \epsilon \\
\implies \quad & (x-\bar{x})^\top \odot \widehat{\mathbf{L}}_{\bar{x}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \epsilon \\
\implies \quad & \widehat{\mathbf{L}}_{\bar{x}} \in \boldsymbol{\partial}_\epsilon \mathbf{F}(\bar{x}) \text{ using Definition 3.2,}
\end{aligned}$$

which completes the proof. □

Theorem 3.11 provides a convenient method for calculating a gH_ϵ-subgradient of $\mathbf{F}$ at $\bar{x}$: once a supremum point $z_{\bar{x}}$ of $\mathbf{\Phi}(\bar{x}, \cdot)$ has been found, we simply use any gH_ϵ-subgradient in $\boldsymbol{\partial}_\epsilon \mathbf{\Phi}(\bar{x}, z_{\bar{x}})$. To understand this, we present an example below.

Example 3.6 *Let us consider* $\boldsymbol{\Phi}(x,z)$=$[|x|-2z^2,\ 2|x|-z^2]$ *on* $\mathbb{R}\times\mathbb{R}$*. Then,*

$$\boldsymbol{F}(x)=\sup_{z\in\mathbb{R}}\boldsymbol{\Phi}(x,z)=\sup_{z\in\mathbb{R}}\left[|x|-2z^2,2|x|-z^2\right].$$

It is easy to see that for a fixed $\bar{x}\in\mathbb{R}$, $\boldsymbol{\Phi}(\bar{x},z)$ *attains supremum at* $z=0$*. Let us try to find* gH_ϵ*-subgradient of* $\boldsymbol{F}$ *at* $\bar{x}=0$ *by using Theorem 3.11. In this context, let* $\boldsymbol{L}$ *be a* gH_ϵ*-subgradient of* $\boldsymbol{\Phi}(\bar{x},0)$*. Then, for all* $x\in\mathbb{R}$*, by Definition 3.2,*

$$\begin{aligned}&(x-0)\odot\boldsymbol{L}\preceq\boldsymbol{\Phi}(x,0)\ominus_{gH}\boldsymbol{\Phi}(0,0)\oplus\epsilon\\ \Longrightarrow\ &x\odot\boldsymbol{L}\preceq[|x|,2|x|]\ominus_{gH}[0,0]\oplus\epsilon\\ \Longrightarrow\ &(-1)\odot[1,2]\preceq\boldsymbol{L}\preceq[1,2]\ \textit{using Remark 3.4},\end{aligned}\tag{3.31}$$

which is also a gH_ϵ*-subgradient of* $\boldsymbol{F}$ *at* 0 *(see Remark 3.4).*

Figure 3.2 represents the graph of the IVF $\boldsymbol{\Phi}(x,z)$*. In which cyan color indicates the upper function of* $\boldsymbol{\Phi}(x,z)$*, i.e.,* $2|x|-z^2$*, and pink color indicates the lower function of* $\boldsymbol{\Phi}(x,z)$*, i.e.,* $|x|-2z^2$*. Figure 3.3 shows the graph of* $\boldsymbol{F}(x)=\sup_{z\in\mathbb{R}}\boldsymbol{\Phi}(x,z)=\boldsymbol{\Phi}(x,0)$ *in cyan and pink colors. Blue color represents the graph of* $x\odot\boldsymbol{L}_1$*, where* $\boldsymbol{L}_1=[0.75,1.5]$ *is a* gH_ϵ*-subgradient of* $\boldsymbol{\Phi}(x,0)$ *at* $\bar{x}=0$ *in Figure 3.3.*

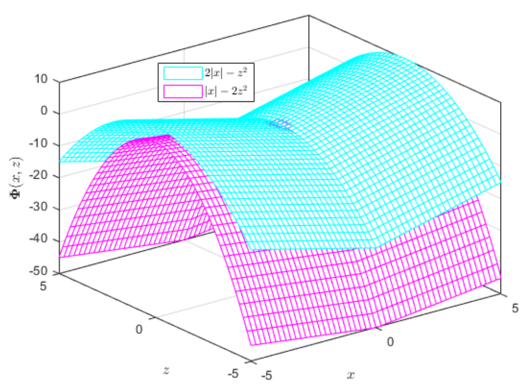


Figure 3.2: Graph of $\boldsymbol{\Phi}(x,z)$ in Example 3.6

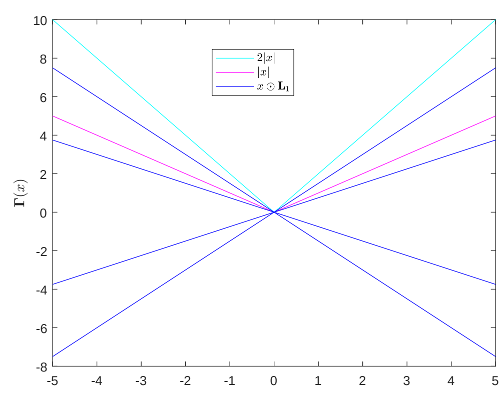


Figure 3.3: Graphs of $\mathbf{F}(x)$ and $x\odot\mathbf{L}_1$, where $\mathbf{L}_1=[0.75,1.5]$

Remark 3.9 *One can notice from* (3.31) *that* $\boldsymbol{0} \in \boldsymbol{\partial}_{\epsilon}\boldsymbol{F}(0)$. *Therefore,* $\bar{x} = 0$ *is an* ϵ*-solution of* (3.30) *by Theorem* 3.9*. Thus, the optimality condition given in Theorem* 3.9 *can also be used to find an* ϵ*-solution of interval minimax optimization problem* (3.30).

3.6 Conclusion

In this chapter, we have proposed the concepts of gH_{ϵ}-subgradient and gH_{ϵ}-directional derivative for a convex IVF with several essential characteristics. It has been shown that the gH_{ϵ}-subdifferential set of convex IVF is nonempty (Remark 2.1), closed (Theorem 2.2), and convex (Theorem 3.3). The boundedness of gH_{ϵ}-subdifferential set has been proved on the interior of the effective domain set (Theorem 3.5). A relation between gH-subdifferential set and gH_{ϵ}-subdifferential set has been given as well (Theorem 3.6). It has also been observed that a convex IVF is gH-Lipschitz continuous if it is gH_{ϵ}-subdifferentiable at each point in its domain (Theorem 3.7). Furthermore, notion of ϵ-solution (Definition 1.22) and two optimality conditions (Theorem 3.9 and Theorem 3.10) have been proposed. Lastly, we have proved a theorem (Theorem 3.11) to solve interval minimax optimization problems using gH_{ϵ}-subdifferentiability.

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