Chapter 1

Introduction

1.1 The Uncertainty Principle

In 1927, Heisenberg introduced the uncertainty principle. In physics and mathematics, there are several versions of this principle. In harmonic analysis, a branch of mathematics, the uncertainty principle usually involves the *Fourier transform*, which is defined for a function $f \in L^1(\mathbb{R})$ by

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-i\xi x}\, dx.$$

It states that "a nonzero function and its Fourier transform cannot both be sharply localized". There are many different precise formulations of it, depending on the way in which localization is quantified. Broadly speaking, uncertainty principles in harmonic analysis are categorized into the quantitative and qualitative uncertainty principles. An example of a quantitative uncertainty principle is the Heisenberg-Pauli-Weyl inequality.

Theorem 1.1. *If* $f \in L^2(\mathbb{R})$, *then*

$$\left(\int x^2|f(x)|^2dx\right)\left(\int \xi^2|\hat{f}(\xi)|^2d\xi\right) \geq \frac{\|f\|_2^4}{16\pi^2}.$$

For a proof, see [5]. An example of a qualitative uncertainty principle is the one due to Matolcsi and Szücs (see [9]). It can be stated as follows. Let G be a locally compact abelian group with Haar measure μ. Let $\hat{G}$ be the Pontryagin dual of G (the group of continuous group homomorphisms from G to the circle group) with Haar measure ν. Assume that the Haar measures of G and $\hat{G}$ are normalized so that the Fourier transform is an isometry from $L^2(G) \to L^2(\hat{G})$. Let $f \in L^2(G)$. Put

$$\Sigma(f) = \{x \in G : f(x) \neq 0\}, \quad \text{and} \quad \Delta(\hat{f}) = \{\xi \in \hat{G} : \hat{f}(\xi) \neq 0\}.$$

Then

$$\mu(\Sigma(f))\,\nu(\Delta(\hat{f})) \geq 1.$$

However, Benedicks proved a much stronger result on $\mathbb{R}^d$ (see [1]). It can be stated as follows.

Theorem 1.2. *Let* λ *be the Lebesgue measure on* $\mathbb{R}^d$. *If* $f \in L^1(\mathbb{R}^d)$ *and* $\lambda(\Sigma(f))\,\lambda(\Delta(\hat{f})) < \infty$, *then* $f = 0$.

Another qualitative uncertainty principle is due to Hardy (see [7]). For $a > 0$, let $g_a(x) = e^{-ax^2/2}$. For $a, b > 0$, let

$$E(a,b) = \{f \in L^1(\mathbb{R}) \mid |f(x)| \leq Cg_a(x) \text{ and } \left|\hat{f}(\xi)\right| \leq Cg_b(\xi) \quad \text{for some } C \in \mathbb{R}\}.$$

Then, Hardy's uncertainty principle can be stated as follows.

Theorem 1.3. *If $ab > 1$ then $E(a,b) = \{0\}$. If $ab = 1$ then $E(a,b) = \mathbb{C}g_a$. If $ab < 1$ then $\dim E(a,b) = \infty$.*

1.2 Discussion of the main results

We can see from the Cauchy-Schwarz inequality and Mehler's formula (Lemma 1.9) that if for some $t > 0$

$$\langle f, \varphi_n \rangle = O\left(e^{-2nt}\right), \tag{1.1}$$

then $f \in E(\tanh 2rt, \tanh 2rt)$ for $0 < r < 1$ (see e.g. [13]). Using other methods, Radha and Thangavelu [10] proved that under the hypothesis (1.1), f extends to $\mathbb{C}$ as an entire function and satisfies the estimate

$$f(x+iy) = O\left(e^{-\frac{1}{2}\tanh(2rt)x^2 + \frac{1}{2}\coth(2rt)y^2}\right),$$

for $0 < r < 1$, and a similar estimate holds for $\widehat{f}$ as well.

Conversely, what can we say about the Hermite coefficients of a function in $E(a,b)$?

It is clear that Hardy's theorem characterizes the Hermite coefficients of the functions in $E(a,b)$ when $ab \geq 1$. Theorem 1.3 has been proved in many different settings (see [12] and the references therein). All these results are concerned with the case $ab \geq 1$. In [13], Vemuri gave closer attention to the case $ab < 1$, where he proved the following theorem.

Theorem 1.4. *Let $a \in (0,1)$. If $f \in E(a,a)$ then*

$$\langle f, \varphi_n \rangle = O\left[n^{-1/4}\left(\frac{1-a}{1+a}\right)^{n/4}\right]$$

for $n = 1, 2, \ldots,$ and this estimate is sharp.

This theorem can be restated in the following way. Let $t > 0$. If $f \in E(\tanh 2t, \tanh 2t)$, then

$$\langle f, \varphi_n \rangle = O\left(n^{-1/4} e^{-nt}\right), \tag{1.2}$$

for $n = 1, 2, \ldots$, and this estimate is sharp.

Shortly thereafter, Garg and Thangavelu generalized Vemuri's result to the several variable case [6]. Combining the results of Vemuri with those of Radha and Thangavelu leads to a certain loss of information about the decay properties of f and $\widehat{f}$. Indeed, if we start with $f \in E(\tanh 2t, \tanh 2t)$, then by Vemuri's result $\langle f, \varphi_n \rangle = O(n^{-1/4} e^{-nt})$, and so by Radha and Thangavelu's result, we get $f \in E(\tanh rt, \tanh rt)$ for $0 < r < 1$. Therefore, it stands to reason that the condition $f \in E(\tanh 2t, \tanh 2t)$ has consequences on the Hermite coefficients of f other than mere exponential decay. Indeed, in this thesis, our first result is a sort of pair correlation between certain Hermite coefficients (see Theorem 2.1). Theorem 1.4 characterizes the Hermite coefficients of functions in $E(a, b)$ for $a = b$. Our second result is a generalization of Theorem 1.4 for $a \neq b$ (see Theorem 3.1).

In the next few sections, we shall discuss some tools which are necessary to prove our main results.

1.3 The Hermite functions

In order to define the Hermite functions, we first define a function space and some operators.

Definition 1.3.1. The *Schwartz space* is

$$\mathcal{S} = \left\{ f : \mathbb{R} \to \mathbb{C} \, : \, \sup_{t \in \mathbb{R}} \left| \left(t^m \frac{d^n}{dt^n} f \right)(t) \right| < \infty, m, n = 1, 2, 3 \ldots \right\}.$$

Note: The operators will act on the Schwartz space unless stated otherwise.

Definition 1.3.2. If $f : \mathbb{R} \to \mathbb{C}$, we define

$$\begin{aligned} (Qf)(t) &= tf(t) \quad \text{(Position operator)} \\ (Pf)(t) &= -if'(t) \quad \text{(Momentum operator)} \\ A &= Q + iP \quad \text{(Annihilation operator)} \\ A^\star &= Q - iP \quad \text{(Creation operator)}. \end{aligned}$$

It is noteworthy that $A^\star$ is the formal adjoint of A. Indeed, we have

$$\begin{aligned} \langle Af, g\rangle &= \int (Af)(t)\,\overline{g(t)}\,dt \\ &= \int (tf(t) + f'(t))\,\overline{g(t)}\,dt \\ &= \int tf(t)\,\overline{g(t)}\,dt + \int f'(t)\,\overline{g(t)}\,dt \\ &= \int tf(t)\,\overline{g(t)}\,dt - \int f(t)\,\overline{g'(t)}\,dt \quad \text{(integration by parts)} \\ &= \int f(t)\,\overline{(tg(t) - g'(t))}\,dt \\ &= \int f(t)\,\overline{(A^\star g)(t)}\,dt \\ &= \langle f, A^\star g\rangle. \end{aligned}$$

The *harmonic oscillator* is

$$H = P^2 + Q^2.$$

Definition 1.3.3. The *commutator* of two operators X, and Y (defined on the suitable spaces) is defined by

$$[X, Y] = XY - YX.$$

Definition 1.3.4. The *Hermite functions* $\{\varphi_n\}_{n=0}^{\infty}$ are defined by

$$\varphi_0(x) = \pi^{-1/4} e^{-x^2/2}, \quad \text{and} \quad \varphi_n = \frac{1}{\sqrt{2n}} A^\star \varphi_{n-1}, \qquad n = 1, 2, \ldots.$$

The following are some basic facts about Hermite functions, see [11].

Lemma 1.5.

$$H\varphi_n = (2n+1)\varphi_n, \qquad n = 0, 1, 2, \ldots.$$

Proof. We first compute the commutators of the operators P and Q, and H and $A^\star$. Let $f \in \mathcal{S}$. Then $QPf(t) = -itf'(t)$, and $PQf(t) = -i(tf'(t) + f(t))$, so

$$[P, Q] = PQ - QP = -iI.$$

We have

$$A^\star A = (Q - iP)(Q + iP) = Q^2 + P^2 - i[P, Q] = H - I,$$

and

$$AA^\star = (Q + iP)(Q - iP) = Q^2 + P^2 + i[P, Q] = H + I.$$

It is clear from the above two equations that H is formally self-adjoint, and

$$[H, A^\star] = (A^\star A + 1)A^\star - A^\star(AA^\star - 1) = 2A^\star.$$

We now proceed by induction. Since $A\varphi_0 = 0$, it follows that $H\varphi_0 = (A^\star A + 1)\varphi_0 = \varphi_0$. Now assume $n \geq 1$ and $H\varphi_{n-1} = (2n-1)\varphi_{n-1}$. Then

$$\begin{aligned} H\varphi_n &= \frac{1}{\sqrt{2n}} H A^\star \varphi_{n-1} \\ &= \frac{1}{\sqrt{2n}} ([H, A^\star] + A^\star H)\varphi_{n-1} \\ &= \frac{1}{\sqrt{2n}} (2 + (2n-1)) A^\star \varphi_{n-1} \\ &= (2n+1)\varphi_n. \end{aligned}$$

□

Lemma 1.6.

$$\mathcal{F}\varphi_n = (-i)^n \varphi_n, \qquad n = 0, 1, 2, \ldots.$$

Proof. We compute

$$\begin{aligned} (\mathcal{F}Qf)(\xi) &= \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} x f(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int \frac{\partial}{\partial \xi} \left(i e^{-i\xi x} \right) f(x) dx \\ &= i \frac{d}{d\xi} \left(\frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x) \right) dx \\ &= (-P\mathcal{F}f)(\xi). \end{aligned}$$

$$\begin{aligned} (\mathcal{F}Pf)(\xi) &= \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f'(x) dx \\ &= i \frac{1}{\sqrt{2\pi}} \int -i\xi e^{-i\xi x} f(x) dx \quad \text{(integration by parts)} \\ &= \xi \left(\frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x) \right) dx \\ &= (Q\mathcal{F}f)(\xi). \end{aligned}$$

It follows that $\mathcal{F}A^\star = -iA^\star\mathcal{F}$, and $\mathcal{F}A = iA\mathcal{F}$. We now proceed by induction. We know that $\mathcal{F}\varphi_0 = \varphi_0$. Assume $\mathcal{F}\varphi_{n-1} = (-i)^{n-1}\varphi_{n-1}$. Thus, we have

$$\mathcal{F}\varphi_n = \frac{1}{\sqrt{2n}}\mathcal{F}A^\star\varphi_{n-1} = \frac{1}{\sqrt{2n}}(-iA^\star\mathcal{F}\varphi_{n-1}) = (-i)^n\varphi_n.$$

□

Lemma 1.7. $\|\varphi_n\| = 1$ *for* $n = 0, 1, 2, \ldots$.

Proof. Again we use induction.

$$\|\varphi_0\|^2 = \frac{1}{\sqrt{\pi}}\int e^{-x^2}dx = 1.$$

Now assume $n > 0$ and $\|\varphi_{n-1}\| = 1$. Then

$$\begin{aligned}\|\varphi_n\|^2 &= \frac{1}{2n}\langle A^\star\varphi_{n-1}, A^\star\varphi_{n-1}\rangle \\ &= \frac{1}{2n}\langle AA^\star\varphi_{n-1}, \varphi_{n-1}\rangle \\ &= \frac{1}{2n}\langle (H+1)\varphi_{n-1}, \varphi_{n-1}\rangle \\ &= \frac{1}{2n}\langle 2n\varphi_{n-1}, \varphi_{n-1}\rangle \\ &= 1.\end{aligned}$$

□

Theorem 1.8. $\{\varphi_n\}_{n=0}^\infty$ *is a complete orthonormal set in* $L^2(\mathbb{R})$.

Proof. By Lemma 1.5 and the formal self-adjointness of H, we have

$$(2k+1)\langle\varphi_k, \varphi_l\rangle = \langle H\varphi_k, \varphi_l\rangle = \langle\varphi_k, H\varphi_l\rangle = (2l+1)\langle\varphi_k, \varphi_l\rangle.$$

So $k \neq l \implies \langle \varphi_k, \varphi_l \rangle = 0$. Together with Lemma 1.7, orthonormality follows. Let S be the linear span of $\{\varphi_n\}_{n=0}^{\infty}$. Since $\varphi_n(x)e^{x^2/2}$ is a polynomial of degree n, it follows that S is also the linear span of the functions $\{f_n\}_{n=0}^{\infty}$, where $f_n(x) = x^n e^{-x^2/2}$. Now

$$\begin{aligned} \|f_n\|^2 &= \int x^{2n} e^{-x^2} dx \\ &= \int_0^\infty y^{n-1/2} e^{-y} dy \quad (y = x^2, dy = 2x dx) \\ &= \Gamma(n + 1/2) \leq n!, \end{aligned}$$

so

$$\sum_{n=0}^{\infty} \left\| \frac{(i\xi)^n}{n!} f_n \right\| < \infty.$$

Since absolutely convergent series in Banach spaces are convergent, it follows that for each $\xi \in \mathbb{R}$, the series

$$e^{i\xi x - x^2/2} = \sum_{n=0}^{\infty} \frac{(i\xi)^n}{n!} f_n$$

converges in $L^2(\mathbb{R})$. In particular, for each $\xi \in \mathbb{R}$, the function $e^{i\xi x - x^2/2}$ belongs to the closure of S. So if $f \in S^\perp$ then for all $\xi \in \mathbb{R}$,

$$\int f(x) e^{-x^2/2 - i\xi x} dx = 0.$$

By the Fourier inversion formula, it follows that $f(x)e^{-x^2/2} = 0$, and hence $f = 0$. This proves completeness. □

A useful identity for the Hermite functions is *Mehler's formula*.

Lemma 1.9. *For $|w| < 1$ we have*

$$\sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(y) w^n = \pi^{-\frac{1}{2}} (1 - w^2)^{-\frac{1}{2}} \, e^{-\frac{1}{2}\frac{1+w^2}{1-w^2}(x^2+y^2) + \frac{2w}{1-w^2} xy}.$$

For the proof, see [11].

1.4 The Bargmann transform and its properties

The *Fock space* is the Hilbert space $\mathcal{H}$ of all entire functions F on $\mathbb{C}$ such that

$$\|F\|_{\mathcal{H}}^2 = \int_{\mathbb{C}} |F(w)|^2 \frac{e^{-|w|^2/2}\,du\,dv}{\sqrt{4\pi}} < \infty \quad (w = u + iv).$$

Let

$$e_n = \frac{w^n}{\sqrt{2^n n! \pi^{1/2}}}.$$

Then $\{e_n\}$ forms a complete orthonormal set in $\mathcal{H}$. For the proof, see [4].

For a Schwartz class function f, the *Bargmann transform* of f is defined by

$$Bf(w) = \frac{e^{-w^2/4}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{xw} e^{-x^2/2} f(x)\,dx.$$

Let $f \in \mathcal{H}$, we define

$$Zf(z) = \frac{df}{dz} \quad \text{and} \quad Z^\star f(z) = zf(z).$$

Observe that $Z^\star e_{n-1} = \sqrt{2n} e_n$.

Lemma 1.10.

$$ZB = BA \quad \textit{and} \quad Z^\star B = BA^\star.$$

Proof. Clearly

$$ZBf(w) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \left(x - \frac{w}{2}\right) e^{xw - \frac{w^2}{4} - \frac{x^2}{2}} f(x)\,dx. \tag{1.3}$$

This gives

$$ZBf(w) = BQf(w) - \frac{1}{2} Z^\star Bf(w). \tag{1.4}$$

Equation (1.3) can also be written as

$$ZBf(w) = -\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x) \frac{d}{dx} e^{xw - \frac{w^2}{4} - \frac{x^2}{2}}\, dx + \frac{1}{2} Z^\star Bf(w).$$

Integration by parts yields

$$ZBf(w) = iBPf(w) + \frac{1}{2} Z^\star Bf(w). \tag{1.5}$$

Combining equations (1.4) and (1.5) gives us the desired result. □

Lemma 1.11.

$$B\varphi_n(w) = e_n. \tag{1.6}$$

Proof. Observe that $B\varphi_0 = e_0$. Assume $B\varphi_{n-1}(w) = e_{n-1}$. Therefore, by the previous lemma

$$\begin{aligned} B\varphi_n(w) &= \frac{1}{\sqrt{2n}} BA^\star \varphi_{n-1} \\ &= \frac{1}{\sqrt{2n}} Z^\star B\varphi_{n-1} \\ &= e_n. \end{aligned}$$

□

This also shows that B is an isometry from $L^2(\mathbb{R}) \to \mathcal{H}$. Therefore, by the Cauchy integral formula for derivatives, we have $Bf(w) = \sum_{n=0}^{\infty} c_n w^n$ where

$$c_n = \frac{1}{2\pi i} \int_{\gamma_n} \frac{Bf(w)}{w^{n+1}}\, dw,$$

and

$$\begin{aligned}
\langle f, \varphi_n \rangle &= \langle Bf, B\varphi_n \rangle \\
&= \int\int \left(\sum_{k=0}^{\infty} c_k w^k \right) \overline{\left(\frac{w^n}{\sqrt{2^n n! \pi^{1/2}}} \right)} \frac{e^{-r^2/2}\, du\, dv}{\sqrt{4\pi}} \\
&= \frac{c_n}{\sqrt{2^n n! \pi^{1/2}}} \int\int r^{2n} \frac{e^{-r^2/2}\, du\, dv}{\sqrt{4\pi}} \\
&= \sqrt{2^n n! \pi^{1/2}} c_n.
\end{aligned} \tag{1.7}$$

Lastly, in this section, we point out some estimates for B which will be used in the next chapter. Let $a \in (0, 1)$. Let $w = re^{i\theta}$ and $\mu = \frac{1-a}{1+a}$. It is shown in [13] that if $f \in E(a, a)$ then

$$|Bf(w)| \leq C\sqrt{\frac{2}{1+a}} \exp \frac{(\mu + (1-\mu)\sin^2\theta)r^2}{4}, \tag{1.8}$$

$$|Bf(w)| \leq C\sqrt{\frac{2}{1+a}} \exp \frac{(\mu + (1-\mu)\cos^2\theta)r^2}{4}. \tag{1.9}$$

Furthermore, it is shown, using the Phragmén-Lindelöf principle that

$$|Bf(w)| \leq C\sqrt{\frac{2}{1+a}} \exp \left(\sqrt{\mu} \sin(|2\theta|) \frac{r^2}{4} \right), \tag{1.10}$$

for $\theta_0 \leq \theta - \frac{k\pi}{2} \leq \frac{\pi}{2} - \theta_0$, $k = 0, 1, 2, 3$, where $\theta_0 = \tan^{-1}(\sqrt{\mu})$. In Chapter 3 we shall generalize the estimates (1.8-1.10) for $f \in E(a, b)$.

1.5 Laplace's method

Laplace's method can be stated as follows (see [3, Section 2.4]).

Theorem 1.12. *Let $g : [\alpha, \beta] \to \mathbb{R}$ be a continuous function, and let $h : [\alpha, \beta] \to \mathbb{R}$ be a twice continuously differentiable function. If $t_0 \in [\alpha, \beta]$ is the unique point of* *maxima of* h, *and* $h''(t_0) < 0$. *Then*

$$\int_\alpha^\beta g(t)e^{xh(t)}dt \sim g(t_0)e^{xh(t_0)}\left[\frac{-2\pi}{xh''(t_0)}\right]^{1/2}, \quad x \to \infty.$$

1.6 The Theorem of Phragmén and Lindelöf

Let G be a bounded domain in the complex plane, ∂G the boundary of G, and $z_0 \in \partial G$. Let $U(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. Let f be an analytic function on G. We denote the quantity $\lim_{\delta\to 0}\sup_{z\in U_\delta(z_0)\cap G}|f(z)|$ by $\limsup_{z\to z_0}|f(z)|$. If $\limsup_{z\to z_0}|f(z)| \leq M$, for some $M > 0$ and all $z_0 \in \partial G$, then we shall say that $|f(z)| \leq M$ on the boundary of G. The *maximum modulus principle* for functions analytic in G can be stated as follows.

If $|f(z)| \leq M$ on the boundary of G, then $|f(z)| \leq M$ on G.

The following example shows that the boundedness of the domain in the *maximum modulus principle* is crucial. Let

$$G = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \in (-\pi/2, \pi/2)\} \text{ and } f(z) = \exp(\exp z).$$

Then f is analytic on G. If $z \in \partial G$ then

$$|f(z)| = |\exp(\exp(x \pm i\pi/2))| = |\exp(\pm ie^x)| = 1.$$

However, f is unbounded on the real axis.

A *maximum modulus principle* for a function analytic in a sector of the complex plane is due to Phragmén and Lindelöf, and can be stated as follows (see [2, Chapter XI, Corollary 4.2]).

Theorem 1.13. *Let $a \geq \frac{1}{2}$ and put*

$$G = \{z : |\arg z| < \frac{\pi}{2a}\}.$$

Suppose that f is analytic on G and there is a constant M such that $|f(z)| \leq M$ on ∂G. If there are positive constants P and $b < a$ such that

$$|f(z)| \leq P \exp(|z|^b)$$

for all z with $|z|$ sufficiently large, then $|f(z)| \leq M$ for all z in G.

1.7 Outline of the thesis

The rest of the thesis is organized as follows.

In ***Chapter 2***, under the same assumptions as in Theorem 1.4, we show that a certain combination of Hermite coefficients has a better rate of decay. To prove this result, we use the estimates (1.8), (1.9), (1.10), and Theorem 1.12.

In ***Chapter 3***, an extension of Theorem 1.4 is discussed. To prove Theorem 1.3, Hardy applied the Phragmén-Lindelöf principle to the Fourier transform (of f). Inspired by Hardy, Vemuri applied the Phragmén-Lindelöf principle to the Bargmann transform to prove Theorem 1.4. Garg and Thangavelu proved their results using the Fourier-Wigner transform and the vector valued Bargmann transform. To prove our result, we adapt Vemuri's technique. As a byproduct, we obtain a strong Gaussian bound for the Bargmann transform of a function $f \in E(a, b)$.