# Chapter 4

## Mess-Less Numerical Scheme Based on Matrices for Integro-Differential Equation

### 4.1 Introduction

In this chapter, we propose a numerical wavelet and polynomials method for solving system of volterra integro-differential equations (SSVIDEs). System of singular voltera integro-differential equations (SSVIDEs) arises in many branches of science and engineering such as biological models, electromagnetism, fluid dynamics, financial economics, heat and mass transfer etc. The main application of system of integro-differential equations (SIDEs) in Wilson-Cowan model ((1972), (1973)) which describes the evolution of excitatory and inhibitory activities in a coupled neuronal network. First Amari (1977) gave the following mathematical expression for the Wilson-Cowan model:

$$\frac{\partial E}{\partial t} = -E + \int w_{ee}(x-x')f(E-\theta_1)dx' - \int w_{ie}(x-x')f(E-\theta_2)dx' + \psi_1(x,t)$$

$$\tau\frac{\partial I}{\partial t} = -I + \int w_{ei}(x-x')f(E-\theta_1)dx' - \int w_{ii}(x-x')f(E-\theta_2)dx' + \psi_2(x,t)$$

with initial conditions $E(x,0)=E_0(x),\ I(x,0)=I_0(x)$ and $x\in\mathbb{R},\ t\geq 0$. Here $E(x,t)$ and $I(x,t)$ represent the activities of a population of excitatory and inhibitory neurons respectively. The function $w_{ij}$ describes the strength of connection from cell type $k$ to $j$ and $f$ is firing rate function or Heaviside function (in some case). The positive parameters $\theta_1$, $\theta_2$ and $\tau$ denote the inhibitory time inputs. Pinto and Ermentrout (2001 a, 2001 b) modified the Wilson-Cowan model as follows:

$$\frac{\partial u}{\partial t} = -u - v + \int w(x - x')H(u - \theta)dx'$$

$$\tau \frac{\partial v}{\partial t} = \in (\beta u - v),$$

where $u$ is the activity excitatory neurons and $v$ could represent spike frequency adaption, synaptic depression or some other slow process that the excitation of the network. Existence of solutions to the above systems using topological shooting and Evans function method is already discussed in Gutkin et al. (2003). In this chapter, we will not discuss exactly the numerical solutions of Wilson-Cowan model but in general to SSVIDEs.

Singh et al. (2009, 2010) have developed numerical schemes for integro-differential equations (IDEs) using operational matrix concepts based on wavelets and orthogonal polynomial approximations as well. System of voltera integro-differential equations (SVIDEs) are discused mainly for non-singular and linear type using hybrid functions, Haar wavelets and Runge-Kutta methods in Maleknejad and Mirzaee (2003), Maleknejad and Kajani (2004), Maleknejad and Shahrezee (2004). In Maleknejad and Shamloo (2008), numerical solution of system of singular volterra integral equations (SVIEs) of convolution type are considered using operational matrices . In this chapter, we promote and analyse the mess-less scheme based on Legendre wavelet operational matrices and Bernstein polynomials operational matrices for solving the following SSVIDEs.

$$\sum_{j=1}^{2} a_{ij} D^{\alpha} y_j = \sum_{k=1}^{2} b_{ik} f_k(x) + \lambda_i \sum_{l=1}^{2} \int_0^x \frac{c_{il} y_l(t)}{\sqrt{x-t}} \quad (4.1.1)$$

with initial conditions $y_i(0) = d_i$, where $i = 1,2$. We assume that the functions $f_1(x)$ and $f_2(x)$ are integrable functions.

## 4.2 Operational Matrices

### 4.2.1 Bernstein Polynomials and Operational Matrices

The Bernstein basis polynomials of degree $n$ form a complete basis over the interval $[0,1]$, are defined by

$$B_{j,n}(t) = \binom{n}{j} t^j (1-t)^{n-j} \qquad \text{for} \quad j = 0,1,...,n, \tag{4.2.1}$$

where $t$ is a parameter. However, the Bernstein basis polynomials can be generalized to cover an arbitrary interval [a, b] by normalizing $t$ over the interval [a, b], i.e. $t = \frac{(x-a)}{(b-a)}$, which lead to the following

$$B_{j,n}(t) = \binom{n}{j} \frac{(x-a)^j (b-x)^{n-j}}{(b-a)^n} \qquad \text{for} \quad j = 0,1,...,n. \tag{4.2.2}$$

These polynomials satisfy symmetry $B_{j,n}(t) = B_{n-j,n}(1-x)$, positivity $B_{j,n}(t) \geq 0$, form a partition of unity $\sum_{j=0}^{n} B_{j,n}(x) = 1$ on the defining interval [a, b]. The explicit representation of the orthogonal Bernstein basis polynomial, denoted by $\hat{B}(x) = \phi_{j,n}(x)$ is obtained by using Gram Schmidt orthonormalization process on $B_{j,n}(t)$.

If function $f \in L^2[0,1]$, then

$$f(t) = \lim_{n\to\infty} \sum_{j=0}^{n} c_{jn} \hat{b}_{jn}(t). \tag{4.2.3}$$

Here $c_{jn} = \left\langle f, \hat{b}_{jn} \right\rangle$, where $\langle , \rangle$ is the inner product over $L^2\left[0,1\right]$. If the series (4.2.3) is truncated at $n = m$, then we denote

$$f(t) \approx \sum_{j=0}^{m} c_{jn}\hat{b}_{jn}(t) = C^T \hat{B}(t), \tag{4.2.4}$$

where $C = [c_{0m}, c_{1m}, ..., c_{mn}]^T$, $\hat{B}(t) = [c_{0m}, c_{1m}, ..., c_{mn}]^T$. In this case the orthogonal Bernstein operational matrix of integration and almost operational matrix of integration of order $(m+1)\times(m+1)$ are given by

$$\int_0^x \hat{B}(t)dt = P_6^B \hat{B}(x), \tag{4.2.5}$$

$$\int_0^x \frac{\hat{B}(t)}{\sqrt{x-t}}dt = S_6^B \hat{B}(x), \tag{4.2.6}$$

respectively. For $n = 5$ the explicit expressions for $P_6^B$ and $S_6^B$ via five orthogonal polynomials for Eqs. (4.2.5) and (4.2.6) are presented in the chapters (Singh et. al. (2009), Singh et. al. (2010), Vineet and Postnikov (2013)) respectively.

### 4.2.2 Legendre Wavelets Polynomials and Operational Matrices

The Legendre wavelet basis functions

$$\psi_{nm}(t) = \begin{cases} \sqrt{2m+1}\, 2^{k/2} P_m\left(2^k t - \hat{n}\right), & for\ \frac{\hat{n}-1}{2^n} \le t < \frac{\hat{n}+1}{2^n}, \\ 0, & otherwise, \end{cases} \tag{4.2.7}$$

where $P_m(t)$ are Legendre polynomials of degree $m$ defined on [-1, 1] by the following recurrence relation for $m = 1, 2, 3, ...$:

$$P_0(t) = 1, P_1(t) = t \text{ and } P_{m+1}(t) = \frac{2m+1}{m+1} t P_m(t) - \frac{m}{m+1} P_{m-1}(t).$$

The other arguments take the values within the following range: $\hat{n} = 2n-1,\ n = 1, 2, 3, ..., 2^{k-1},\ k = 1, 2, 3, ..., m$. A function $g(x)$ defined over [0, 1] may be expanded by Legendre wavelets as:

$$g(x) \approx \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\Psi_{nm}(x), \tag{4.2.8}$$

where $C_{nm} = \langle g, \Psi_{nm} \rangle$ and $\langle , \rangle$ is inner product on $L^2[0,1]$. If the infinite series Eq. (4.2.8) is truncated, then it can be written as

$$g(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} C_{nm}\Psi_{nm}(x) = C^T\Psi(x), \tag{4.2.9}$$

where the matrices $C$ and $\Psi$ are given by,

$$C = [C_{10}, C_{11}, ..., C_{1M}, C_{20}, C_{21}, ..., C_{2M}, ..., C_{2^{k-1}0}, ..., C_{2^{k-1}M}],$$

and

$$\Psi(x) = [\Psi_{10}, \Psi_{11}, ..., \Psi_{1M}, \Psi_{20}, \Psi_{21}, ..., \Psi_{2M}, ..., \Psi_{2^{k-1}0}, ..., \Psi_{2^{k-1}M}].$$

Similarly Legendre wavelets operational matrices and almost operational matrices of integration are defined as:

$$\int_0^x \Psi(t)dt = P_6^L\Psi(x), \tag{4.2.10}$$

and

$$\int_0^x \frac{\Psi(t)}{\sqrt{x-t}}dt = S_6^L\Psi(x) \tag{4.2.11}$$

respectively. These both matrices are of order $(2^{k-1}M)\times(2^{k-1}M)$.

The explicit expressions for the Legendre wavelet functions and operational matrices can be found in Yousefi (2006), Razzaghi and Yousefi (2011) for $M = 3$ and $k = 2$.

## 4.3 Mess-Less Scheme Derivation

We started this chapter with the objective of solving the system of singular voltera integro-differential (SSVIDEs) given by Eq. (4.1.1). Let both the known functions $f_1(x)$ and $f_2(x)$ involved in Eq. (4.1.1) are approximated in terms of orthogonal Bernstein's polynomials as

$$f_1(x) \approx F_1^T \Phi(x) \text{ and } f_2(x) \approx F_2^T \Phi(x), \tag{4.3.1}$$

where

$\Phi(x) = [\Phi_0(x), \Phi_1(x), \Phi_2(x), \Phi_3(x), \Phi_4(x), \Phi_5(x)]^T$, $F_i = [F_{i0}, F_{i1}, F_{i2}, F_{i3}, F_{i4}, F_{i5}]^T$, $i \in \{1,2\}$ and the components $F_{ij}$ in column vector $F_i$ are determined by the expression:

$$F_{ij} = \int_0^1 f_i(x)\Phi_j(x)dx, \quad 0 \le j \le 5.$$

With similar approach, if we approximate derivative terms as

$$\frac{dy_1}{dx} \simeq C_1^T \Phi(x) \text{ and } \frac{dy_2}{dx} \simeq C_2^T \Phi(x) \tag{4.3.2}$$

then integration of the Eq. (4.3.2) between $0$ and $x$ gives

$$y_1(x) - d_1 = C_1^T \int_0^x \Phi(x)dx \text{ and } y_2(x) - d_2 = C_2^T \int_0^x \Phi(x)dx.$$

Approximating the constants $d_1$ and $d_2$ as $d_1 \simeq A^T \Phi(x) = \sum_{i=0}^{5} A_i \Phi_i$ and $d_2 \simeq B^T \Phi(x) = \sum_{i=0}^{5} B_i \Phi_i$ with $A_i = \int_0^1 d_1 \ \Phi_i(x)\ dx$ and $B_i = \int_0^1 d_2 \ \Phi_i(x)\ dx$ , $0 \le i \le 5$, the above equation may be written as

$$y_1(x) \simeq A^T \Phi(x) + C_1^T P\Phi(x), \text{ and } y_2(x) \simeq B^T \Phi(x) + C_2^T P\Phi(x), \tag{4.3.3}$$

with $\int_0^x \Phi(x)\,dx \simeq P\Phi(x)$.

Using the Eqs. (4.3.1), (4.3.2) and (4.3.3) in Eq. (4.1.1) with $\alpha=1$, $\lambda_1=1, i=1$ and $\int_0^x \frac{\Phi(x)}{\sqrt{x-t}} \simeq S\Phi(x)$ we have,

$$\begin{aligned} C_1^T(a_{11}\Phi(x)-c_{11}PS\Phi(x))+C_2^T(a_{12}\Phi(x)-c_{12}PS\Phi(x)) \\ =b_{11}F_1^T\Phi(x)+b_{12}F_2^T\Phi(x)+c_{11}A^TS\Phi(x)+c_{12}B^TS\Phi(x). \end{aligned} \tag{4.3.4}$$

Since $\Phi(x)$ is non zero function, we can write Eq. (4.3.4) as

$$C_1^T(a_{11}I_{6\times6}-c_{11}PS)+C_2^T(a_{12}I_{6\times6}-c_{12}PS)=b_{11}F_1^T+b_{12}F_2^T+c_{11}A^TS+c_{12}B^TS. \tag{4.3.5}$$

Similarly for $\lambda_2=1, i=2$, the Eq. (4.1.1) becomes

$$C_1^T(a_{21}I_{6\times6}-c_{21}PS)+C_2^T(a_{22}I_{6\times6}-c_{22}PS)=b_{21}F_1^T+b_{22}F_2^T+c_{21}A^TS+c_{22}B^TS. \tag{4.3.6}$$

Solving Eqs. (4.3.5) and (4.3.6) for $C_1^T, C_2^T$ we obtain:

$$\begin{aligned} C_1^T =& [(b_{11}F_1^T+b_{12}F_2^T+c_{11}A^TS+c_{12}B^TS)(a_{12}I_{6\times6}-c_{12}PS)^{-1} \\ &-(b_{21}F_1^T+b_{22}F_2^T+c_{21}A^TS+c_{22}B^TS)(a_{22}I_{6\times6}-c_{22}PS)^{-1}] \\ &\times[(a_{11}I_{6\times6}-c_{11}PS)(a_{12}I_{6\times6}-c_{12}PS)^{-1}-(a_{21}I_{6\times6}-c_{21}PS)(a_{22}I_{6\times6}-c_{21}PS)^{-1}]^{-1}, \end{aligned} \tag{4.3.7}$$

and

$$\begin{aligned} C_2^T =& [(b_{11}F_1^T+b_{12}F_2^T+c_{11}A^TS+c_{12}B^TS)(a_{11}I_{6\times6}-c_{11}PS)^{-1} \\ &-(b_{21}F_1^T+b_{22}F_2^T+c_{21}A^TS+c_{22}B^TS)(a_{21}I_{6\times6}-c_{21}PS)^{-1}] \\ &\times[(a_{12}I_{6\times6}-c_{12}PS)(a_{11}I_{6\times6}-c_{11}PS)^{-1}-(a_{22}I_{6\times6}-c_{22}PS)(a_{21}I_{6\times6}-c_{21}PS)^{-1}]^{-1}. \end{aligned} \tag{4.3.8}$$

The approximated numerical solutions $y_1(x)$ and $y_2(x)$ for the system of Eqs. (4.1.1) can be easily achieved by substitution of $C_1^T$ and $C_2^T$ (obtained from Eqs. (4.3.7) and (4.3.8)) in Eq. (4.3.3).

**Table 4.1.**

| List of notations | | |
|---|---|---|
| General symbol | Usage of Legendre wavelets | Usage of Bernstein Polynomials |
| $\Phi(x)$ | $\Psi_{nm}(x)$ | $\hat{B}(x)$ |
| $P$ | $P_6^L$ | $P_6^B$ |
| $S$ | $S_6^L$ | $S_6^B$ |
| $F_1$ | $F_1^L$ | $F_1^B$ |
| $F_2$ | $F_2^L$ | $F_2^B$ |
| $A$ | $A_6^L$ | $A_6^B$ |
| $y_1$ | $y_1^L$ | $y_1^B$ |
| $y_2$ | $y_2^L$ | $y_2^B$ |
| $C_1^T$ | $C_1^{T,L}$ | $C_1^{T,B}$ |
| $C_2^T$ | $C_2^{T,L}$ | $C_2^{T,B}$ |

## 4.4 Convergence Analysis of Legendre Wavelets and Bernstein Polynomials

**Theorem 4.4.1.** Let $D^{\alpha} y_M(x)$ is the Legendre wavelets approximation of $D^{\alpha} y(x)$. If $D^{\alpha} y(x)$ is bounded together with its second order derivative , then the upper bound of error may be given by

$$\left\| D^{\alpha} y(x) - D^{\alpha} y_M(x) \right\|_E^2 \leq \frac{A^2}{2^{k+1}} F_3\left(\frac{-1}{2} + M\right),$$

where, $\|y(x)\|_E = (\int_0^1 |y(x)|^2\, dx)^{\frac{1}{2}}$, $D = \dfrac{d}{dx}$ and $A$ is positive constant.

**Proof.**

Let

$$D^\alpha y(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{nm}\Psi_{nm}(x), \tag{4.4.1}$$

$$D^\alpha y_M(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} C_{nm}\Psi_{nm}(x), \tag{4.4.2}$$

where $D^\alpha y_M(x)$ be the following approximation of $D^\alpha y(x)$.

Grouping Eqs. (4.4.1) and (4.4.2), we get

$$D^\alpha y(x) - D^\alpha y_M(x) = \sum_{n=(2^{k-1}+1)}^{\infty}\ \sum_{m=(M+1)}^{\infty} C_{nm}\Psi_{nm}(x). \tag{4.4.3}$$

Let $D^\alpha y(x)$ be a function defined on [0, 1), such that

$$\left|D^{\alpha+2} y(x)\right| \le A, \tag{4.4.4}$$

where $A$ is a positive constant.

Since $\{\Psi_{nm}(x)\}$ are orthonormal on [0, 1) so we have

$$\int_0^1 \Psi_{nm}(x)\Psi_{nm}(x)^T\, dx = I, \tag{4.4.5}$$

where $I$ is identity matrix.

$$\begin{aligned}\left\|D^\alpha y(x) - D^\alpha y_M(x)\right\|_E^2 &= \int_0^1 (D^\alpha y(x) - D^\alpha y_M(x))^2\, dx \\ &= \int_0^1 \Big(\sum_{n=(2^{k-1}+1)}^{\infty}\ \sum_{m=(M+1)}^{\infty} C_{nm}\Psi_{nm}(x)\Big)^2 dx \\ &= \sum_{n=(2^{k-1}+1)}^{\infty}\ \sum_{m=(M+1)}^{\infty} C_{nm}^2.\end{aligned}$$

Where,

$$
\begin{aligned}
C_{nm} &= \left\langle D^{\alpha} y(x), \Psi_{nm}(x) \right\rangle \\
&= \int_0^1 D^{\alpha} y(x) \Psi_{nm}(x) dx \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} D^{\alpha} y(x) (2m+1)^{\frac{1}{2}} 2^{\frac{k}{2}} P_m(2^k x - \hat{n}) dx \\
&= (2m+1)^{\frac{1}{2}} 2^{\frac{k}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} D^{\alpha} y(x) P_m(2^k x - \hat{n}) dx \\
&= \frac{(2m+1)^{\frac{1}{2}} 2^{\frac{k}{2}}}{2^k} \int_{-1}^{1} D^{\alpha} y(t) P_m(t) dt \\
&= \frac{(2m+1)^{\frac{1}{2}} 2^{\frac{k}{2}}}{2^k} \int_{-1}^{1} D^{\alpha} y(t) \frac{P_{m+1}(t) - P_{m-1}}{2m+1} dx \\
&= [\frac{1}{2^k (2m+1)}]^{\frac{1}{2}} \int_{-1}^{1} D^{\alpha} y(t) d(P_{m+1}(t) - P_{m-1}(t)) dt \\
&= -[\frac{1}{2^k (2m+1)}]^{\frac{1}{2}} \int_{-1}^{1} D^{\alpha+1} y(t) (P_{m+1}(t) - P_{m-1}(t)) dt \\
&= -[\frac{1}{2^k (2m+1)}]^{\frac{1}{2}} \int_{-1}^{1} D^{\alpha+1} y(t) d(\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}}{2m-1}) dt \\
&= [\frac{1}{2^k (2m+1)}]^{\frac{1}{2}} \int_{-1}^{1} D^{\alpha+2} y(t) (\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}}{2m-1}) dt.
\end{aligned}
$$

Thus

$$\begin{aligned}
\left|C_{nm}\right|^2 &= \frac{1}{2^k(2m+1)}\left|\int_{-1}^{1} D^{\alpha+2}y(t)\left(\frac{(2m-1)P_{m+2}(t)-(4m+2)P_m(t)+(2m+3)P_{m-2}(t)}{(2m+3)(2m-1)}dt\right)\right|^2 \\
&\le \frac{1}{2^k(2m+1)}\int_{-1}^{1}\left|D^{\alpha+2}y(t)\right|^2 dt\int_{-1}^{1}\left|\left(\frac{(2m-1)P_{m+2}(t)-(4m+2)P_m(t)+(2m+3)P_{m-2}(t)}{(2m+3)(2m-1)}\right|^2 dt \\
&\le \frac{A^2}{2^k(2m+1)}\int_{-1}^{1}\left|\left(\frac{(2m-1)P_{m+2}(t)-(4m+2)P_m(t)+(2m+3)P_{m-2}(t)}{(2m+3)(2m-1)}\right|^2 dt \\
&< \frac{A^2}{2^k(2m+1)}\int_{-1}^{1}\frac{(2m-1)^2P_{m+2}^2(t)+(4m+2)^2P_m^2(t)+(2m+3)^2P_{m-2}^2(t)}{(2m+3)^2(2m-1)^2}dt \\
&= \frac{A^2}{2^k(2m+1)(2m-1)^2(2m+3)^2}\left[\frac{2(2m-1)^2}{2m+5}+\frac{8(2m+1)^2}{2m+1}+\frac{2(2m+3)^2}{2m-3}\right] \\
&< \frac{A^2}{2^k(2m+1)(2m-1)^2(2m+3)^2}\frac{12(2m+3)^2}{(2m-3)} \\
&< \frac{3A^2}{2^{k-2}(2m-3)^4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{m=M+1}^{\infty} C_{nm}^2 &< \sum_{m=M+1}^{\infty}\frac{3A^2}{2^{k-2}(2m-3)^4} \\
&= \frac{3A^2}{2^{k-2}}\sum_{m=M+1}^{\infty}\frac{1}{(2m-3)^4} \\
&= \frac{3A^2}{2^{k-2}}\left(\frac{1}{96}\right)F_3\left(\frac{-1}{2}+M\right).
\end{aligned}$$

Finally, we get

$$\left\|D^{\alpha}y(x)-D^{\alpha}y_M(x)\right\|_E^2 \le \frac{A^2}{2^{k+1}}F_3\left(\frac{-1}{2}+M\right).$$

**Theorem 4.4.2.** Let $f$ be a real valued function bounded by $M$ on the interval [0, 1]. Then for each point $x$ of continuity of $f$, $B_n(f)(x) \to f(x)$ as $n \to \infty$. If $f$ is continuous on [0, 1], then the Bernstein polynomial $B_n(f)$ tends uniformly to $f$ as $n \to \infty$ with $x$ a point of differentiability of $f$, $B_n^{'}(f)(x) \to f^{'}(x)$ as $n \to \infty$. If $f$ is continuously differentiable on [0, 1] then $B_n^{'}(f)$ tends to $f^{'}$ uniformly as $n \to \infty$.

**Proof.** See (Richard and Liu (2014)).

## 4.5 Numerical Examples

In this section we consider the proposed method to find the numerical solution of Eq. (4.1.1) . We solve Eq. (4.1.1) as a special case for which $\alpha = 1$, $\lambda_i = 1, i \in \{1, 2\}$, $d_1 = 0$, $d_2 = 0$, and the coefficients are taken as:

$$\begin{array}{l} a_{11} = 1,\ a_{12} = -1,\ b_{11} = 1,\ b_{12} = 0,\ c_{11} = 1,\ c_{12} = 1,\ \lambda_1 = 1 \\ a_{21} = -1,\ a_{22} = -2,\ b_{21} = 0,\ b_{22} = 1,\ c_{11} = 1,\ c_{12} = 1,\ \lambda_1 = 1. \end{array} \tag{4.5.1}$$

In each graphs $E1L(t)$ and $E1B(t)$ are errors corresponding to the function $Y_1$ by using Legendre wavelets and Bernstein polynomials. $E2L(t)$ and $E2B(t)$ are errors corresponding to the function $Y_2$ by using Legendre wavelets and Bernstein polynomials.

The accuracy of the proposed method is demonstrated by calculating the errors $E_i(t)$ and average deviations $\sigma_i$ (also known as root mean square error (RMS)), for $i = 1, 2$ . They have been calculated by using the following rules:

$$E_i(t) = \text{ Exact solutions } (Y_i(t)) - \text{ Approximate solutions } (y_i(t)) \tag{4.5.2}$$

and

$$\sigma_i = \left( \frac{1}{N} \sum_{j=0}^{N} \left[ y_i(t) - Y_i(t) \right]^2 \right)^{\frac{1}{2}} = \left\| E_i(t) \right\|_{l^2}, \tag{4.5.3}$$

where $y_i(t)$ and $Y_i(t)$ are the approximate and exact values calculated at points $t_j$. Note that $\sigma_i$ is discrete $l^2$- norm of the error $E_i$ denoted by $\|E_i\|_{l^2}$. We also compute the continuous $L^2$- norm of the error $E_i$ by

$$\left\| E_i(t) \right\|_{L^2} = [\int_0^1 E_i^2(t)]^{\frac{1}{2}}. \tag{4.5.4}$$

For examples 4.5.1. and 4.5.2. we have calculated continuous error norms $\|E_i\|_{L^2}^L$, $\|E_i\|_{L^2}^B$ and discrete error norms $\|E_i\|_{l^2}^L$, $\|E_i\|_{l^2}^B$ and listed them in Table 4.2.. Here the symbol $\|E_i\|_{(*)}^L$ denotes the errors calculated by using Legendre wavelet and $\|E_i\|_{(*)}^B$ denotes the errors caculated by using Bernstein polynomial respectively with their respective norms for $i = 1, 2$.

**Example 4.5.1.** We choose $f_1 = \frac{3}{2} t^{\frac{1}{2}} - 2t - (\frac{3\pi t^2}{8} + \frac{16 t^{\frac{5}{2}}}{15})$ and $f_2 = \frac{-3}{2} t^{\frac{1}{2}} + 4t - \frac{16 t^{\frac{5}{2}}}{15}$ so that exact solutions for Eq. (4.1.1) by $Y_1 = t^{\frac{3}{2}}$ and $Y_1 = t^2$. The approximate solutions using Legendre wavelets and Bernstein polynomials are given by Eq. (4.3.3) with coefficients.

$C_1^{T,L} = [0.500051\ 0.17293\ -0.032067\ 0.915018\ 0.089367\ -0.003808]$,

$C_2^{T,L} = [0.353591\ 0.203994\ -0.000063\ 1.061117\ 0.204259\ 0.000064]$,

$C_1^{T,B} = [0.281393\ 0.508667\ 0.525384\ 0.507306\ 0.408898\ 0.245642]$,

$C_2^{T,B} = [0.157126\ 0.426911\ 0.57684\ 0.613314\ 0.539461\ 0.329795]$,

$C_1^{T,L}P_6^L(x) = [0.100053\ \ 0.074246\ \ 0.011163\ \ 0.465881\ \ 0.132317\ \ 5.768584\times10^{-3}]$,

$C_2^{T,L}P_6^L(x) = [0.058954\ \ 0.05104\ \ 0.013168\ \ 0.412592\ \ 0.153155\ \ 0.013185]$,

$C_1^{T,B}P_6^B(x) = [0.037304\ \ 0.135718\ \ 0.231778\ \ 0.277455\ \ 0.261821\ \ 0.164741]$

and

$C_2^{T,B}P_6^B(x) = [0.019342\ \ 0.088886\ \ 0.185052\ \ 0.2517\ \ 0.254279\ \ 0.165658]$.

Therefore, the approximate solutions corresponding to Legendre wavelets and Bernstein polynomials are given as:

$$y_1^L(x) = C_1^{T,L}P_6^L(x)\psi_{nm}(x), \quad y_2^L(x) = C_2^{T,L}P_6^L(x)\psi_{nm}(x).$$
$$y_1^B(x) = C_1^{T,B}P_6^B(x)\hat{B}(x), \quad y_2^B(x) = C_2^{T,B}P_6^B(x)\hat{B}(x).$$

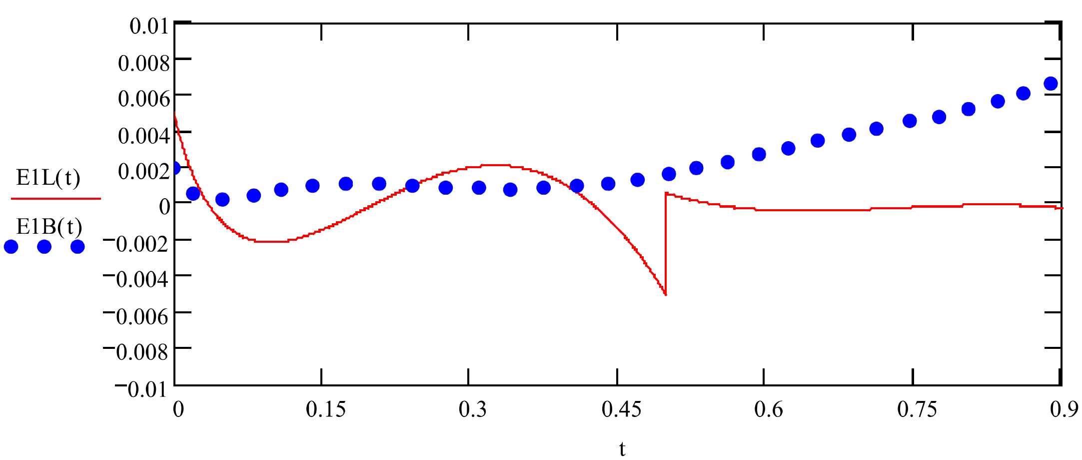


**Figure 4.1.1** Comparison of approximation errors for the $y_1(t)$ by using Legendre wavelets (solid red line) and Bernstein polynomials (blue dotted line).

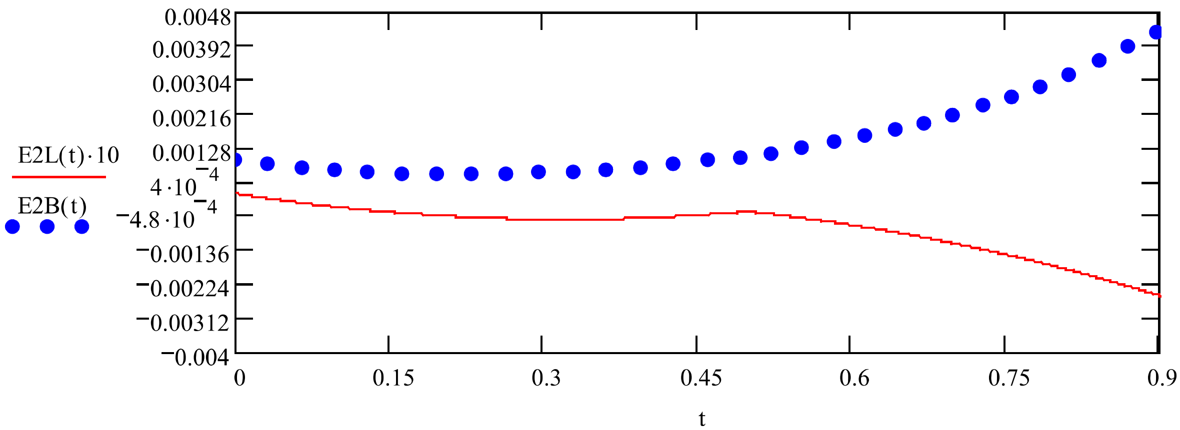


**Figure 4.1.2** Comparison of approximation errors for the $y_2(t)$ by using Legendre wavelets (solid red line) and Bernstein polynomials (blue dotted line).

**Example 4.5.2.** We choose $f_1 = -1 - \frac{4}{15} t^{\frac{3}{2}} (-5 + 8t)$ and $f_2 = -2t + 1 - \frac{16}{15} t^{\frac{5}{2}}$ so that exact solutions for Eq. (4.1.1) by $Y_1 = t(t-1)$ and $Y_1 = t^2$. The approximate solutions using Legendre wavelets and Bernstein polynomials are given by Eq. (4.3.3) with coefficients.

$C_1^{T,L} = [-0.353554\ \ 0.204124\ \ 0\ \ 0.353555\ \ 0.204124\ \ 0]$,

$C_2^{T,L} = [0.353553\ \ 0.204124\ \ 0\ \ 1.060661\ \ 0.204124\ \ 0]$,

$C_1^{T,B} = [-0.396037\ \ -0.07393\ \ 0.134368\ \ 0.238363\ \ 0.248062\ \ 0.161138]$,

$C_2^{T,B} = [0.1572\ \ 0.427085\ \ 0.577087\ \ 0.613782\ \ 0.539943\ \ 0.33013]$,

$C_1^{T,L} P_6^L(x) = [-0.117851\ \ -0.051031\ \ 0.013176\ \ -0.117851\ \ 0.051031\ \ 0.013176]$,

$C_2^{T,L} P_6^L(x) = [0.058926\ \ 0.051031\ \ 0.013176\ \ 0.412479\ \ 0.153093\ \ 0.013176]$,

$C_1^{T,B} P_6^B(x) = [-0.059527\ \ -0.125595\ \ -0.105125\ \ -0.05772\ \ -0.018648\ \ -1.553198 \times 10^{-3}]$
and

$C_2^{T,B} P_6^B(x) = [0.019351\ \ 0.088925\ \ 0.185127\ \ 0.251817\ \ 0.254422\ \ 0.165759]$.

Therefore, the approximate solutions corresponding to Legendre wavelets and Bernstein polynomials are given as:

$$y_1^L(x) = C_1^{T,L} P_6^L(x) \psi_{nm}(x), \quad y_2^L(x) = C_2^{T,L} P_6^L(x) \psi_{nm}(x).$$
$$y_1^B(x) = C_1^{T,B} P_6^B(x) \hat{B}(x), \quad y_2^B(x) = C_2^{T,B} P_6^B(x) \hat{B}(x).$$

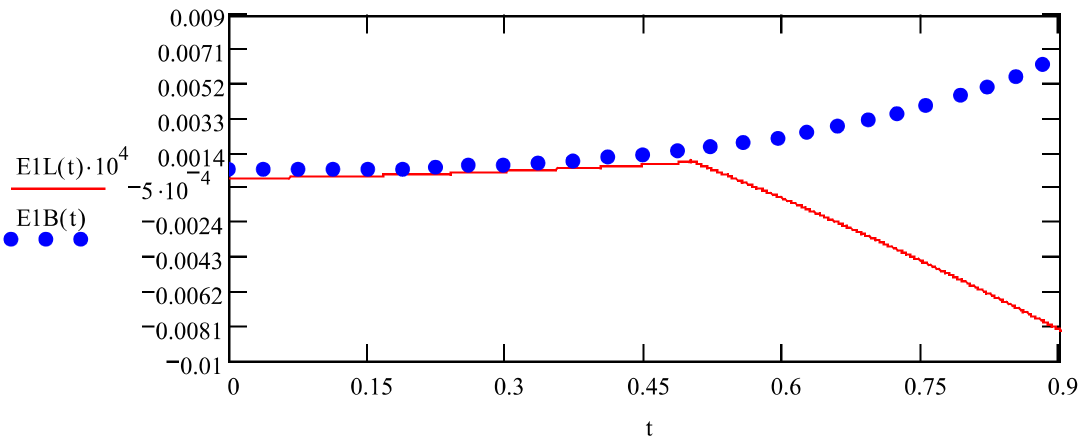


**Figure 4.2.1** Comparison of approximation errors for the $y_3(t)$ by using Legendre wavelets (solid red line) and Bernstein polynomials (blue dotted line).

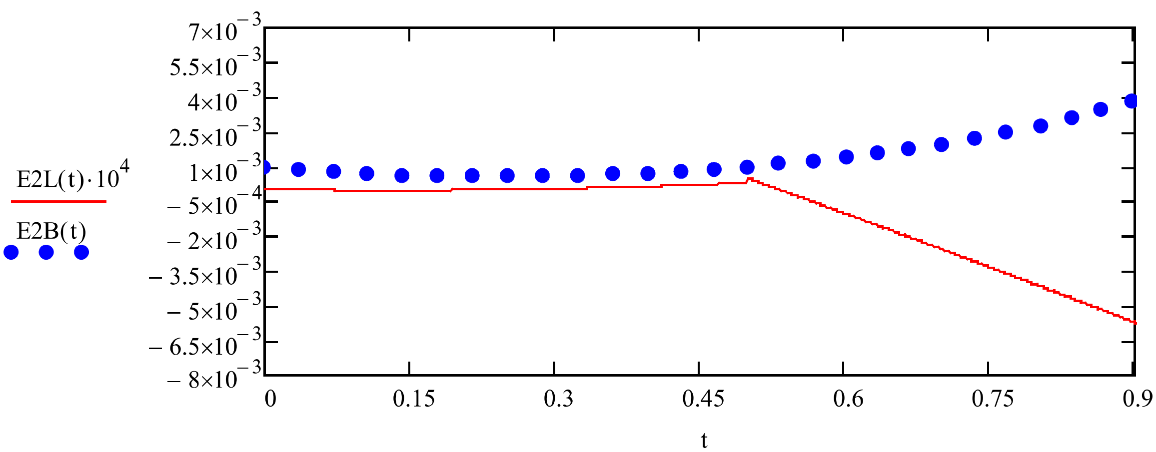


**Figure 4.2.2** Comparison of approximation errors for the $y_4(t)$ by using Legendre wavelets (solid red line) and Bernstein polynomials (blue dotted line).

**Table 4.2.**

Continuous and discrete error norms

| Error norms | Example 4.5.1. | Example 4.5.2. |
|---|---|---|
| $\|E_1\|_{L^2}^{L}$ | $1.33\times10^{-3}$ | $4.13\times10^{-7}$ |
| $\|E_1\|_{L^2}^{B}$ | $3.92\times10^{-3}$ | $3.49\times10^{-3}$ |
| $\|E_1\|_{l^2}^{L}$ | $3.17\times10^{-2}$ | $4.12\times10^{-7}$ |
| $\|E_1\|_{l^2}^{B}$ | $3.93\times10^{-3}$ | $3.50\times10^{-3}$ |
| $\|E_2\|_{L^2}^{L}$ | $1.33\times10^{-4}$ | $2.87\times10^{-7}$ |
| $\|E_2\|_{L^2}^{B}$ | $2.31\times10^{-3}$ | $2.08\times10^{-3}$ |
| $\|E_2\|_{l^2}^{L}$ | $3.16\times10^{-2}$ | $3.16\times10^{-2}$ |
| $\|E_2\|_{l^2}^{B}$ | $2.31\times10^{-3}$ | $2.08\times10^{-3}$ |

## 4.6 Conclusions

The main aim of this chapter was to demonstrate that the wavelet mess-less approximation method is a more powerful numerical tool than the orthogonal polynomial mess-less approximation for solving SSVIDEs. The computer simulation is carried out for the problems of SSVIDEs, this allow us to estimate the precision of the mess-less scheme based on operational matrices for wavelet and orthogonal polynomial respectively. High accuracy of the results even in the case of smaller number of wavelet basis function compare to Bernstein polynomial is observed. The numerical schemes of fractional order derivatives and non-linear problems are topics of further study.