Chapter 3

Quasi Projective-synchronization of complex-valued recurrent neural networks with proportional delay and mismatched parameters via matrix measure approach

3.1 Introduction

This chapter concerns the quasi-projective synchronization of non-identical complex-valued recurrent neural networks (CVRNNs) with proportional delays and mismatched parameters. So far, research has extensively explored quasi-projective and projective synchronization in both traditional RVNNs and fractional order NNs [94, 95, 96, 97, 98, 99, 100], primarily employing the Lyapunov function approach. However, an alternative approach known as the matrix measure method offers a distinct advantage over the Lyapunov function by considering both the positive and negative aspects of the connection weights matrix. While limited studies focus on quasi-synchronization and projective synchronization [7], the growing recognition that CVNNs offer superior data representation compared to RVNNs has spurred a surge in research dedicated to CVNNs. Only a handful of studies have centered on quasi and quasi-projective synchronizations [101, 102, 56, 103]. Moreover, research on projective synchronization of complex-valued NNs with mismatched parameters remains sparse. The inherent complexity escalates due to the coexistence of the projective coefficient and parameter mismatch, posing challenges in deriving solutions. Proportional delays exist widely in the real world, such as routing decisions and web quality of service [104]. The researchers have introduced the dynamic nature of the NNs with proportional delay terms. Most of the works concerning synchronizations of NNs include asymptotic proportional delay terms. Many researchers have introduced synchronization schemes, which assume that the drive-response systems are identical in CVNNs with or without time delay. Although there are always mismatches between drive and response systems in practical applications. If the drive and response systems are different, then the synchronization error will generally not tend to zero asymptotically [105, 106]. Most synchronization results are mainly based on the Lyapunov functional method. The matrix measure method is an alternative method that provides many potential advantages. The matrix measure has other benefits, considering both the positive and negative components of the connection weights matrix. Each of these components corresponds to the excitation or inhibition of neurons. Most of the synchronization results in CVNNs have identical drive and response systems. However, in practical applications, mismatches between drive and response systems are always present. This chapter employs non-linear Lipschitz activation functions under Lyapunov stability criteria and matrix measure approach. By designing a suitable controller, a sufficient condition for projective quasi-synchronization criteria of the non-identical CVRNNs model has been derived through the proper description of the matrix measure approach. A significant result for the CVRNNs with mismatched parameters and proportional delays is provided. Finally, a numerical simulation result is given to validate the usefulness and persistence of the theoretical results. The results for different particular cases are displayed graphically.

3.2 Model Description and Preliminaries

Consider the following CVRNNs, including proportional delay, as

$$\dot{\gamma}(t) = -R\gamma(t) + Pg(\gamma(t)) + Qh(\gamma(qt)) + S(t), \tag{3.1}$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t), ..., \gamma_n(t))^T \in C^n$ represents state of n-th neurons at time t, $R = diag(r_1, r_2, ..., r_n) \in R^{n\times n}$ and $r_n > 0$ represent the n-th neuron self-inhibition. $P \in C^{n\times n}$ and $Q \in C^{n\times n}$ denote the connection weight matrices, respectively. $S(t) = (s_1(t), s_2(t), ..., s_n(t))^T$ denotes the external input vector
$g(\gamma(t)) = (g_1(\gamma_1(t)), g_2(\gamma_2(t)), ..., g_n(\gamma_n(t)))^T : C^n \to C^n$,
$h(\gamma(qt)) = (h_1(\gamma_1(qt)), h_2(\gamma_2(qt)), ..., h_n(\gamma_n(qt)))^T : C^n \to C^n$,
represents activation function without and with proportional delays respectively. The proportional delay term is $q \in (0, 1)$ and $qt = t - \tau(t)$, with $\tau(t) = (1 - q)t$ is a continuous time varying function which satisfies $\tau(t) \to \infty$ as $t \to +\infty$.
The initial condition of system (3.1) is given as
$\gamma(s) = \phi(s), s \in [t_0 - \tau, t_0]$,

where $\phi(s) = (\phi_1(s), \phi_2(s), .., \phi_n(s)) \in C^n$, the real parts $\phi^R(s)$ and imaginary parts $\phi^I(s)$ of $\phi(s)$ are continuous on $[t_0 - \tau, t_0]$.

Let us assume that $\gamma(t) = \alpha(t) + i\beta(t)$, where $\alpha(t), \beta(t) \in R^n$, then the equation (3.1) can be rewritten as

$$\begin{aligned}
\dot{\alpha}(t) =& - R\alpha(t) + P^R g^R(\gamma(t)) - P^I g^I(\gamma(t)) + Q^R h^R(\gamma(qt)) \\
& - Q^I h^I(\gamma(qt)) + S^R(t), \\
\dot{\beta}(t) =& - R\beta(t) + P^I g^R(\gamma(t)) + P^R g^I(\gamma(t)) + Q^I h^R(\gamma(qt)) \\
& + Q^R h^I(\gamma(qt)) + S^I(t).
\end{aligned} \tag{3.2}$$

The initial conditions of the equation (3.2) will be

$$\begin{cases} \alpha(s) = \phi^R(s), \\ \beta(s) = \phi^I(s), \quad -\tau \leq s \leq 0, \end{cases}$$

where $\|\phi^R\|_w = sup_{t_0-\tau\leq s\leq t_0}\|\phi^R(s)\|_w$ and $\|\phi^I\|_w = sup_{t_0-\tau\leq s\leq t_0}\|\phi^I(s)\|_w$ represent the norms of real and imaginary parts of $\phi(s)$, respectively.

Corresponding to the drive system (3.1), let us construct the response system as

$$\dot{\widetilde{\gamma}}(t) = -R'\widetilde{\gamma}(t) + P'g(\widetilde{\gamma}(t)) + Q'h(\widetilde{\gamma}(qt)) + S'(t) + D(t), \tag{3.3}$$

where $D(t) = (D_1(t), D_2(t), .., D_n(t))^T \in C^n$ denotes the coupling control, $\widetilde{\gamma}(t) = (\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t), ..., \widetilde{\gamma}_n(t))^T \in C^n$ represents the state of nth neurons at time t, $R' = diag(r'_1, r'_2, ..., r'_n) \in R^{n\times n}$ and $r'_n > 0$ represents nth neuron self-inhibition. P' and $Q' \in C^{n\times n}$ represent weight matrices without and with delayed . $S'(t) = (s'_1(t), s'_2(t), ..., s'_n(t))^T$ denotes the external input vector.

The corresponding initial condition is

$\widetilde{\gamma}(s) = \widetilde{\phi}(s), s \in [t_0 - \tau, t_0]$,

where $\widetilde{\phi}(s) = (\widetilde{\phi}_1(s), \widetilde{\phi}_2(s), ..., \widetilde{\phi}_n(s)) \in C^n, \widetilde{\phi}^R(s)$ and $\widetilde{\phi}^I(s)$ denote continuous functions on interval $[t_0 - \tau, t_0]$. Corresponding to the master system (3.1), the response system is given as

Now, the real and imaginary parts of the equation (3.3) are given as

$$\begin{aligned}
\dot{\widetilde{\alpha}}(t) =& - R'\widetilde{\alpha}(t) + P'^R g^R(\widetilde{\gamma}(t)) - P'^I g^I(\widetilde{\gamma}(t)) + Q'^R h^R(\widetilde{\gamma}(qt)) \\
& - Q'^I h^I(\widetilde{\gamma}(qt)) + S'^R(t) + D^R(t), \\
\dot{\widetilde{\beta}}(t) =& - R'\widetilde{\beta}(t) + P'^I g^R(\widetilde{\gamma}(t)) + P'^R g^I(\widetilde{\gamma}(t)) + Q'^I h^R(\widetilde{\gamma}(qt)) \\
& + Q'^R h^I(\widetilde{\gamma}(qt)) + S'^I(t) + D^I(t),
\end{aligned} \tag{3.4}$$

and the initial conditions are given by

$$\begin{cases} \widetilde{\alpha}(s) = \widetilde{\phi}^R(s), \\ \widetilde{\beta}(s) = \widetilde{\phi}^I(s), -\tau \leq s \leq 0, \end{cases}$$

where $D^R(t), D^I(t)$ are the control input vectors.

Now, consider the controllers as

$$D^R(t) = \Omega e^R(t) \tag{3.5}$$

$$and \;\; D^I(t) = \Omega e^I(t), \tag{3.6}$$

where

$D^R(t) = [D_1^R(t), D_2^R(t), ..., D_n^R(t)]^T, D^I(t) = [D_1^I(t), D_2^I(t), ..., D_n^I(t)]^T$,

$e^R(t) = [e_1^R(t), e_2^R(t), ..., e_n^R(t)]^T, e^I(t) = [e_1^I(t), e_2^I(t), ..., e_n^I(t)]^T$ and the controller gain matrix is defined by $\Omega \in R^{n\times n}$.

Throughout this chapter, the following assumptions have been considered.

Assumption 3.2.1. *Suppose $w = u + iv$, where $u, v \in R$. $f_l(w)$ and $g_l(w)$ are described by*

$f_l(w) = f_l^R(u) + if_l^I(v)$ *and* $g_l(w) = g_l^R(u) + ig_l^I(v)$,

where $l = 1, 2, ..., n$ and $f_l^R(.), f_l^I(.), g_l^R(.), g_l^I(.) : R \to R$ satisfy the Lipschitz conditions as

$\left\|f_l^R(\nu) - f_l^R(\eta)\right\|_w \leq r_l\|\nu - \eta\|_w$,

$\left\|f_l^I(\nu) - f_l^I(\eta)\right\|_w \leq s_l\|\nu - \eta\|_w$,

$\left\|g_l^R(\nu) - g_l^R(\eta)\right\|_w \leq m_l\|\nu - \eta\|_w$,

$\left\|g_l^I(\nu) - g_l^I(\eta)\right\|_w \leq n_l\|\nu - \eta\|_w$,

where r_l, s_l, m_l, and n_l are the Lipschitz constants, ν and $\eta \in R^n$.

Assumption 3.2.2. *Let's suppose that for any initial function $v(t) \in C([-\tau, 0], R^n)$, the solution of the system is bounded, i.e., $\exists$ positive constant $h \in R^+$ and time instant t_0 s.t. $\|v(t)\|_w \leq h$, $\forall\ t \geq t_0$.*

The error system of master and response systems (3.2) and (3.4) are defined as $e^R(t) = \widetilde{\alpha}(t) - r\alpha(t), e^I(t) = \widetilde{\beta}(t) - r\beta(t)$, where r is the projective coefficient or scaling factor. If $e^R(t) \to 0$ and $e^I(t) \to 0$ as $t \to \infty$, the systems (3.2) and (3.4) will be synchronized. The error systems from the controllers (3.5) and (3.6) are obtained as

$$\begin{aligned}
\dot{e}^R(t) =& -R'e^R(t) + P'^R\widetilde{g}^R(e(t)) - P'^I\widetilde{g}^I(e(t)) + Q'^R\widetilde{h}^R(e(qt)) \\
&- Q'^I\widetilde{h}^I(e(qt)) + H(t) + \Omega e^R(t), \\
\dot{e}^I(t) =& -R'e^I(t) + P'^R\widetilde{g}^I(e(t)) + P'^I\widetilde{g}^R(e(t)) + Q'^R\widetilde{h}^I(e(qt)) \\
&+ Q'^I\widetilde{h}^R(e(qt)) + H'(t) + \Omega e^I(t),
\end{aligned} \tag{3.7}$$

where

$e^R(t) = (e_1^R(t), e_2^R(t)...e_n^R(t))^T \in R^n, e^I(t) = (e_1^I(t), e_2^I(t)...e_n^I(t))^T \in R^n.$

$\widetilde{g}^R(e(t)) = g^R(\widetilde{\gamma}(t)) - g^R(\gamma(t)), \widetilde{g}^I(e(t)) = g^I(\widetilde{\gamma}(t)) - g^I(\gamma(t)),$

$\widetilde{h}^R(e(t-\tau(t))) = h^R(\widetilde{\gamma}(t-\tau(t))) - h^R(\gamma(t-\tau(t))),$

$\widetilde{h}^I(e(t-\tau(t))) = h^I(\widetilde{\gamma}(t-\tau(t))) - h^I(\gamma(t-\tau(t))),$

and

$$\begin{aligned}
H(t) &= r(R-R')\alpha(t) + r(P'^R - P^R)g^R(\gamma(t)) + r(P^I - P'^I)g^I(\gamma(t)) \\
&\quad + r(Q'^R - Q^R)h^R(\gamma(qt)) + r(Q^I - Q'^I)h^I(\gamma(qt)) + S'^R(t) - rS'^R(t), \\
H'(t) &= r(R-R')\beta(t) + r(P'^R - P^R)g^I(\gamma(t)) + r(P'^I - P^I)g^R(\gamma(t)) \\
&\quad + r(Q'^R - Q^R)h^I(\gamma(qt)) + r(Q'^I - Q^I)h^R(\gamma(qt)) + S'^I(t) - rS'^I(t).
\end{aligned}$$

It is important to show that $H(t)$ and $H^{'}(t)$ are bounded before getting to the main results. By applying the w-norm in both sides of $H(t)$ and $H^{'}(t)$ and using Assumptions 3.2.1 and 3.2.2, we get

$$\begin{aligned}
\|H(t)\|_w \leq & |r|\|R' - R\|_w\|\alpha(t)\|_w + |r|\big\|P'^R - P^R\big\|_w\big\|g^R(\gamma(t))\big\|_w \\
& + |r|\big\|P'^I - P^I\big\|_w\big\|g^I(\gamma(t))\big\|_w + |r|\big\|Q'^R - Q^R\big\|_w\big\|h^R(\gamma(qt))\big\|_w \\
& + |r|\big\|Q'^I - Q^I\big\|_w\big\|h^I(\gamma(qt))\big\|_w + \big\|S'^R - rS^R\big\|_w \\
\leq & |r|\big(\|R'\|_w + \|R\|_w\big)h + |r|\big(\big\|P'^R\big\|_w + \big\|P^R\big\|_w\big)hL_g \\
& + |r|\big(\big\|P'^I\big\|_w + \big\|P^I\big\|_w\big)hL_{g'} + |r|\big(\big\|Q'^R\big\|_w \\
& + \big\|Q^R\big\|_w\big)hL_h + |r|\big(\big\|Q'^I\big\|_w + \big\|Q^I\big\|_w\big)hL_{h'} + \nu_1, \\
\|H(t)\|_w \leq & \zeta_1, \qquad (3.8) \\
\|H'(t)\|_w \leq & |r|\|R' - R\|_w\|\beta(t)\|_w + |r|\big\|P'^I - P^I\big\|_w\big\|g^R(\gamma(t))\big\|_w \\
& + |r|\big\|P'^R - P^R\big\|_w\big\|g^I(\gamma(t))\big\|_w + |r|\big\|Q'^I - Q^I\big\|_w\big\|h^R(\gamma(t-\tau(t)))\big\|_w \\
& + |r|\big\|Q'^R - Q^R\big\|_w\big\|h^I(\gamma(t-\tau(t)))\big\|_w + \big\|S'^I - rS^I\big\|_w
\end{aligned}$$

$$\begin{aligned}
&\leq |r|\big(\|R'\|_w + \|R\|_w\big)h + |r|\big(\left\|P'^I\right\|_w + \left\|P^I\right\|_w\big)hL_g \\
&\quad + |r|\big(\left\|P'^R\right\|_w + \left\|P^R\right\|_w\big)hL_{g'} + |r|\big(\left\|Q'^I\right\|_w + \left\|Q^I\right\|_w\big)hL_h \\
&\quad + |r|\big(\left\|Q'^R\right\|_w + \left\|Q^R\right\|_w\big)hL_{h'} + \nu_2, \\
\|H'(t)\|_w &\leq \zeta_2. \qquad (3.9)
\end{aligned}$$

Lemma 3.2.1. *[7] Consider that a continuous function $\omega(t) : [-\tau, +\infty) \to [0, +\infty)$ such that $\forall\ t > 0$,*

$D^+(\omega(t)) \leq -k_1\omega(t) + k_2\bar{\omega}(t) + \zeta,$

where $\bar{\omega}(t) \triangleq sup_{-\tau\leq s\leq 0}\omega(s)$, if $k_1 > k_2 > 0, \zeta > 0$,

then the inequality $\omega(t) \leq sup_{-\tau\leq s\leq 0}\omega(s)e^{-\delta t} + \frac{\zeta}{\delta}$ holds for $t \geq t_0$,

where $D^+\omega(t) = \overline{\lim}_{h\to 0^+}\frac{\omega(t+h)-\omega(t)}{h}$ is the upper-right derivative of $\omega(t)$ and $\delta > 0$ is the unique solution of $\delta = k_1 - k_2e^{\delta\tau}$.

Definition 3.2.1. *[105] Under the controllers given in (3.5) and (3.6), the drive system (3.1) and the response system (3.3) will be quasi-projective synchronized for initial conditions $\gamma(s) = \phi(s)$ and $\widetilde{\gamma}(s) = \widetilde{\phi}(s)$, if $\exists$ a small error bound $\bar{\epsilon} > 0$ s.t.,*

$$\lim_{t\to\infty} |\widetilde{\gamma}(t) - r\gamma(t)| \leq \bar{\epsilon}, \quad \forall t \geq s,$$

where r denotes the projective coefficient and $r \neq 0$.

3.3 Main results

Theorem 3.3.1. *Suppose that Assumption 3.2.1 holds. Then the master and response systems (3.1) and (3.3) with mismatched parameters and proportional delays will be quasi-projective synchronized in the presence of the controllers (3.5) and (3.6),* *if there exists controller gain matrix Ω s.t., the following inequality is satisfied.*

$$0 < k_2 < k_1, \tag{3.10}$$

with

$$\begin{aligned} k_1 =& -\Big\{\mu_w(-R' + \Omega) + (r_k + s_k)(\|P'^R\|_w + \|P'^I\|_w)\Big\}, \\ k_2 =& (m_k + n_k)(\|Q'^R\|_w + \|Q'^I\|_w) > 0, \qquad w = 1, 2, \infty. \end{aligned}$$

Moreover, the synchronization error exponentially converges in the region

$$\bar{\omega} = \Big\{e(t) \in R^n : \|e(t)\| \leq \frac{\zeta}{\delta}\Big\}. \tag{3.11}$$

Proof. Construct the Lyapunov function as

$$V_1(e(t)) = \|e^R(t)\|_w + \|e^I(t)\|_w.$$

The upper right Dini derivative of $V_1(e(t))$ along the error systems (3.7) is

$$\begin{aligned} D^+V_1(e(t)) =& \overline{\lim}_{\epsilon\to 0^+} \frac{\|e^R(t+\epsilon)\|_w + \|e^I(t+\epsilon)\|_w - \|e^R(t)\|_w - \|e^I(t)\|_w}{\epsilon} \\ =& \overline{\lim}_{\epsilon\to 0^+} \frac{1}{\epsilon}\Big\{ \|e^R(t) + \epsilon\dot{e}^R(t) + \mathcal{O}(\epsilon)\|_w + \|e^I(t) + \epsilon\dot{e}^I(t) + \mathcal{O}(\epsilon)\|_w \\ & - \|e^R(t)\|_w - \|e^I(t)\|_w\Big\} \\ =& \overline{\lim}_{\epsilon\to 0^+} \frac{1}{\epsilon}\Big(||e^R(t) + \epsilon\Big(- R'e^R(t) + P'^R\widetilde{g}^R(e(t)) - P'^I\widetilde{g}^I(e(t)) \\ & + Q'^R\widetilde{h}^R(e(qt)) - Q'^I\widetilde{h}^I(e(qt)) + H(t) + \Omega e^R(t)\Big) \\ & + \mathcal{O}(\epsilon)||_w - \|e^R(t)\|_w) + \overline{lim}_{\epsilon\to 0^+} \frac{1}{\epsilon}\Big(||e^I(t) + \epsilon\Big(- R'e^I(t) \\ & + P'^R\widetilde{g}^I(e(t)) + P'^I\widetilde{g}^R(e(t)) + Q'^R\widetilde{h}^I(e(qt)) + Q'^I\widetilde{h}^R(e(qt)) \end{aligned}$$

$$
\begin{aligned}
&+ H'(t) + \Omega e^I(t)\Big) + \mathcal{O}(\epsilon)\|_w - \left\|e^I(t)\right\|_w\Big) \\
\leq& \overline{\lim}_{\epsilon\to 0^+} \frac{\|I+\epsilon(-R'+\Omega)\|_w - 1}{\epsilon}\left\|e^R(t)\right\|_w + \left\|P'^R\right\|_w \left\|\widetilde{g}^R(e(t))\right\|_w \\
&+ \left\|P'^I\right\|_w \left\|\widetilde{g}^I(e(t))\right\|_w + \left\|Q'^R\right\|_w \left\|\widetilde{h}^R(e(qt))\right\|_w \\
&+ \left\|Q'^I\right\|_w \left\|\widetilde{h}^I(e(qt))\right\|_w + \|H(t)\|_w \\
&+ \overline{\lim}_{\epsilon\to 0^+} \frac{\|I+\epsilon(-R'+\Omega)\|_w - 1}{\epsilon}\left\|e^I(t)\right\|_w + \left\|P'^R\right\|_w \left\|\widetilde{g}^I(e(t))\right\|_w \\
&+ \left\|P'^I\right\|_w \left\|\widetilde{g}^R(e(t))\right\|_w + \left\|Q'^R\right\|_w \left\|\widetilde{h}^I(e(qt))\right\|_w \\
&+ \left\|Q'^I\right\|_w \left\|\widetilde{h}^R(e(qt))\right\|_w + \|H'(t)\|_w. \qquad (3.12)
\end{aligned}
$$

From Assumption 3.2.1, we have

$$
\begin{aligned}
\left\|\widetilde{g}^R(e(t))\right\|_w &\leq r_k \|e(t)\|_w \\
&= r_k \left\|e^R(t) + ie^I(t)\right\|_w \\
&\leq r_k(\left\|e^R(t)\right\|_w + \left\|e^I(t)\right\|_w), \\
\left\|\widetilde{g}^I(e(t))\right\|_w &\leq s_k \|e(t)\|_w \\
&= s_k \left\|e^R(t) + ie^I(t)\right\|_w \\
&\leq s_k(\left\|e^R(t)\right\|_w + \left\|e^I(t)\right\|_w). \qquad (3.13)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left\|\widetilde{h}^R(e(qt))\right\|_w &\leq m_k \|e(qt)\|_w = m_k \left\|e^R(qt) + ie^I(qt)\right\|_w \\
&\leq m_k(\left\|e^R(qt)\right\|_w + \left\|e^I(qt)\right\|_w), \\
\left\|\widetilde{h}^I(e(qt))\right\|_w &\leq n_k \|e(qt)\|_p = n_k \left\|e^R(qt) + ie^I(qt)\right\|_w \\
&\leq n_k(\left\|e^R(qt)\right\|_w + \left\|e^I(qt)\right\|_w). \qquad (3.14)
\end{aligned}
$$

Under the inequalities (3.13) and (3.14), the inequality (3.12) becomes

$$
\begin{aligned}
D^+V_1(e(t)) \leq & \overline{\lim}_{\epsilon\to 0^+} \frac{\|I+\epsilon(-R'+\Omega)\|_w - 1}{\epsilon}\big\|e^R(t)\big\|_w + \big\|P'^R\big\|_w\big\|\widetilde{g}^R(e(t))\big\|_w \\
& + \big\|P'^I\big\|_w\big\|\widetilde{g}^I(e(t))\big\|_w + \big\|Q'^R\big\|_w\Big\|\widetilde{h}^R(e(qt))\Big\|_w \\
& + \big\|Q'^I\big\|_w\Big\|\widetilde{h}^I(e(qt))\Big\|_w + \|H(t)\|_w \\
& + \overline{\lim}_{\epsilon\to 0^+} \frac{\|I+\epsilon(-R'+\Omega)\|_w - 1}{\epsilon}\big\|e^I(t)\big\|_w \\
& + \big\|P'^R\big\|_w\big\|\widetilde{g}^I(e(t))\big\|_w + \big\|P'^I\big\|_w\big\|\widetilde{g}^R(e(t))\big\|_w \\
& + \big\|Q'^R\big\|_w\Big\|\widetilde{h}^I(e(qt))\Big\|_w + \big\|Q'^I\big\|_w\Big\|\widetilde{h}^R(e(qt))\Big\|_w + \|H'(t)\|_w \\
\leq & \overline{\lim}_{\epsilon\to 0^+} \frac{\|I+\epsilon(-R'+\Omega)\|_w - 1}{\epsilon}\big\|e^R(t)\big\|_w + \big\|P'^R\big\|_w r_k(\big\|e^R(t)\big\|_w \\
& + \big\|e^I(t)\big\|_w) + \big\|P^I\big\|_w s_k(\big\|e^R(t)\big\|_w + \big\|e^I(t)\big\|_w) \\
& + \big\|Q'^R\big\|_w m_k(\big\|e^R(qt)\big\|_w + \big\|e^I(qt)\big\|_w) \\
& + \big\|Q'^I\big\|_w n_k(\big\|e^R(qt)\big\|_w + \big\|e^I(qt)\big\|_w) + \zeta_1 \\
& + \overline{\lim}_{\epsilon\to 0^+} \frac{\|I+\epsilon(-R'+\Omega)\|_w - 1}{\epsilon}\big\|e^I(t)\big\|_w + \big\|P'^R\big\|_w s_k(\big\|e^R(t)\big\|_w \\
& + \big\|e^I(t)\big\|_w) + \big\|P'^I\big\|_w r_k(\big\|e^R(t)\big\|_w + \big\|e^I(t)\big\|_w) \\
& + \big\|Q'^R\big\|_w n_k(\big\|e^R(qt)\big\|_w + \big\|e^I(qt)\big\|_w) \\
& + \big\|Q'^I\big\|_w m_k(\big\|e^R(qt)\big\|_w + \big\|e^I(qt)\big\|_w) + \zeta_2. \qquad (3.15)
\end{aligned}
$$

Using the definition of matrix measure defined in Chapter 1, the equation (3.15) becomes

$$
\begin{aligned}
D^+V_1(e(t)) \leq & \mu_w(-R'+\Omega)\big\|e^R(t)\big\|_w + (r_k\big\|P'^R\big\|_w + s_k\big\|P'^I\big\|_w)(\big\|e^R(t)\big\|_w \\
& + \big\|e^I(t)\big\|_w) + (m_k\big\|Q'^R\big\|_w + n_k\big\|Q'^I\big\|_w)(\big\|e^R(qt)\big\|_w \\
& + \big\|e^I(qt)\big\|_w) + \mu_w(-R'+\Omega)\big\|e^I(t)\big\|_w + (s_k\big\|P'^R\big\|_w \\
& + r_k\big\|P'^I\big\|_w)(\big\|e^R(t)\big\|_w + \big\|e^I(t)\big\|_w) + (n_k\big\|Q'^R\big\|_w
\end{aligned}
$$

$$
\begin{aligned}
&+ m_k\|Q'^I\|_w)(\|e^R(qt)\|_w + \|e^I(qt)\|_w) + \zeta_1 + \zeta_2 \\
\leq&\mu_w(-R' + \Omega)(\|e^R(t)\|_w + \|e^I(t)\|_w) + (r_k\|P'^R\|_w + s_k\|P'^I\|_w \\
&+ s_k\|P'^R\|_w + r_k\|P'^I\|_w)(\|e^R(t)\|_w + \|e^I(t)\|_w) \\
&+ (m_k\|B^R\|_w + n_k\|Q'^I\|_w \\
&+ n_k\|Q'^R\|_w + m_k\|Q'^I\|_w)(\|e^R(qt)\|_w + \|e^I(qt)\|_w) + \zeta_1 + \zeta_2 \\
\leq&\mu_w(-R' + \Omega)(\|e^R(t)\|_w + \|e^I(t)\|_w) + \Big\{(r_k + s_k)(\|P'^R\|_w \\
&+ \|P'^I\|_w)\Big\}(\|e^R(t)\|_w + \|e^I(t)\|_w) + (m_k + n_k)(\|Q'^R\|_w \\
&+ \|Q'^I\|_w)\|e^R(qt)\|_w + \|e^I(qt)\|_w) + \zeta_1 + \zeta_2 \\
\leq&\Big\{\mu_w(-R' + \Omega) + (r_k + s_k)(\|P'^R\|_w + \|P'^I\|_w)\Big\}(\|e^R(t)\|_w \\
&+ \|e^I(t)\|_w) + (m_k + n_k)(\|Q'^R\|_w + \|Q'^I\|_w)(\|e^R(qt)\|_w \\
&+ \|e^I(qt)\|_w) + \zeta_1 + \zeta_2 \\
\leq&\Big\{\mu_w(-R' + \Omega) + (r_k + s_k)(\|P'^R\|_w + \|P'^I\|_w)\Big\}V_1(e(t)) \\
&+ (m_k + n_k)(\|Q'^R\|_w + \|Q'^I\|_w)V_1(e(qt)) + \zeta_1 + \zeta_2 \\
\leq&\Big\{\mu_w(-R' + \Omega) + (r_k + s_k)(\|P'^R\|_w + \|P'^I\|_w)\Big\}V_1(e(t)) \\
&+ (m_k + n_k)(\|Q'^R\|_p + \|Q'^I\|_w)sup_{-\tau\leq s\leq 0}V_1(e(s)) + \zeta. \qquad (3.16)
\end{aligned}
$$

Let, $k_1 = -\Big\{\mu_w(-R' + \Omega) + (r_k + s_k)(\|P'^R\|_w + \|P'^I\|_w)\Big\}$,
$k_2 = (m_k + n_k)(\|Q'^R\|_w + \|Q'^I\|_w)$ and $\zeta = \zeta_1 + \zeta_2$.

Then from the conditions given in equation (3.10), we have $0 < k_2 < k_1$ and using Lemma 3.2.1, we get

$$
V_1(e(t)) \leq \ sup_{-\tau\leq s\leq 0}V_1(e(s))e^{-\delta t} + \frac{\zeta}{\delta},
$$

where solution δ is calculated as

$$\begin{aligned}\delta =& k_1 - k_2 e^{\delta\tau} \\ =& - \left\{ \mu_w(-R' + \Omega) + (r_k + s_k)(\left\|P'^R\right\|_w + \left\|P'^I\right\|_w) \right\} \\ & - (m_k + n_k)(\left\|Q'^R\right\|_w + \left\|Q'^I\right\|_w) e^{\delta\tau}. \end{aligned} \tag{3.17}$$

So, the error $e(t)$ converges to the domain $\bar{\omega} = \{e(t) : \|e(t)\| \leq \frac{\zeta}{\delta}\}$. Hence, the Definition 3.2.1 concludes that the proposed systems (3.1) and (3.3) of CVRNNs will be quasi-projective synchronized in the sufficient region $\bar{\omega}$. □

Corollary 3.3.1. *Assuming that the Assumption 3.2.1 holds. The systems (3.1) and (3.3) with mismatched parameters and proportional delay will be quasi-projective synchronized if there exists the controller gain matrix Ω such that*

$$\begin{aligned} 0 <& (m_k + n_k)(\left\|Q'^R\right\|_w + \left\|Q'^I\right\|_w) \\ <& -\left\{ \mu_w(-R') + \mu(\Omega) + (r_k + s_k)(\left\|P'^R\right\|_w + \left\|P'^I\right\|_w) \right\}, \end{aligned} \tag{3.18}$$

where $w = 1, 2, \infty$.

Proof. Now, using properties of matrix measure given in section 1.5.5, we get

$$\mu_w(-R' + \Omega) \leq \mu_w(-R') + \mu_w(\Omega),$$

which gives

$$\begin{aligned} & -\{\mu_w(-R') + \mu_w(\Omega) + (r_k + s_k)(\left\|P'^R\right\|_w + \left\|P'^I\right\|_w)\} \\ & < -\{\mu_w(-R' + \Omega) + (r_k + s_k)(\left\|P'^R\right\|_w + \left\|P'^I\right\|_w)\}. \end{aligned}$$

Therefore from the equation (3.18), we have

$$\begin{aligned} 0 <&(m_k + n_k)(\left\|Q'^R\right\|_w + \left\|Q'^I\right\|_w) \\ &< -\left\{\mu_w(-R') + \mu(\Omega) + (r_k + s_k)(\left\|P'^R\right\|_w + \left\|P'^I\right\|_w)\right\}. \end{aligned}$$

Therefore, from the above Theorem 3.3.1, it may be concluded that the proposed master system (3.1) and response system (3.3) will be quasi-projective synchronized in the presence of the controllers (3.5) and (3.6). □

Remark 3.3.1. *The errors are characterized by the expressions* $e^R(t) = \widetilde{\alpha}(t) - r\alpha(t)$ *and* $e^I(t) = \widetilde{\beta}(t) - r\beta(t)$*. Specifically, when* $r = 1$*, the scenario extends to the quasi-complete synchronization of CVRNNs with proportional delay and mismatched parameters. Again, when* $r = -1$*, the system achieves quasi-anti-synchronization. These conditions highlight the versatility of the proposed approach, allowing for variations in synchronization behavior based on the choice of the parameter* r*.*

Remark 3.3.2. *This chapter is concerned with the quasi-projective synchronization of CVRNNs with proportional delays and mismatched parameters by using the matrix measure method [107]. Here, the proposed results depend on the matrix measures, which are more efficient and compact. Because it has no restriction on the values of the parameters of the real and imaginary connection weights matrices, it can be taken as negative and positive values. Moreover, the matrix measure is sign-sensitive as* $\mu_w(N) \neq \mu_w(-N)$*. But in the direct Lyapunov function method [101], most of the researchers have discussed synchronization with the help of matrix norms, which have limitations on the values of parameters. This case holds only for non-negative values i.e.,* $\|-N\|_w = \|N\|_w$*. Therefore, the above discussion confirms that the Lyapunov function method is a more conservative approach than the matrix measure technique.*

Remark 3.3.3. *As compared with the research work given in [7, 95, 108], the conditions in Theorem 3.3.1 are rendered in the form of matrix measures, which make the conditions more general and broad. In addition, compared with the articles [109, 110, 101], the control gain fluctuation is displayed in the controller, making the obtained results more significant in practice.*

Remark 3.3.4. *The drive and response systems are identical in the synchronization results [95, 99, 107]. However, In practical applications, there are always mismatches between master and response systems, which can destroy the synchrony state of drive and response systems. Due to this, in the present chapter, the quasi-synchronization of mismatched parameters is shown and found that the drive and response systems are synchronized up to an error bound.*

Remark 3.3.5. *In this chapter, Theorem 3.3.1 is proved by using the matrix measure method to ensure the quasi-projective synchronization of the systems (3.1) and (3.3). Based on Lemma 3.2.1, Assumptions 3.2.1 and 3.2.2 and Definitions 3.2.1, the Theorem 3.1 is derived. But Lemma 3.2.1 is utilized in Corollary 3.3.1. And its conditions are the same, which provides more choices and flexibility during applications.*

Remark 3.3.6. *RVNNs and CVNNs have had remarkable advancements in research since the 1980's. In contrast to RVNNs, the research on CVNNs has not been as active during the development of deep learning. The complex-valued activation function is not complex-differentiable and bounded at the same time, which is the cause of this limitation. Many researchers [111] have claimed that the restrictions are not necessary for a complex-valued activation to be both bounded and complex differentiable, and thus, it is suggested to have differentiability of the activation function with respect to the real and imaginary components. This remains an open area of research.*

Remark 3.3.7. *Complex parameters raise the number of operations needed to increase the computational complexity. Complex-valued parameters typically require up to four real multiplications and two real additions, as opposed to real-valued parameters, which only take a single real-valued multiplication. This indicates that raising the number of real-valued parameters in each layer does not produce the same result as seen in a complex-valued neural network [112]. The number of (real-valued) network parameters can also be used to measure a network's capacity to approximate structurally complex functions. Thus, each layer's number of real parameters is doubled by describing a complex number* $a + ib$ *using real integers* (a, b)*. This suggests that utilizing a complex-valued network may boost its expressiveness and increase its possibility of over-fitting because the network's parameters increase as it becomes more complicated. Also, to choose a complex activation function, one has to sacrifice one of the fundamental characteristics of activation functions, viz. boundedness or analyticity. These are some limitations of using CVNNs during the dynamical study.*

3.4 Numerical Example

In this section, one numerical simulation result is given to demonstrate the effectiveness and viability of the proposed results of the quasi-projective synchronization scheme.

Example 3.4.1. *Consider the two-dimensional CVRNNs in the presence of proportional delay and mismatched parameters as*

$$\dot{\gamma}(t) = -R\gamma(t) + Pg(\gamma(t)) + Qh(\gamma(qt)) + S(t), \tag{3.19}$$

with the following parameters as

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 3+5i & -0.2-0.3i \\ -6-0.2i & 2+3i \end{pmatrix},$$

$$Q = \begin{pmatrix} -0.5+1.1i & -0.4-0.3i \\ -0.3-0.4i & -3+0.5i \end{pmatrix}, \Omega = \begin{pmatrix} -125 & 10 \\ -9.5 & -115 \end{pmatrix}, S(t) = \begin{pmatrix} sin(t)+icos(t) \\ sin(2t)+icos(2t) \end{pmatrix},$$
$q = 0.1,\ \tau(t) = 0.9exp(t).$

The activation functions are taken as

$$g_p(\gamma_p(t)) = \frac{1}{1+e^{(-\alpha_p+2\beta_p)}} + i\frac{1-e^{(-2\alpha_p-\beta_p)}}{1+e^{(-2\alpha_p-\beta_p)}},$$
$$h_p(\gamma_p(t)) = \frac{1-e^{(-\beta_p)}}{1+e^{(-\beta_p)}} + i\frac{1}{1+e^{(-\alpha_p)}},\ (p=1,2).$$

Figure 3.1 depicts the plots of the trajectories $\alpha_1(t), \beta_1(t)$ *and* $\alpha_2(t), \beta_2(t)$ *in three-dimensions at time* t *of the system (3.19) in the presence of mismatched parameters and proportional delay which show the chaotic behaviour of the real and imaginary parts of trajectories with respect to time* t.

The corresponding response system is

$$\dot{\tilde{\gamma}}(t) = -R'\tilde{\gamma}(t) + P'g(\tilde{\gamma}(t)) + Q'h(\tilde{\gamma}(qt)) + S'(t) + D(t), \tag{3.20}$$

with the following parameters

$$R' = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad P' = \begin{pmatrix} 2+4i & -1.2-0.4i \\ 4-0.3i & 8+4i \end{pmatrix},$$

$$Q' = \begin{pmatrix} -0.7+1.2i & -0.5-0.4i \\ -0.2-0.5i & 2+0.6i \end{pmatrix},\ S'(t) = \begin{pmatrix} cos(t)+isin(t) \\ cos(2t)+isin(2t) \end{pmatrix},$$

where $D(t) = \Omega(\widetilde{\gamma}(t) - \gamma(t))$ *denotes the controller matrix. Suppose that Assumption 3.2.1 holds. Then the other parameters of the systems are given as* $r_k = 0.5$*,* $s_k = 0.25$*,* $m_k = 0.707$ *and* $n_k = 1.414$*. The initial conditions are considered as*

$w_1(s) = -0.15 + 0.45i, w_2(s) = -0.25 + 0.15i,$

$\widetilde{w}_1(s) = -0.30 + 0.30i,\ \widetilde{w}_2(s) = 0.30 + 0.25i.$

Case I

When $0 < r < 1 \neq 1$*, say* $r = 0.2$*, then it can be found that*

$k_1 = 110.8352$*,* $k_2 = 3.2334$*, i.e.,* $k_1 > k_2$*,* $\zeta = 7.6532$*.*

Thus all the conditions of Theorem 3.3.1 are satisfied. Hence the system (3.19) will be quasi-projective synchronized with the system (3.20) having estimated error level $= 0.2481$.

Figures 3.2 and 3.3 represent the state evaluation curves with respect to time t *of the systems (3.19) and (3.20) under the controllers having mismatched parameters. It could be seen that the state variables of the drive and response systems cannot achieve an identical behavior with time flowing and, therefore, are always different. In other words, the system cannot achieve complete synchronization. The evolution curves of the synchronization errors under no controller are depicted in Figure 3.4(a), which indicates that quasi-projective synchronization is not achieved. Figure 3.4(b) shows the quasi-projective synchronization in the presence of the controller with the error bound = 0.2481.*

Case II

When $r = 1$*, it can be verified that*

$k_1 = 110.8352$*,* $k_2 = 3.2334$*, i.e.,* $k_1 > k_2$*,* ζ *= 38.2662.*

Thus, all the conditions of Theorem 3.3.1 are satisfied here. Hence the drive system (3.19) and response system (3.20) will be quasi-synchronized with an estimated error level = 1.2409.

Figures 3.5 and 3.6 show the state evaluation curves with respect to time t *of the systems (3.19) and (3.20) under the controllers with mismatched parameters at* r *= 1. It could be seen that the state variables of drive and response systems cannot achieve an identical behavior with time flowing and always be a little different that is the system cannot achieve complete synchronization. Figure 3.7(a) shows the evaluation curves of the synchronization error without controllers, indicating that the quasi-synchronization is not achieved, while Figure 3.7(b) shows the quasi-synchronization with proportional delay and mismatched parameters of the error system (3.7) in the presence of controllers having error bound = 1.2409.*

Case III

If one chooses $r > 1$*, more precisely* $r = 3.5$*, then after performing the calculation, it is obtained that* $k_1 = 110.8352$*,* $k_2 = 3.2334$*, i.e.,* $k_1 > k_2$*, and* $\zeta = 133.9316$.

This shows that all conditions of Theorem 3.3.1 are satisfied here. Hence, the drive system (3.19) and response system (3.20) will be quasi-synchronized with an estimated error bound $= 4.1925$.

Figure 3.8(a) shows the evaluation curves of the synchronization error without controllers, indicating that the quasi-synchronization is not achieved, while Figure 3.8(b)

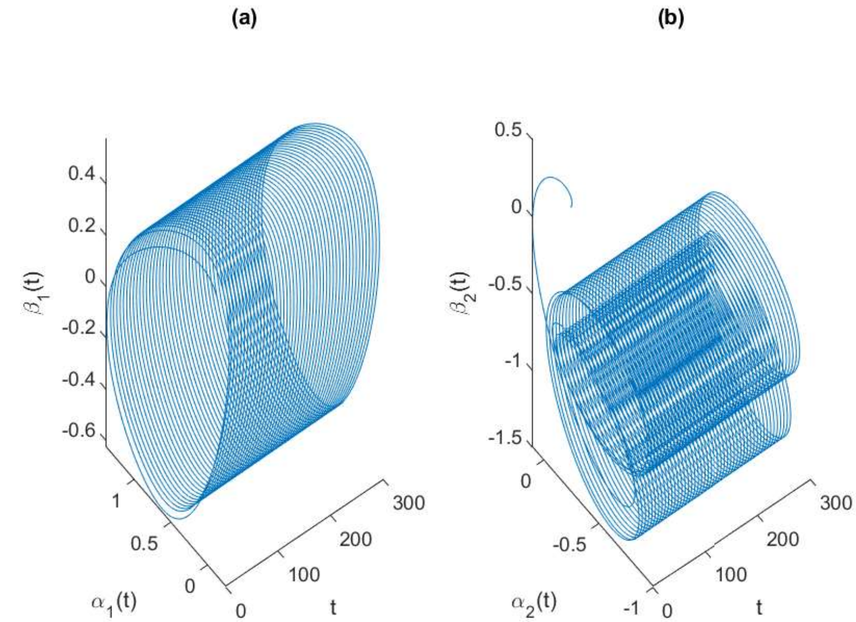

FIGURE 3.1: The state trajectories of the system (3.19) in three-dimensional space for **(a)** $\alpha_1(t)$, $\beta_1(t)$, and **(b)** $\alpha_2(t)$ and $\beta_2(t)$.

shows the quasi-synchronization with proportional delay and mismatched parameters of the error system (3.7) in the presence of controllers having error bound $= 4.1925$.

Remark 3.4.1. *Based on the analyses of Cases I, II, and III, it is seen that the estimated error bounds are* $0.2481, 1.2409,$ *and* 4.1925*, respectively, for the projective coefficient* $r = 0.1, 1,$ *and* 3.5*. Therefore, it may be concluded that the error bound increases as the value of the projective coefficient increases.*

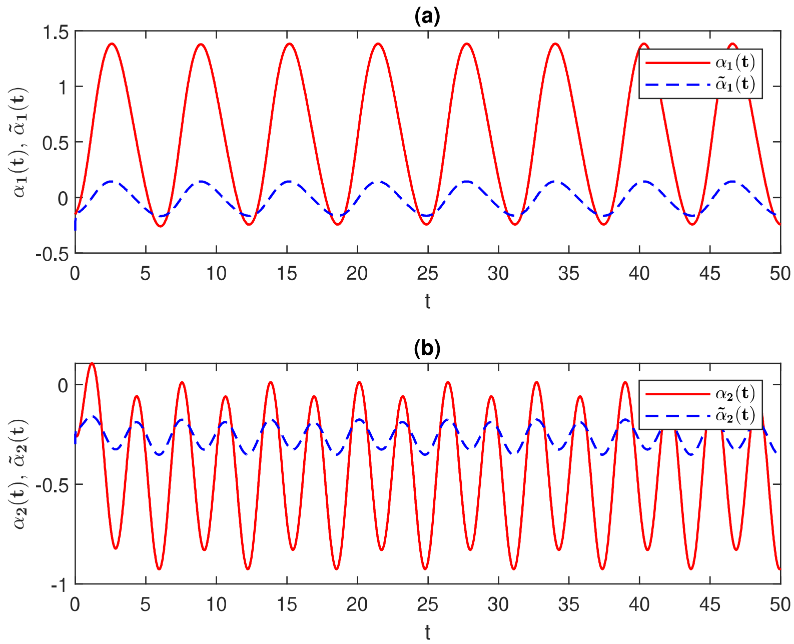

FIGURE 3.2: The state trajectories of the systems (3.19) and (3.20) for **(a)** $\alpha_1(t)$, $\widetilde{\alpha}_1(t)$, and **(b)** $\alpha_2(t)$, $\widetilde{\alpha}_2(t)$ with time t for the case I.

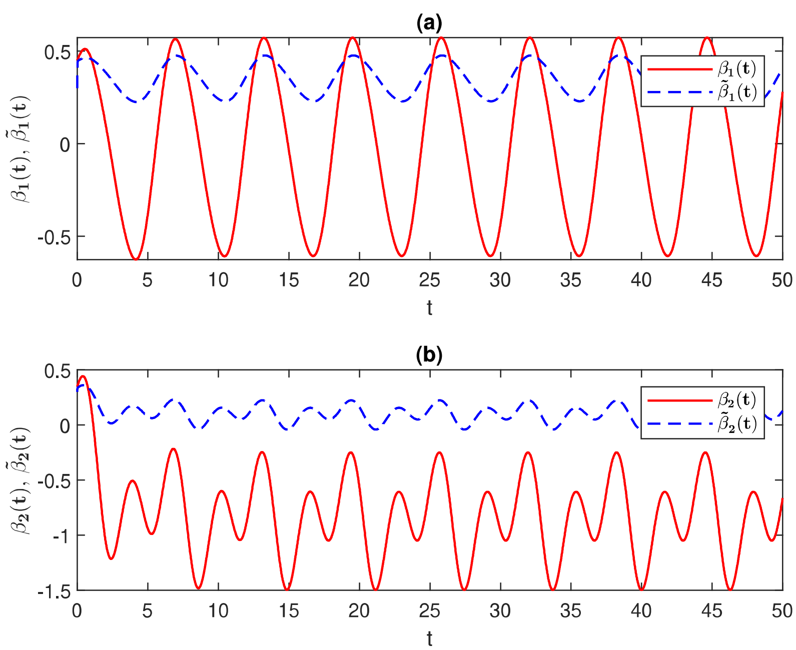

FIGURE 3.3: The state trajectories of the systems (3.19) and (3.20) for **(a)** $\beta_1(t)$, $\widetilde{\beta}_1(t)$, and **(b)** $\beta_2(t)$, $\widetilde{\beta}_2(t)$ with time t for the case I.

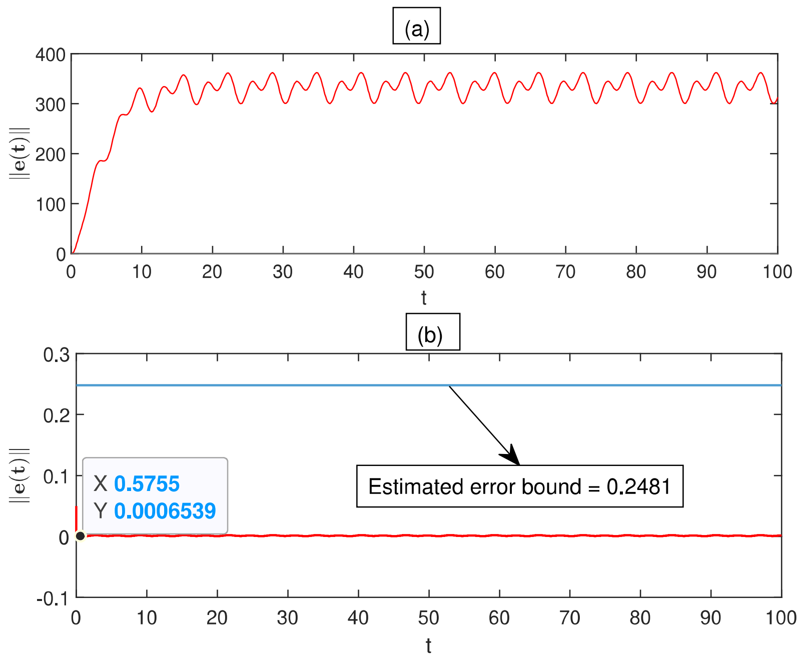

FIGURE 3.4: Plots of the error system (3.7) for **(a)** without controllers and **(b)** with controllers for case I.

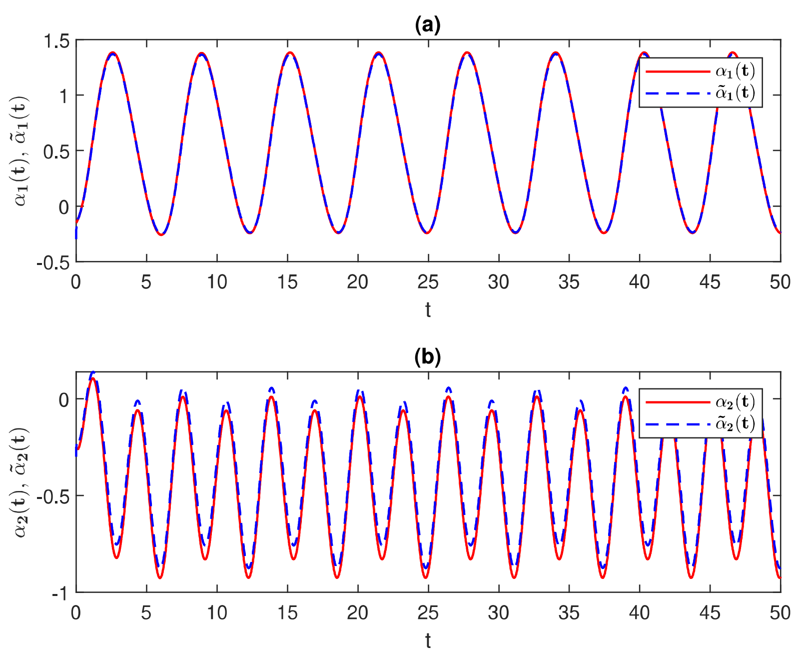

FIGURE 3.5: The state trajectories of the systems (3.19) and (3.20) for **(a)** $\alpha_1(t)$, $\widetilde{\alpha}_1(t)$, and **(b)** $\alpha_2(t)$, $\widetilde{\alpha}_2(t)$ with time t for the case II.

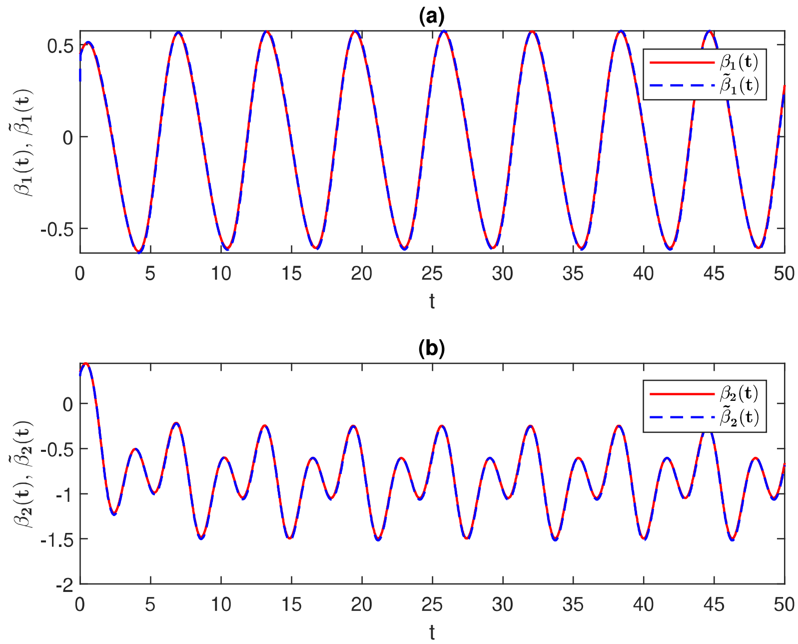

FIGURE 3.6: The state trajectories of the systems (3.19) and (3.20) for **(a)** $\beta_1(t)$, $\widetilde{\beta}_1(t)$, and **(b)** $\beta_2(t)$, $\widetilde{\beta}_2(t)$ with time t for the case II.

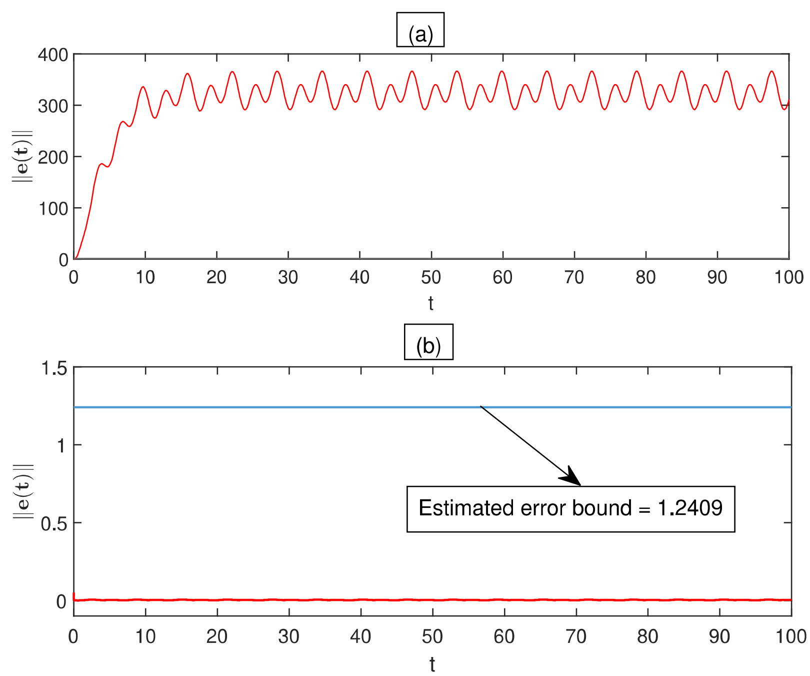

FIGURE 3.7: Plots of the system (3.7) for **(a)** without controllers and **(b)** with controllers for case II.

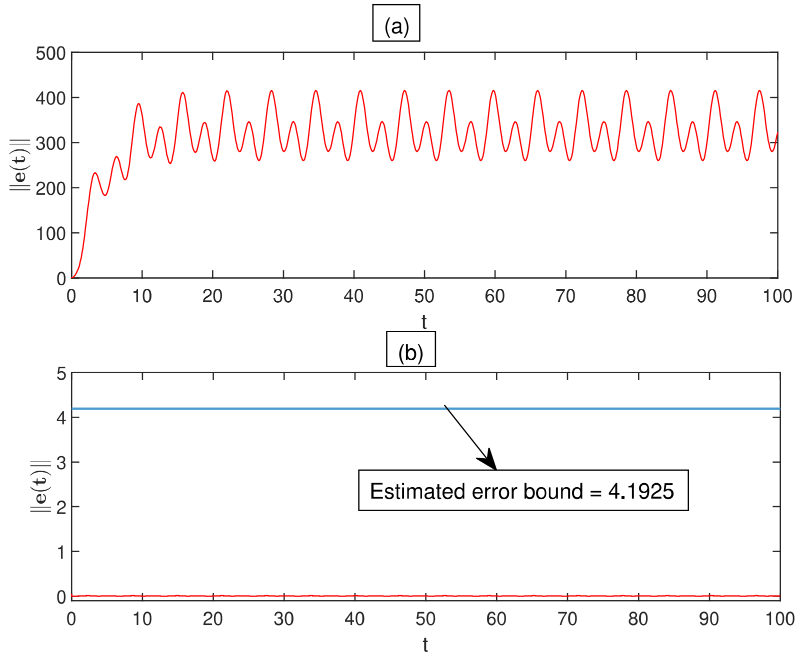

FIGURE 3.8: Plots of the system (3.7) for **(a)** without controllers and **(b)** with controllers for case III.

3.5 Conclusion

This chapter introduces the problem of quasi-projective synchronization in CVRNNs with proportional delay and mismatched parameters, considering two non-identical systems. The quasi-projective synchronization criterion is achieved by defining a simple Lyapunov function and employing a matrix measure approach with nonlinear Lipschitz activation functions. A suitable controller is constructed to facilitate this synchronization, and graphical representations illustrate the upper bounds of synchronization errors. This chapter also presents sufficient conditions for specific cases, and a numerical simulation validates the unwavering accuracy of the theoretical results. Innovatively, a keen interest in addressing a CVRNNs optimization problem has been expressed. The input parameters are considered decision variables, aiming to minimize the error bound for the lower value of the projective coefficient.

This forward-looking perspective underscores the commitment to advance the understanding and practical applications of CVRNNs.

****