Chapter 1

Introduction

1.1 Background

It is well-known that uncertainties including parametric variations, time delay and external disturbances are usually unavoidable in practical control systems due to data transfer, signal transformation, modeling inaccuracies, linearizion approximations, unknown inputs and measurement errors. and they can quite often be a source of instability or oscillations and poor control performance in control systems. Therefore, the design of any control schemes for dynamic systems subject to time delays and disturbances, must take into account these influences on the closed-loop performance.

For system with disturbances and uncertainties, sliding mode control has been recognized as a control methodology, belonging to the variable structure systems which are characterized by their robustness with respect to parameter variations and external disturbances [21]. The basic idea of the sliding mode is to drive the system trajectories into a predetermined hyperplane or surface in finite time, and maintain the trajectory on it for all subsequent time. During the ideal sliding motion, the system is completely insensitive to uncertainties or external disturbances. The dynamics and performance of the system then depend on the selection of the sliding surface. In sliding mode control, a sliding surface is first constructed to meet existence conditions of the sliding mode. Then, a discontinuous control law is synthesized to drive the system state towards the sliding surface in a finite time and maintain it thereafter on that surface. With the widespread use of digital controllers, many researchers have focused on discrete-time sliding mode control. For discrete-time sliding mode control, the state trajectory of the system will reach the switching surface and cross it in a finite time. When the trajectory has crossed the switching surface the first time, it will cross the surface again in every successive sampling period which leads to a zigzag motion about the switching surface. The size of each successive zigzagging step is stable and the trajectory stays within a specified band which is called a quasi-sliding mode band [51]. For discrete-time sliding mode control, a reaching law is first constructed. Then, a control law is often synthesized from the reaching law, in conjunction with a specified quasi-sliding mode band.

Sliding mode control is one of the most promising control technique for controlling plants under uncertain conditions. However, this controller is still not quite popular in industries because of their discontinuous nature. In the last two decades, some methods have been proposed to construct the continuous control action based on sliding mode. One of such algorithm is proposed by Prof. Levant known as higher order sliding mode [62,63]. The main idea is to introduce one or more integrators in the system such that the control signal becomes a continuous function. For instance, to obtain the absolute continuous control signal for the system, an integrator is introduced to increase the order of the system by one and then discontinuous higher order sliding mode algorithm can be used.

Many systems with time delay which are represented by functional differential equation has application in different areas [70]. Thus, analysis of stability of systems with time delay becomes important. A stable feedback system without delay may become unstable with some delays, whereas judicious introduction of delay may stabilize an unstable system [71] . In this work an artificially delayed output feedback based controller which is structurally similar to the twisting algorithm, but instead of using the information of output and its derivative, uses the information of output and an artificially delayed version of it. For example, in metal milling machines, spindle speed is appropriately adjusted to achieve chattering stability and in supply chain management, by adding delays for decision-making may be benefited for purchasing and stocking decision. The approach is otherwise known as wait-and-act control strategy.

Stability of time-delay systems is a basic issue in control theory and there have been considerable efforts devoted to the problem of stability analysis [3]. First of all, frequency-domain methods based on the distribution of the roots of its characteristic equation or the solutions of a complex Lyapunov matrix function equation, were used to determine the stability of time-delay systems [1]. However, the disadvantage of this method is that it is only suitable for systems with constant delays, since the solutions of systems with time-varying delay are much more complicated, due to its transcendental characteristic equations. Therefore, the application of frequency-domain methods for stability analysis of time-delay systems has serious limitations. Then, time-domain methods, in which the most common approaches to the stability analysis of time-delay systems are the Lyapunov-Krasovsky functional and Razumikhin function methods, were introduced. The main idea of these methods is to construct a set of appropriate Lyapunov-Krasovsky functionals or an appropriate Lyapunov function to obtain the stability criteria of time-delay systems in the form of existence conditions. At that time, this idea was theoretically very important, however, it was impossible to derive a general solution. However, this problem was solved when Riccati equations, linear matrix inequalities [123] and Matlab toolboxes came into use. Consequently, time-domain methods have been widely used to address the problem of stability analysis of time-delay systems.

1.2 Dynamical system

In general, dynamical systems can be modeled as

$$\dot{x}(t) = f(t, x(t)),\ \ t \in \mathbb{R}^+ \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ are the state variables. Let x_e be the equilibrium state in the sense that $f(t, x_e) = 0$, $\forall\ t \geq 0$ and the differential equation (1.1) characterize the evolution of the state variables with respect to time. The system remains at equilibrium point as long as there is no external action on it. The question is, if there is an external action, will the system remain near the equilibrium point or will it move farther and farther away? Shifting the origin of the system allows us to move the equilibrium point to $x_e = 0$. If there are multiple equilibrium points, the stability of each must be studied by an appropriate shift of the origin. It is assumed that the future evolution of the systems can be determined on the basis of current value of the state variables.The values of the state variables $x(t), t_0 < t < \infty$ for any initial time t_0 can be obtained by using the initial condition $x(t_0) = x_0$.

In reality, due to modeling errors, aging, uncertainties and disturbances in any practical problem, there exist a perturbation term $g(t, x(t))$. In a typical situation, one do not know the information about this term, but one can estimate some information such as upper bound of $\|g(t, x(t))\|$. Generally, the perturbation is considered as additive term on right hand of the state equation. Uncertainties that do not change the order of the system are always expressed in this form. Therefore, the dynamical system with perturbation term can be represented as

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t)), \; t \in \mathbb{R}^{+} \tag{1.2}$$

In addition, there exist some systems in which future evolution of the state of a dynamic system depends not only on current values, but also on past ones, then the system is called a time-delay system [1,2]. Time-delay systems are also known as systems with after-effect or delay [3,4]. Mainly, the delay phenomena appear in almost all the dynamics such as biology, chemistry, economics, mechanics, physics, psychology and in engineering.

For time-delay system, the evolution of states are expressed in finite dimensional Euclidean space or in functional space. The most extensively used representation is based on functional differential equations [1,5,6]. The retarded functional differential equation takes the form as

$$\dot{x}(t) = f(t, x_t), \; t \in \mathbb{R}^{+} \tag{1.3}$$

where $x(t) \in \mathbb{R}^n$ are state variables: $x_t = x(t+\theta)$, $-h \leq \theta \leq 0$, $h > 0$ is the time delay; $f(t, x_t) : \mathbb{R}^{+} \times \mathbb{C} \to \mathbb{R}^n$, where $\mathbb{C}$ are the set of continuous functions mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ in the time interval $t - h \leq \varphi \leq t$. Clearly, to obtain the evolution of $x(t)$ at any instant of time $t \geq t_0$, then one must know x_t for $-h \leq \theta_0$, which is therefore defined as the initial condition and is denoted by $x_{t_0} \in \mathbb{C}$.

1.3 Classification of systems with disturbance

According to commonly accepted denomination introduced in [8,51], uncertain system can be represented based on how the disturbance affects the evolution of the states, as the following.

1.3.1 Systems with vanishing and non vanishing disturbance

Consider a system as

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t)),\ t \in \mathbb{R}^+ \tag{1.4}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ and $g : [0, \infty) \times D \to \mathbb{R}^n$ are piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. It is also considered that $x = 0$ is an equilibrium point of (1.4).

A disturbance is vanishing or non vanishing depends on whether $g(t, x(t))$ is vanishing at the origin or not. If $g(t, 0) = 0$, the system has an equilibrium point at the origin. So the syatem is called as vanishing disturbance system. However, if $g(t, 0) \neq 0$, the origin will not be an equilibrium point and it is called as non-vanishing disturbance.

1.3.2 Systems with matched and unmatched disturbance

Consider the system as

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + g(t, x(t))(u + \delta_1(t, x(t))) + \delta_2(t, x(t)),\ t \in \mathbb{R}^+ \\ y &= h(x, t) \end{aligned} \tag{1.5}$$

where f and g are smooth known functions. $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state vector and control input respectively. $y = h(x, t)$ is a sufficiently smooth output function. $\delta_1(t, x(t))$ is the matched disturbance because it enters through the control channel, whereas $\delta_2(t, x(t))$ is the unmatched disturbance.

1.4 Classification of systems with time-delay

Apart from disturbance, time-delays are often encountered in system models. According to commonly accepted denomination introduced by [1, 5], linear system with time delay can be represented on the basis that how the delay affects the evolution of the states, as the following.

1.4.1 Systems with discrete delays

In discrete delay systems, the evolution of states depend on states at some specific past time instants and can be represented as:

$$\dot{x}(t) = Ax(t) + A_1x(t - \tau_x) + Bu(t) + B_1u(t - \tau_u) \tag{1.6}$$
$$y(t) = Cx(t - \tau_y)$$

where $x(t)$, $u(t)$ and $y(t)$ are the state, input and output respectively. τ_x, τ_u and τ_y are state, control input and output delays respectively.

1.4.2 Systems with distributed delays

In these systems, the delay acts in state $x(t)$ and in input $u(t)$ in a distributed manner as shown below

$$\dot{x}(t) = Ax(t) + \int_{-\tau_x}^{0} A_1(\theta)x(t+\theta)d\theta + Bu(t) + \int_{-\tau_u}^{0} B_1(\theta)u(t+\theta)d\theta \tag{1.7}$$

In distributed delay systems, the delay does not have local effect as in point wise delay but acts in distributed fashion over the entire delayed interval.

1.4.3 Neutral delay systems

A neutral system is a system with a delay in both the state and the derivative of the state, with the one in the derivative being called a neutral delay. That makes it more complicated than a system with a delay in only the state. Neutral delays occur not only in physical systems, but also in control systems, where they are sometimes artificially added to boost the performance. In neutral delay systems, the delay acts in the state derivative terms and can be represented as

$$\dot{x}(t) = f(x_t, \dot{x}(t), u_t, t) \tag{1.8}$$

or

$$F\dot{x}(t) = f(x_t, u_t, t) \tag{1.9}$$

where $F \in \mathbb{R}^n$ is regular operator. A neutral time delay equations are called as neutral functional differential equations which describes a system in which the rate of change of the state depends not only on the present and past values of the system state but also on the earlier values of the state rate change. The modeling of transmission lines and population dynamics is done using neutral time-delay systems.

1.5 Lyapunov stability Definitions and Theorems

For the stability analysis of systems without delay, Lyapunov based method has been widely investigated in the literature. In this section the stability definitions and Lyapunov stability theorems [51] are recalled.

1.5.1 Stability definition for system without delay

The equilibrium point $x = 0$ of (1.1) is

- stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon, \qquad \forall\, t \geq t_0 \geq 0 \tag{1.10}$$

- uniformly stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of t_0, such that (6.4) is satisfied.

- unstable if it is not stable.

- asymptotically stable if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$.

- uniformly asymptotically stable if it is uniformly stable and there is a positive constant c independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; for each $\eta > 0$, there is $T = T(\eta) > 0$ such that

$$\|x(t)\| < \eta, \qquad \forall\, t \geq t_0 + T(\eta), \qquad \forall\, \|x(t_0)\| < c \tag{1.11}$$

- globally uniformly asymptotically stable if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $lim_{\epsilon\to\infty}\ \delta(\epsilon) = \infty$, and each pair of positive numbers η and c, there is $T = T(\eta, c) > 0$ such that

$$\|x(t)\| < \eta, \qquad \forall\, t \geq t_0 + T(\eta, c), \qquad \forall \|x(t_0)\| < c \tag{1.12}$$

1.5.2 Stability Definitions for system with delay

Defining a state norm as $\|x_t\|_c = max_{t-h\leq\varphi\leq t}\|x(\varphi)\|$, the stability definitions for (1.3) in the sense of Lyapunov are as follows.

- The system (1.3) is stable if for a $\epsilon > 0$ there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta$ implies $\|x_t\|_c < \epsilon$ for all $t > t_0$.

- It is uniformly stable if for a $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta$ implies $\|x_t\|_c < \epsilon$ for all $t > t_0$.

- It is asymptotically stable if there exists a $\delta(t_0)$ such that $\|x_{t_0}\|_c < \delta(t_0)$ implies $lim_{t\to\infty}x(t) = 0$.

- It is uniformly asymptotically stable if for every $\epsilon > 0$ there exists a $\delta > 0$ and a $T(\epsilon) > 0$ such that $\|x_t\|_c < \epsilon$ for all $t > t_0 + T(\epsilon)$ whenever $\|x_{t_0}\|_c < \delta$.

- It is bounded if there exists a $\beta > 0$ such that $\|x_t\|_c < \beta$, where β may depend on each solution.

- It is uniformly bounded if for any $\alpha > 0$ there exists a $\beta = \beta(\alpha)$ independent of t_0 such that if $\|x_{t_0}\|_c < \alpha$, then $\|x_t\|_c < \beta$ for all $t \geq t_0$.

- It is uniformly ultimately bounded if there exists a $\gamma > 0$ and if corresponding to any $\alpha > 0$ there exists a $T(\alpha) > 0$ such that $\|x_{t_0}\|_c < \alpha$ implies $\|x_t\|_c < \gamma$ for all $t \geq t_0 + T(\alpha)$.

On the basis of stability definitions defined above, one can ascertain the stability of systems with and without delay using the Lyapunov stability theorems. For the stability analysis of systems without delay generally Lyapunovs direct method has been considered, whereas for stability of delayed systems mainly two different approaches are used such as (a) an evolution in the Euclidean space (Lyapunov-Razumikhin (LR) functions) (b) as an evolution in a functional space (Lyapunov-Krasovskii (LK) functionals). It is demonstrated in literature that LK method yields less conservative results than LR one. Next, we describe the LR and LK theorems [1, 5] developed in the literature.

1.5.3 Lyapunov stability theorem

Lyapunov used classical mechanics to investigate how the distribution of the energy field of a system influences its stability, and then devised a method based on the above definition of stability to determine the stability of a system without explicitly integrating a differential equation. This is called Lyapunovs direct method or the second method of Lyapunov. Let $x = 0$ be an equilibrium point for (1.1) for $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{1.13}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0 \tag{1.14}$$

$\forall\ t \geq 0$ and $\forall\ x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite function on D. Then $x = 0$ is uniformly stable.

1.5.4 Lyapunov-Razumikhin Theorem

Suppose the vector field f of (1.3) is bounded for bounded values of its arguments and $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, is strictly increasing. The trivial solution of (1.3) is uniformly stable if there exists a differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$, which is positive-definite, then

$$u(\|x(t)\|) \leq V(t, x(t)) \leq v(\|x(t)\|) \tag{1.15}$$

and such that the derivative of V along the solution x(t) of (1.3) satisfies

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|) \text{ if } V(t+\theta, x(t+\theta)) \leq V(t, x(t))\ \forall\ \theta \in [-h, 0] \tag{1.16}$$

If, in addition, $w(s) > 0$ for $s > 0$ and there exists a continuous nondecreasing function $\rho(s) > 0$ for $s > 0$ such that condition (1.16) is strengthened to

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|) \text{ if } V(t+\theta, x(t+\theta)) \leq \rho(V(t, x(t)))\ \forall\ \theta \in [-h, 0] \tag{1.17}$$

then the trivial solution of (1.3) is uniformly asymptotically stable. If, in addition, $lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

1.5.5 Lyapunov-Krasovskii Theorem

The LK method is an extension of the LR method dedicated to the stability analysis of functional differential equations. It consists in selecting energy functionals, i.e. functions of the functional state x_t of the form $V(t, x_t)$, that are positive definite and decreasing along the trajectories of system (1.3). The LK theorem is stated next.
Suppose the vector field f of (1.3) is bounded for bounded values of its arguments and $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, is strictly increasing. if there exists a positive differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$, then

$$u(\|x(t)\|) \leq V(t, x_t) \leq v(\|x_t\|_c) \tag{1.18}$$

and

$$\dot{V}(t, x_t) \leq -w(\|x(t)\|) \tag{1.19}$$

If $w(s) > 0$ for $s > 0$ then it is uniformly asymptotically stable and if, in addition, $lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.
The primary concept behind the statements of these theorems are to determine positive definite function and functional respectively, such that its derivative with respect to time along the solutions of the system are negative definite. The key problem within the application of these theorems are in the design of functions and then to obtain conditions that ensure its positive definiteness and the negative definiteness of their derivatives. Although the LR approach generally leads to more conservative results than those of LK theorem, but it allows taking into account variable delays without restriction on derivative of the delay function and leads to delay-independent stability conditions. It has also been shown that for constant delays, the existence of a LR function is equivalent to the existence of a LK functional. A brief literature survey based on LR method for stability analysis of systems with delay is presented next.

1.6 Literature Review on stability analysis

When we look back into the history of stability analysis, there are enormous ways to discuss the stability of a nonlinear system. But, we emphasize on the Lyapunov's method [33], where a sufficient condition of positive definite function with a negative semidefinite derivative is discussed. Although, the Lyapunov method is motivated by the problems of astronomy in which the stability of the motion of plants is considered to decide the boundaries of a stable region. The concept of a Lyapunov function has been turned out to be extremely fruitful in various branches of science as meteorology, biology, computer science, and economics. The Lyapunov function is considered as one of the most useful general theory for the stability of dynamical systems. Classical mechanics tells us that, in a physical system, a mass is less stable when it has a high energy than when it has a low energy. Thus, when a particle moves from an unstable state towards a stable state, its energy must necessarily continuously decrease.

1.6.1 Literature Review on Lyapunov direct method

The widely used Lyapunov direct methods or second Lyapunov method has its foundation on the so called energy approach to stability analysis. It consider a positive definite function as a energetic characteristic (energy) of a dynamic system and studies growth of this energy in time. If a dynamical system has an energy function, which is decreasing or bounded along the trajectories of the system, then the system has a stability property and the corresponding energy function is called Lyapunov function. As an example, for analyzing the asymptotic stability of the origin of the system (1.1), it is sufficient to select a continuous positive definite function $V(x(t))$ such that for any solution $x(t)$ of the system (1.1), the function $V(x(t))$ is decreasing and tending to zero for $t \to \infty$. The existence of such function guarantees asymptotic stability of the origin of the system (1.1) due to Zubov theorem [34].

If the function $V(x(t))$ is continuously differentiable then the required monotonicity property can be rewritten in the form of the classical condition [33]

$$\dot{V}(x(t)) = \nabla^T V(x(t) f(t, x(t)) < 0 \tag{1.20}$$

The inequality (1.20) is quite useful, since it does not require knowing the solutions of (1.20) in order to check the asymptotic stability. From practical point of view, it is important to represent monotonicity conditions in the form of differential or algebraic inequalities like (1.20).

The converse of Lyapunov direct method for nonlinear systems is a topic of substantial ongoing research in the Lyapunov theory research area. It states that if the origin of an differential equation is asymptotically stable, then the function defined by a semidefinite integral of an suitable selected function of the norm of the solution is a continuous differentiable Lyapunov function (LF). It is known that to find an explicit form of a LF for nonlinear systems is a tough task. One of the constructive result on answering the converse problem has been introduced in [35], for autonomous systems and an analytic formula of LF is introduced. As per the recent works, for the cases of differential inclusion, a converse theorem for uniform asymptotic stability (AS) of a compact set was proposed in [36]. For the cases of homogeneous systems, it was illustrate that AS provides the existence of smooth homogeneous LF. Further converse result for differential inclusion for stability with two measure were considered in [37]. If control inputs are to be taken, an existence results of control LF under the assumptions of asymptotic controllability was introduced in [38].

To simplify the construction of LF for the analysis and controller design of nonlinear systems, a new form of LF have been introduced such as vector LF as an alternative to scalar LF. Vector LF were introduced by Bellman [9], and further developed in [11][12], exploiting their utility for analysis of large scale systems. The use of vector LF in dynamical system theory offer a flexible framework since each components of the vector LF can satisfy less rigid requirements in comparison to a single scalar LF. In particular, each component of vector LF need not to be positive definite with a negative or even negative semidefinite derivative. On the other hand, time derivative of vector LF need only to satisfy an element by element inequality involving a vector field of a comparison system. Since in these cases the stability property of the comparison system imply the stability property of the dynamical system. The usage of vector LF can significantly decrease the complexity of the dynamical system being analyzed. Extension of vector LF theory that includes relaxed condition on standard vector LF as well as matrix LFs proposed in [39].

A strong (or strict) LF, is a positive definite function with negative definite derivative, has been introduced in [40], in which an estimates of convergence time for the perturbed STA is derived. Note that a strict LF is monotonically decreasing along trajectory, in comparison to weak Lyapunov functions which are just non increasing along trajectory of systems. By using the strong Lyapunov functions it is possible to demonnstrate that, in the absence of perturbation, the algorithms can converge globally and in finite time to the origin. In case of bounded perturbations, these functions are robust in nature and monotonically decreases along the trajectory of the systems for all possible perturbation. However, its form makes it difficult to operate with it for application purpose or further development. In comparison, the strict LFs proposed in [41] are simple, because they are Quadratic like functions, similar to the ones employed for linear systems. In that article, modifications to the STA are proposed consisting of the addition of linear correction terms by replacing the nonlinear terms of the STA. this extra degree of freedom provides a stronger attractive force to the algorithms, when the trajectories are far away from the origin. These are the cases since the added linear terms are stronger as compared to the nonlinear ones far from the origin. So, faster convergence and increased robustness of the stability to wider classes of perturbation are obtained. Because the modified algorithms are not homogeneous, so it is not possible to use in homogeneity theorem [42] for proving of convergence and stability robustness.

1.6.2 Literature Review on Lyapunov-Razumikhin Method

It has been observed that stability problem for time-varying time-delays can be mainly dealt with the LR methodology better than the LK methodology, because, in the former case, the derivative of the time delay is not present in the derivation of the Lyapunov functions on the system solutions [43]. It is also known fact that, the use of Lyapunov function constructed on Euclidean space may be easier than the usage of Lyapunov functional defined on the infinite dimensional Banach space of the system. LR method is considered as a particular cases of the methods of Lyapunov functional, but it provides us to exploit the Lyapunov functions, in Euclidean space, well demonstrated for systems represented in ordinary differential equations.

The Razumikhin results allows to obtain stability result based on the Lyapunov functions instead of Lyapunov functionals. These set of results leads to more simple and compact stability criteria in the classical context of differential equations with delay. It was recognized in the work of [44], that the Razumikhin approach could be considered as small gain theorem in the context of input to state stability (ISS). The primary idea was to consider the delayed state variables as disturbances, and then the application of small gain criterion provides a desired stability property. The key advantage of the Razumikhin method is that instead of carrying out stability analysis in functional differential equation settings, one can work in the more familiar domain of stability analysis for delay free systems in the context of ISS. One can point out that, although Razumikhin method provides sufficient conditions rather than equivalent conditions for stability properties, however the Razumikhin results in combination with small gain theorem gives a rich collection of tool in robust stability analysis for systems with delays.

A widely used method for stability of systems with delay, which directly deals with the non-augment system, is the Razumikhin approach, see, [45] which employed a class of Lyapunov function that requires to decrease only if some conditions on the past state trajectory and the current state has been satisfied. A control LyapunovRazumikhin function (CLRF) and static state feedback controller synthesis for continuous time system with multiple state delays and actuator saturation has been introduced in [46]. In which, the controller and the CLRF are selected such that the closed loop region of attraction has been maximized in presence of symmetric input constraints. A translation of Razumikhin method to discrete-time systems was proposed in [47] by requiring the candidate Lyapunov function to be less than the maximum over its past value within a time window. So far, the application of CLRF has been limited to some particular sub classes of polytope delay difference equations, see, [47], where a synthesis algorithm has been developed for linear time invariant system subject to input delay with a strictly stable matrix multiplying the current states.

1.6.3 Literature Review on Lyapunov-Krasovskii method

Based on the existing results, the delay dependent stability of linear systems with time delay are of two types : frequency domain framework and time domain framework. The recent developments in the frequency domain is integral quadratic constraint approach [48], which provides the system stability in frequency domain in terms of integral constraint based on Fourier transform of the input and output signals. In time domain framework, the Lyapunov-Kresovskii functional (LKF) method is a widely used tool for studying stability of linear systems with time-delay [1]. In this approach the key problem is the estimation of largest possible upper bound (LPUB) of delay that the system can tolerate by maintaining stability. The LKF method for estimating the LPUB of delay for linear time-delay systems is based on the selection of a proper LKF functional candidate. But, it is a challenging task to construct exact LKF so that some necessary and sufficient stability criteria can be obtained for system with delay. A large number of LyapunovKrasovskii functionals are constructed in order to decrease the conservativeness of the stability condition such as (i) Augmented Lyapunov-Krasovskii functional [88] in which the information of the delayed states is exploited by augmentation in the quadratic terms (ii) LKF with multiple-integrals [87], which introduces of higher order integral terms (iii) Delay product type LKF [88] which exploits the information of delay and its derivative (iv) Delay partitioning LKF [76] which is based on the partition of delay interval into several subintervals.

The double integral based quadratic term has been widely used in the formation of LKF, since its derivative contains the delay terms [83, 85]. The time derivative of this double integral term produces an single integral function that has no equivalent LMI form. Therefore, approximating this integral function is considered as one of the main issue to get LPUB of delay. In earlier works Jensen inequality [1] has been widely used to estimate this integral term. In recent years, many inequalities have been introduced to obtain better results as compared to the Jensen inequality, such as Wirtinger inequality [77], Auxiliary function based inequality [78], Bessel-Legendre inequality [79] and Free matrix based integral inequalities [80, 81].

The bounding integral inequalities such as Jensen inequality, inequality and Bessel-Legendre inequality directly estimates the integral function in the derivative of LKF into quadratic terms of integral quantities. Although these terms have shown a great interest for constant delay systems but these terms appeared as linear combinations of positive functions weighted by inverse of convex parameters in case of time varying delay systems. To relax such linear combinations of positive function with inverse of convex parameters as the coefficients a new reciprocally convexity lemma has been introduced in [49]. This idea has been extended in [50] by including four additional matrix variables. The extended reciprocally convexity lemma is helpful to derive less conservative condition, but introduction of four slack matrix variables enhances the computation burden of the obtained criterion. In [82] an improved reciprocity convex lemma is proposed, which provides a bigger lower bound for the reciprocally convex combination, while less number of slack matrix variables are used as compared with the extended reciprocally convex inequality in [50].

To develop the stability criteria, it requires to find the conditions that ensures the negativity of the derivative of the LKF. Due to the usage of higher order Bessel-Legendre inequality to approximate the derivative of LKF, the stability criteria is in the form of quadratic function with respect to the time-varying delay. It is the key issue to find negative definite conditions of the resulting quadratic function to obtain tractable LMI related stability criteria. So far, some results has been proposed on the negative determination of quadratic function with respect to time-varying delay in [85, 107, 108]. In order to make the quadratic function negative, a set of three inequalities are introduced in [85], whereas in [107] and [108], a scalar tuning parameter has been used to form a new set of inequalities. Recently, two lemmas have been reported in [109] for the negativity requirement of quadratic polynomial by exploiting the property of cross-point between two tangent lines of the finite delay interval.

1.7 Literature survey on controller design

1.7.1 Sliding Mode Control

Several delicate controllers have been developed to enhance control performance and robustness. Sliding mode control (SMC) [56] is one of them. It is one of the most promising control technique for controlling of systems with uncertain condition. However, this controller is not so popular in industries due to its discontinuous nature. In past two decades, some algorithms have been introduced to construct the continuous control law based on sliding mode. One of such idea is known as Higher order sliding mode [62, 63]. The main idea is to introduce one or more integrators in the system such that the control signal becomes a continuous function. Sliding Mode Observers using first order sliding mode are considered as an alternative to the issue of observation of perturbed systems. The issue of unknown input estimation and state observation have been developed based on Sliding Mode approach. The sliding mode observers are extensively used due to the finite time convergence of output states, robust with respect to uncertainty and the possibility of estimation of uncertainities [20, 21]. But these are relevant to the conditions when the relative degree of disturbances with respect to unknown measured output is one. Next generation of observers as per second-order sliding-mode approach have been recently proposed. The Super-Twisting Algorithm is a popular second order sliding mode algorithm developed in [63] and it has been extensively used for observation, control and exact robust differentiation [22].

The presence of delay within a sliding mode control can induce oscillations around the sliding surface. In spite of some extension in infinite dimensional systems and of differential inclusions to after effect system, the concrete control results are not so many [28]. In [29], a controller design has been proposed guaranteeing a robust convergence of sliding mode under state delays. The basic idea was to add integral effect in the input, to get a system with increased order but without state delay, then to combine a normal form for delay system with the use of the Lyapunov-Krasovskii method. However, this strategy may not allow adequate disturbance rejection.

A new approach has been introduced in [30] for control of uncertain linear/nonlinear systems using sliding mode with multiple delayed-partial state feedback. In [31, 32], one can found that for systems, the presence of delay can produce stabilizing effect. This triggers the possibility of taking a system which is not stabilizability using static output feedback without delay and finding a constant delay strictly greater than zero such that the system is stable. In this case, a stabilizing delay is introduced into the dynamics to effect output feedback stability. This design concept is not new, several authors have considered this possibility and it has been shown that introducing a delay in an output feedback controller can stabilize a system which cannot be stabilized without delay.

Most of the articles in literature deal with systems without state delay. In these cases, state predictor based algorithm [27] can be computed from either model order reduction or in the forms of finite spectrum assignments. For without delay cases, it is known that if a complex plant may be written in normal form, then an suitable sliding mode algorithm can be designed to influence the nonlinear terms and the disturbance, provided the disturbances satisfies proper matching conditions.

1.7.2 Proportional and Integral Control

Integral control plays an important role because it ensures disturbance rejection and asymptotic tracking when the exogenous signal approaching constant limits [10, 11]. Also it is robust for system parametric uncertainties such that asymptotic tracking and rejection of disturbance take place for certain uncertainty that preserves the closed loop stability. For general linear system, the design of integral control were done in the works of [12, 13]. Later on Isidori and Byrnes [14] extended the idea of integral control to nonlinear systems. In [15], singular perturbation tools are used to study the disturbance rejection capabilities of multi variable PI controllers. They illustrate that if the nonlinear system is exponentially stable and has strictly increasing steady-state input-output gain, then a PI compensator can be utilized to provide a stable closed-loop system which asymptotically tracks reference signals and rejects disturbances. It is reported in [51], if disturbance vanishes at the equilibrium point, then classical feedback can ensure the asymptotic stability. But for non-vanishing disturbances at the origin, the memoryless state feedback control does not guarantee uniform asymptotic stability of the system. It is reporting in [52–55] that simple PI controller can handle some classes of non-vanishing perturbations, but fails to tackle the time-varying perturbations.

Delays exists naturally in various control applications, such as delay in process and in control interface. In general, delays are practically unavoidable in controlled systems based on communication networks, as data is delayed owing to propagation delay and buffering. The presence of delays also complicates controller design considerably. In the continuous time case, complications are caused by the fact that the delay element is infinite dimensional, so that many classical methods cannot be applied directly.

1.7.3 Observer based Controller Design

Uncertainties and disturbance widely exist in all industrial systems and adversely affects system performancea nd even may lead to instability of control systems [7–9]. Not surprisingly, rejection of disturbance and uncertainty is considered as the main problem in control system design. There are several approaches proposed in the literature to address this problem. In many practical scenario, the main objective is to design a feedback control law such that the output tracks a reference signal, irrespective of the unknown external perturbation or model uncertainty. Many methodologies were reported in control literature to simplify the above problem in the design of feedback control law. When the disturbance can be measured, it is well known that a feed forward approach could eliminate or attenuate the influence of disturbance.

When the external disturbance is too expensive to measure or cannot be directly measured, one way is to deal with this issue is to estimate the disturbance (or the influence of the disturbance) from measurable variables, and then a control action can be designed on the basis of disturbance estimate to compensate for the influence of the disturbance. This idea can be intuitively extended to deal with uncertainties where the influence of the unmodeled dynamics could be considered as a part of the disturbance. Consequently, in a similar fashion, the influence of the uncertainties could be removed, and system robustness could be enhanced [16, 17]. This motivates the development and application of a wide variety of disturbance/uncertainty attenuation algorithms. Although under various names and development from difference prospects, these methods share a similar basic idea, in which observation mechanism is designed to estimate disturbances or uncertainties, and corresponding compensation is then generated by making use of the estimate. In this setting, the disturbances refer to that from the external environment of a control system but also uncertainties of the controlled system including parameter perturbations and unmodeled dynamics [18, 19].

The time-delay control introduces a small delay value in the controller design, so that it reduces the consequence of additive disturbance representing unknown dynamics. This idea appeared in the early 1990s in [23, 24] mainly targeting at practical issues. In spite of adjusting control gains or identification of model parameters, its crucial idea is to use past observations with regard to both the system response and control input. Various results have been proposed for controlling of systems with state delays but without input or output delays. These leads to controllers with or without memory. However a more challenging and interesting problem is to control a system without instantaneous computational access to state variables, or via delayed actuators. This problem was first solved in [25, 26].

1.8 Motivation

In many practical systems, the uncertainty and time-delay are unavoidable because they can quite often be a source of instability or oscillations and poor control performance. So several controllers have been proposed in the literature to improve control performance and robustness. For systems with uncertainty sliding mode control (SMC), which is one of the most popular control algorithm. The other technique is a nonlinear PI controller known as the super twisting algorithm has been introduced which provides finite time stability. But the generalization and practical implementation of Super twisting technique is not so straight forward because one has to maintain the homogeneity in order guarantee finite time stability and it is difficult to implement the fractional power in an industrial environment. This motivates to work on nonlinear controller design to obtain continuous control action by ensuring system stability.

To give mathematical guarantee for the convergence of the designed closed loop control algorithm various forms of Lyapunov functions are proposed in the control theory. Lyapunov functions that has been reported in literature for stability analysis of closed loop systems are scalar Lyapunov functions, vector Lyapunov functions and strict Lyapunov functions. Vector Lyapunov functions were introduced and further developed to exploit the utility for analysis of large scale complex systems. The use of vector Lyapunov functions in dynamical system theory offer a flexible framework since each components of the vector Lyapunov functions can satisfy less rigid requirements in comparison to a scalar Lyapunov functions. The usage of vector LF can significantly decrease the complexity of the analysis. Strict Lyapunov functions are the positive definite functions with negative definite derivative, in which an estimate of convergence time for the perturbed super twisting algorithm is derived. Extension of these theories that includes relaxed condition on standard vector LF as well as matrix LFs motivates for further analysis in this direction.

For stability analysis of systems with time-delay, that deals with the non-augmented system, is the Lyapunov-Razumikhin approach. This approach consist of a class of Lyapunov function that requires to decrease only if some conditions on the past state trajectory and the current state has been satisfied. Although Razumikhin method provides sufficient conditions rather than equivalent conditions for stability properties, its combination with small gain theorem gives a rich collection of tool in robust stability analysis for systems with delays. Further, the condition for the delay has been obtained by using the Razumikhin method by considering the same Lyapunov function that is used for obtaining the controller gains. This motivates to work using Razumikhin approach for the stability analysis of delayed output based twisting algorithms.

For delay dependent stability analysis of linear systems with time varying delay Lyapunov-Krasovskii theorem is best method and provides conditions in the form of linear matrix inequalities. But, this method yields results that are conservative. To deal with conservativeness more general LKF has to be designed. Various methods have been reported such as delay partitioning, augmentation type LKF, inequality based LKF and delay product type LKF. These functionals helped to get enhanced results to certain extent. Still there are scopes to constitute new types of LKFs to improve the existing results and to overcome the conservatism. The other method to get less conservative stability criteria of time-delay systems is to find a precise bound of the quadratic integral function in the derivative of LKF. To find an effective inequality for integral function, some inequalities have been proposed in literature such as the Jensen inequality, Wirtinger inequality, auxiliary function based inequality, Bessel-Legendre inequality and free matrix based inequality. Also, in order to estimate the derivate of LKF, several approaches has been introduced in the form of reciprocal convex lemmas. It should be noted that such quadratic inequalities introduce conservativeness since one has to approximate the integral function in the derivative of LKF.

Due to the above two techniques, namely construction of LKFs and estimation of its derivative, the stability criteria obtained is in the form of quadratic polynomial with respect to time varying delay. So, the third important issue is to get negative definiteness conditions of this quadratic function to formulate it in LMI form. Generally, a few works have been reported on the negative determination of this quadratic function. In some works the quadratic function have been estimated by suitable inequality constraints, but these inequalities are still conservative which motivates the current research to develop a relaxed requirement to overcome conservativeness.

1.9 Thesis Objectives

The main objectives of the thesis are described as follows:

First of all, a brief survey on recent development of various methods in stability analysis such as Lyapunov direct method for uncertain systems, Lyapunov-Razumikhin approach for artificial delayed systems, Lyapunov-Krasovskii for time-delay systems are presented. Further a survey on sliding mode controller design, PI control method and observer based control of uncertain and time-delay system are discussed.

The second objective of the thesis is to design control law such that overall control input is continuous and does not require any extra information other than the state variables. To achieve this proposed nonsmooth PI controller for an uncertain chain of an integrator is established for first order system via new Lyapunov functions for systems with vanishing or non-vanishing disturbance such that these functions can provide mathematical proof for the convergence analysis of the modified algorithm; then it is extended to an uncertain chain of an integrator.

The third objective is to investigate the problem of stabilizing ability of artificial delay for systems with perturbation or disturbance. For this purpose (i) a less conservative stabilization criteria by modifying super twisting algorithms for systems with input-delay to be designed (ii) constructing Lyapunov-Razumikhin functions for obtaining controller gains in the presence of time-varying disturbances and to estimate the upper bound of the artificial delay by ensuring closed loop stability. (iii) Apart from time varying disturbance rejection, the proposed controller can also be used as an observer. Moreover, it can also mitigate the unmatched disturbance for an uncertain second order system.

Finally, the problem of delay-dependent stability analysis of linear systems with time varying delay using Lyapunov-Krasovskii method has been addressed. The main issue in this method is to find larger upper bound delay value by ensuring negative definiteness of derivative of the Lyapunov-Krasovskii functional. For achieving above difficulties some less conservative stability criteria have been proposed by designing new delay product type LKF. The new LKF consists of new states in augmented vectors to achieve less conservative stability criteria. The new delay product LKF proposed are utilized in the stability analysis of neural networks with time-varying delay and to get improvement in the bounds of delay.

1.10 Organization of the Thesis

The present chapter provides a brief review on stability and stabilization of uncertain and time-delay systems. Remaining of the thesis is organized as follows.

Chapter 2 investigates modified classical Proportional-Integral (PI) controller for the problem of nonsmooth feedback stabilization for the higher order uncertain chain of integrators. In order to achieve the specific goal an integral of the discontinuous function has been used in place of the integral term of the PI controller. The stability analysis of closed loop system controlled by nonsmooth PI controller has been carried out by constructing a strict Lyapunov function. The efficacy of the proposed controller is demonstrated using numerical simulation on the magnetic suspension system.

In Chapter 3, on the basis of super twisting algorithm an artificially delayed output feedback controller has been introduced for the uncertain systems with relative degree two.

The proposed controller considers information of output and its artificially delayed version to get practical asymptotic convergence to the origin. A novel continuous, weighted homogeneous and strict Lyapunov function has been proposed for the closed loop system controlled by proposed twisting algorithm.

Chapter 4 is based on the application of Lyapunov-Krasovskii method for delay dependent stability analysis of linear time delay system. Two improved stability criteria are proposed by constructing a novel delay coefficient based Lyapunov-Krasovskii functional. With the aid of improved reciprocal convexity lemma and bounding technique based reciprocal lemma the stability criteria are formulated in terms of linear matrix inequalities. To demonstrate the improvement offered by proposed criteria, two numerical examples are considered.

Continuing in the same direction, Chapter 5 presents two new stability criteria in terms of first and second order polynomial with respect to time varying delay. Both the conditions are derived by constructing a new Lyapunov -Krasovskii functional. Also a zero equality is formulated such that by augmenting it in the derivative of functional helps to overcome the conservativeness. The improvement provided by these criteria are shown by three examples.

In Chapter 6, stability analysis of delayed neural network has been presented by using Lyapunov-Krasovskii approach. First a delay product type Lyapunov -Krasovskii functional and a zero equality are proposed. On the basis of these functional and zero equality, two less conservative stability conditions are derived. A detail comparative study has been provided by demonstrating the trade-off between conservatism reduction and increase in computational complexity by considering two numerical examples.

Finally, Chapter 7 states the overall conclusions of the work with future perspectives led by the present work.