Chapter 4

Finite Direct Projective Covers and Envelopes

In this Chapter, we study finite direct projective covers and envelopes of modules. We find a condition for a finite direct projective cover to be a projective cover of a module. Also, we characterize the finite direct projective cover of a module over semi-perfect and semi-regular rings. In the last of this chapter, we discuss the concept of finite direct projective envelopes and establish its equivalence with semisimple Artinian rings and S-rings.

4.1 Finite Direct Projective Covers

This section deals with a finite direct projective cover of a module and we study some properties related to it. We begin by defining the finite direct projective cover of a module.

Definition 4.1.1. *A finite direct projective R-module S is said to be a finite Direct projective cover of R-module M if there exists epimorphism $f : S \longrightarrow M$ such that $Ker(f)$ is a small submodule of S.*

Note 4.1.2. *This chapter will only focus on module M that is not finite direct projective. This means that we only consider non-trivial coverings of M. It is worth noting that any finite direct projective module is a cover for itself, as the identity R-homomorphism has a kernel zero, which is always minimal.*

Proposition 4.1.3. *Let P be a projective module and $P \bigoplus M$ has a finite direct projective cover where M is finitely generated. If there is an epimorphism $f : P \longrightarrow M$, then M has a projective cover.*

Proof:- Suppose we have an epimorphism $f : P \to M$. Let $g : S \to P \oplus M$ be a finite direct projective cover of $P \oplus M$. We can define an exact sequence $0 \to g^{-1}(M) \to S \to P \to 0$ using the concept of exact sequences, where $\pi : P \oplus M \to M$ is a projection map. By the property of exact sequences, we know that $S \cong P \oplus g^{-1}(M)$. We can define $g' = g|_{g^{-1}(M)}$ and construct another exact sequence $0 \to Ker(g) \to g^{-1}(M) \to M \to 0$. Since $Ker(g)$ is small in S, we can conclude that $Ker(g) = Ker(g')$ is small in $g^{-1}(M)$, and $g' : g^{-1}(M) \to M$ is an epimorphism. We only need to prove that $g^{-1}(M)$ is a projective module. Also, we know that $S = P \oplus g^{-1}(M)$ is a finite-direct-projective module where P is projective. By setting $f = g' \circ h$, where $Ker(g)$ is small in $g^{-1}(M)$, we can conclude that h is an epimorphism such that $h : P \to g^{-1}(M)$, as per [1, 5.15]. By using Lemma 3.2.7, we can conclude that $g^{-1}(M)$ is a projective module. Therefore, M has a projective cover.

Proposition 4.1.4. *Every finitely generated R-module M has a finite direct projective cover if and only if it has a projective cover.*

Proof:- Given a finitely generated module M, we can always find a free module F of finite rank such that there is an epimorphism $f : F \to M$. Since $F \oplus M$ is finitely generated, it has a finite direct projective cover by assumption. By Proposition 4.1.3, we can conclude that M has a projective cover. The converse is obvious.

Corollary 4.1.5. *Let M be a Noetherian module. Then every submodule of M has a finite direct projective cover if and only if it has a projective cover.*

Proof:- The proof follows from the fact that submodules of a Noetherian module are finitely generated.

Proposition 4.1.6. *The following statements are equivalent for a ring R:*

1. *R is a semi-perfect ring;*
2. *Every 2-generated R-module M has a finite direct projective cover;*
3. *Every finitely generated submodule of an R-module M has a finite direct projective cover.*

Proof:- (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious. Since R is a semi-perfect ring, so every finitely generated submodule of an R-module M has a projective cover.

(2) $\Rightarrow$ (1) To prove that R is semi-perfect, we need to prove that every finitely generated module M has a projective cover. Since M is finitely generated, epimorphism $g : F \to M$ exists where F is a free R-module of finite rank. Since $F \oplus M$

is finitely generated, therefore P is a finite direct projective cover of $F \oplus M$. Then using Proposition 4.1.3, M has a projective cover.

(3) $\Rightarrow$ (1) Follows from Proposition 4.1.4.

Corollary 4.1.7. *Every R-module M has a finite direct projective cover if the ring is perfect.*

Proof:- Follows from Proposition 4.1.6, since every perfect ring is semi-perfect.

Let us recall the definition of finitely presented modules. An R-module M is said to be finitely presented or finitely related if there exists an exact sequence $0 \to T \to F \to M \to 0$ of R-modules, where F is free with finite rank and T is finitely generated. A ring is called semi-regular if every finitely presented module has a projective cover.

Proposition 4.1.8. *The following statements are equivalent for a ring R:*

1. *R is a semi-regular ring;*
2. *Every finitely presented R-module M has a finite direct projective cover.*

Proof:- (1) $\Rightarrow$ (2) Since R is semi-regular, every finitely presented module has a projective cover.

(2) $\Rightarrow$ (1) Let R-module M be finitely presented, hence there exists an exact sequence of R-modules $0 \longrightarrow T \longrightarrow F \longrightarrow M \longrightarrow 0$, where T is finitely generated, and F is a free module of finite rank. Let us consider module $F \bigoplus M$ since it is finitely presented and therefore it has a finite direct projective cover. This implies some finite direct projective module S covering $F \bigoplus M$. Using Proposition 4.1.3 and every finitely presented module is finitely generated, M has a projective cover.

Now, we will use the fact that every finitely generated coherent module is a finitely presented module.

Corollary 4.1.9. *Every finitely generated coherent module over semi-regular R has a finite direct projective cover.*

Proof:- Follows from the definition of the coherent module and the Proposition 4.1.8.

It is not necessary that every module has projective cover which is also true in the case of finite direct projective cover. Now we show this in the following example.

Example 4.1.10. *Consider $\mathbb{Z}$-module $M = \mathbb{Z} \bigoplus \mathbb{Z}_2$. We claim that M has no finite direct projective cover. We will prove it by using contradiction. If M has a finite direct projective cover, then from Proposition 4.1.3, $\mathbb{Z}_2$ has a projective cover. Since we know that $\mathbb{Z}_2$ does not have any projective cover as epimorphism $f : \mathbb{Z} \to \mathbb{Z}_2$ and $Ker(f) = 2\mathbb{Z}$ which is not small submodule in$\mathbb{Z}$ as $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$. Hence we got our claim that M does not have a finite direct projective cover.*

Remark 4.1.11. *Based on the above example, we conclude that the direct sum of two finite direct projective modules need not have a finite direct projective cover.*

4.2 Finite Direct Projective Envelopes

Next, we define finite direct projective envelopes and discuss some of their related results.

Definition 4.2.1. *An R-homomorphism $f : M \longrightarrow F$ is called a finite direct projective envelope of M if F is finite direct projective and the diagram below commutes,*

$$\begin{tikzcd} & F' \arrow[d, dashed, "\exists\Phi'"] \\ M \arrow[ur, "f'"] \arrow[r, "f"] & F \end{tikzcd}$$

where F' finite direct projective module and Φ' is an isomorphism,

$$\begin{tikzcd} & F \arrow[d, dashed, "\exists\Phi"] \\ M \arrow[ur, "f"] \arrow[r, "f"] & F \end{tikzcd}$$

and the diagram above can be completed only by an automorphism Φ.

In the next proposition, we use the fact, that if each cyclic module over ring is projective then the ring is semi-simple Artinian [19].

Proposition 4.2.2. *The following statements are equivalent for a ring R:-*

1. *R is a semi-simple Artinian ring;*
2. *Every 2-generated R-module has a finite direct projective envelope.*

Proof:- (1) $\Rightarrow$ (2) Let E be a 2-generated R-module over a semi-simple Artinian ring, then E is a semi-simple module. Hence E is a finite direct projective module and its envelope is trivial.

(2) $\Rightarrow$ (1) We are required to prove that every cyclic submodule N of M is projective. Since $E = N \bigoplus R$ is a two-generated R-module, it has a finite direct projective envelope.

$$\begin{tikzcd} & F \arrow[d, dashed, "\exists\phi"] \\ E \arrow[ur, "f"] \arrow[r, "p_N"] & N \end{tikzcd}$$

$$\begin{tikzcd}
 & F \arrow[d, dashed, "\exists\alpha"] \\
E \arrow[ur, "f"] \arrow[r, "p_R"] & R
\end{tikzcd}$$

with canonical projections p_N and p_R, respectively. Let $f : E \longrightarrow F$ be a finite direct-projective envelope. Since R and N are both finite direct-projective modules, there exist R-homomorphisms $\phi : F \longrightarrow R$ and $\alpha : F \longrightarrow N$ such that $\phi \circ f = p_R$ and $\alpha \circ f = p_N$. Define an R-homomorphism $g : F \longrightarrow N \oplus R$ as $g(x) = \alpha(x) + \phi(x)$ for each $x \in F$. It can be easily verified that g is a well-defined function, and also $g \circ f(x) = \alpha \circ f(x) + \phi \circ f(x) = p_N(x) + p_R(x) = x$. This implies that g is a split R-epimorphism, so we have $E \cong K \leq^{\oplus} F$. Using Proposition 3.1.4, we know that every direct summand of a finite direct-projective module is also a finite direct-projective module. Therefore, E is a finite direct-projective module. Since every onto homomorphism $h : R \longrightarrow N$ splits, we can use Lemma 3.2.7 to conclude that N is also projective since R is projective.

Proposition 4.2.3. *Let R be a ring and I be an ideal of R such that I is not direct summand and R/I is finitely generated and a finite direct projective module. Then R-module $M = R \bigoplus R/I$ has no finite direct projective envelope.*

Proof:- We will prove our claim by contradiction. Let f be R-homomorphism a finite direct projective envelope of M. The above diagrams

$$\begin{tikzcd}
 & F \arrow[d, dashed, "\exists\phi"] \\
M \arrow[ur, "f"] \arrow[r, "p_R"] & R
\end{tikzcd}$$

$$\begin{tikzcd}
& F \arrow[d, dashed, "\exists\alpha"] \\
M \arrow[ur, "f"] \arrow[r, "p_{R/I}"] & R/I
\end{tikzcd}$$

Consider p_R and $p_{R/I}$ be the canonical projections of M. Since both R and R/I are finite direct projective modules and $f : M \longrightarrow F$ is a finite direct projective envelope of M, we can find R-homomorphisms $\phi : F \longrightarrow R$ and $\alpha : F \longrightarrow R/I$ such that $\phi \circ f = p_R$ and $\alpha \circ f = p_{R/I}$. We define a function $g : F \longrightarrow M$ by $g(x) = \alpha(x) + \phi(x)$ for each $x \in F$. We can easily check that g is well-defined and that $g \circ f = i_M$, which implies that g is a split R epimorphism and M is isomorphic to a direct summand K of F, denoted $M \cong K \leq^{\oplus} F$. However, every direct summand of a finite direct projective module is finite direct projective. Every epimorphism $f' : R \longrightarrow R/I$ splits. Hence I is a direct summand of R, which is false. Therefore, there exists no finite direct projective envelope for M.

Example 4.2.4. *For $\mathbb{Z}$ module $M = \mathbb{Z} \bigoplus \mathbb{Z}_n$ for $n \in \mathbb{N}$ and $\mathbb{Z}_n$ is finitely generated and finite direct projective module over $\mathbb{Z}$. So module M has no finite direct projective envelope, and the argument is similar to the above Proposition.*

Remark 4.2.5. *Similarly, as Remark 4.1.11. We assert that the direct sum of two finite direct projective modules may not have a finite direct projective envelope.*

In the following proposition, we prove that every 2-generated flat module possesses a finite direct projective envelope if and only if the ring is S-ring. According to Puninski and Rothmaler [38], a ring R is called a right S-ring if every finitely generated flat right R-module is projective.

Proposition 4.2.6. *The following statements are equivalent for a ring R:*

1. *R is a S ring;*

2. *Every 2-generated flat R-module has finite direct projective envelope.*

Proof:- (1) $\Rightarrow$ (2) If R is S-ring then every finitely generated flat R-module is projective module. Hence has a trivial projective envelope which is also a finite direct projective envelope.

(2) $\Rightarrow$ (1) In particular, we prove this result for cyclic flat R-module, and that can be easily extended for finitely generated flat R-modules. We must prove that every cyclic flat R-module M is projective. Let us consider $T = M \bigoplus R$. Here T is a 2-generated flat R-module with a finite direct projective envelope. Therefore the following diagrams must commute:

$$\begin{tikzcd}
 & F \arrow[d, dashed, "\exists\alpha"] \\
T \arrow[ur, "f"] \arrow[r, "p_M"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
 & F \arrow[d, dashed, "\exists\phi"] \\
T \arrow[ur, "f"] \arrow[r, "p_R"] & R
\end{tikzcd}$$

with p_R and p_M be the canonical projections of T, and R and M are both finite direct-projective modules. Moreover, let $f : T \longrightarrow F$ be a finite direct projective envelope. Then, there exist homomorphisms $\phi : F \longrightarrow R$ and $\alpha : F \longrightarrow M$ such that $\phi \circ f = p_R$ and $\alpha \circ f = p_M$. Now, we define an homomorphism $g : F \longrightarrow M \bigoplus R$ such that $g(x) = \alpha(x) + \phi(x)$ for each $x \in F$. It can be easily verified that g is well-defined and satisfies $(g \circ f)_R = \alpha \circ f(x) + \phi \circ f(x) = p_M(x) + p_R(x) = x$. Thus, g is a split R-epimorphism, and we have $T \cong K \leq^{\oplus} F$. Using Proposition 3.1.4, we know that every direct summand of a finite direct-projective module is also finite direct-projective. Therefore, T is a finite direct-projective module. Additionally, since M is a cyclic module, an epimorphism $h : R \to M$ exists. Using Lemma 3.2.7 h splits since R is a projective module. Hence, M is also a projective module.