Chapter 6

Function projective Mittag-Leffler synchronization of non-identical fractional-order neural networks

6.1 Introduction

In this chapter, the authors addressed the concept of function projective Mittag-Leffler synchronization (FPMLS) among non-identical fractional-order neural networks (FONNs). Stability analysis utilizes an existing lemma for Lyapunov functions in FONN systems. Through the stability theorem of FONN, a non-linear controller was devised to achieve FPMLS. Additionally, the exploration of global Mittag-Leffler synchronization (GMLS) within the realm of other synchronization techniques like projective synchronization (PS), anti-synchronization (AS), and complete synchronization (CS) is undertaken. Leveraging the Caputo derivative definition, the Mittag-Leffler function, and the Lyapunov stability theory, various stability results were discussed for the FPMLS scheme in FONNs. Finally, a numerical example was employed to validate the efficacy of the proposed technique and the robustness of the applied synchronization conditions.

- Utilization of Mittag-Leffler stability theory, Lyapunov stability theory, and additional criteria to analyze FPMLS among non-identical FONNs.
- Achievement of FPMLS through designing a non-linear feedback controller and applying suitable lemmas and assumptions.
- Mittag-Leffler PS and AS are examples of instances of FPMLS among non-identical FONNs, with appropriate formulation of error functions.

6.2 Fractional-order system

Consider the n-dimensional fractional order system with the initial condition $\chi(t_0)$ as

$$D^q\chi(t) = \hat{\phi}(t, \chi(t)), \tag{6.1}$$

where $q \in (0,1)$ and $\chi(t) \in (\chi_1(t), \chi_2(t), \ldots, \chi_\Omega(t)^T \in \mathbb{R}^\Omega$, $t_0 \geq 0$ is the initial time. The system (6.1) is asymptotically stable at its equilibrium points if

$$|arg(eig(J))| = |arg(\lambda_\hbar)| > \frac{\pi q}{2},\ \hbar = 1, 2, \ldots, n, \tag{6.2}$$

is satisfied for all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_\Omega$ of the Jacobian matrix $J = \frac{\partial \hat{\phi}}{\partial \chi}$, where $\hat{\phi} = [\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_\Omega]^T$ is evaluated at the equilibrium points of the system (6.1) [117].

Theorem 6.2.1. [118] Consider the equilibrium point of the system (6.1) as $\chi = 0$, and $V(t, \chi(t)) : [0, \infty) \times D \longrightarrow \mathbb{R}$ be a continuous positive definite Lyapunov function given on a region $D \subset \mathbb{R}^\Omega$ that contains the equilibrium point. Supppose $\mathcal{B}_{\hat{r}} \subset D$ for some $\hat{r} > 0$. Then there exit $\delta_1, \delta_2 \in K$ (defined as class K function) on $[0, a]$ such that

$$\delta_1(\|\chi\|) \leq V(t, \chi(t)) \leq \delta_2(\|\chi\|), \forall \chi \in \mathcal{B}_{\hat{r}} \tag{6.3}$$

and

$$D^q V(t, \chi(t)) \leq -\delta_3(\|\chi\|). \tag{6.4}$$

If $D = \mathbb{R}^\Omega$, δ_1 and δ_2 are defined on $[0, \infty)$, the system (6.1) will be asymptotically stable. If $V(t, \chi(t))$, is radially unbounded, then $\delta_1, \delta_2 \in K_\infty$.

Theorem 6.2.2. [119] Take the equilibrium point of the system (6.1) to be $\chi = 0$. Assume that $V(t, \chi(t)) : [0, \infty) \times D \longrightarrow \mathbb{R}$ is a continuously differentiable function in a region $D \subset \mathbb{R}^\Omega$ containing the origin and be locally Lipschitz w.r.t. x such that

$$\delta_1(\|\chi\|) \leq V(t, \chi(t)) \leq \delta_2(\|\chi\|), \tag{6.5}$$

and

$$D^q V(t, \chi(t)) \leq 0, \tag{6.6}$$

where $\delta_1, \delta_2 > 0$ are constants, for any $\|\chi_0\| \leq \delta$, then $\chi = 0$ is asymptotically stable.

Theorem 6.2.3. [119] Suppose equilibrium point of the system (6.1) is $\chi = 0$ and $V(t, \chi(t)) : [0, \infty) \times D \longrightarrow \mathbb{R}$ be a continuously differentiable function in a region $D \subset \mathbb{R}^{\Omega}$ containing the origin and be locally Lipschitz with respect to χ s.t.

$$\delta_1(\|\chi\|) \leq V(t, \chi(t)) \leq \delta_2(\|\chi\|), \tag{6.7}$$

and

$$D^q V(t, \chi(t)) \leq -\delta_3(\|\chi\|), \tag{6.8}$$

where $\delta_1, \delta_2, \delta_3 > 0$ are constants, for any $\|\chi_0\| \leq \hat{r}$, then $\chi = 0$ is asymptotically stable.

Remark 6.2.1. According to the Theorem 6.2.3 equilibrium point $\chi = 0$ of the system (6.1) will be exponentially stable if there exist constants $c, \gamma, \lambda > 0$ s.t.

$$\|\chi(t)\| \leq \gamma \|\chi(t_0)\|(t - t_0)^{q-1} e^{-\lambda(t-t_0)}, \quad \forall\ \|\chi(t_0)\| \leq c, 0 \leq q \leq 1. \tag{6.9}$$

Remark 6.2.2. The equation (6.9) will be changed in the following form if we take $q = 1$,

$$\|\chi(t)\| \leq k \|\chi(t_0)\| e^{-\lambda(t-t_0)}, \quad \forall\ \|\chi(t_0)\| \leq c. \tag{6.10}$$

Remark 6.2.3. In the vicinity of $t = 0$, the exponential stability converges more quickly for fractional order systems than integer order systems, i.e.,

$$\left[\frac{d}{dt}(e^{-\alpha t})\right]_{t=0} = \left[-\alpha e^{-\alpha t}\right]_{t=0} = -\alpha, \tag{6.11}$$

and

$$\left[\frac{d}{dt}(t^{-\beta}e^{-\alpha t})\right]_{t=0} = \big[-\beta t^{-\beta-1}e^{-\alpha t} - \alpha t^{-\beta}e^{-\alpha t}\big]_{t=0} = \infty, \tag{6.12}$$

where $\alpha > 0$ and $\beta > 0$ are constants.

Remark 6.2.4. Let $\chi = 0$ is an equilibrium point of the system $D^q\chi(t) = \hat{\phi}(\chi(t))$, $\chi(t) \in \mathbb{R}$. Then $\chi = 0$ is stable if the condition $\chi\hat{\phi}(\chi(t)) \leq 0$, $\forall\chi$ is satisfied, and $\chi = 0$ is asymptotically stable if $\chi\hat{\phi}(\chi(t)) < 0$, $\forall\chi \neq 0$.

6.2.1 Problem formulation

Let us consider the following class of FONN as the master system as

$$D^q\chi_\hbar(t) = -c_\hbar\chi_\hbar(t) + \sum_{\mu=1}^{\Omega} a_{\hbar\mu}f_\mu(\chi_\hbar(t)) + \hat{L}_\hbar, \tag{6.13}$$

which can be written in matrix form as

$$D^q\chi(t) = -C\chi(t) + Af(\chi(t)) + \hat{L}, \tag{6.14}$$

where $q \in (0,1)$, $\chi(t) = (\chi_1(t), \chi_2(t), \ldots, \chi_\Omega(t))^T \in \mathbb{R}^\Omega$ is the state vector, and $\chi_\hbar(t)$ denotes the state of the $\hbar$-th unit at the time t. $C = diag(c_1, c_2, \ldots, c_\Omega)$ with, $c_\hbar > 0, \hbar = 1, 2, \ldots, n$ is a diagonal self-connection weight matrix, and $A = (a_{\hbar\mu})$ is the interconnection weight matrix. $f(\chi) = (f_1(\chi_1), f_2(\chi_2), \ldots, f_\Omega(\chi_\Omega))^T$ is a diagonal mapping, where $f_\hbar(\chi_\hbar) : \mathbb{R}^\Omega \longrightarrow \mathbb{R}^\Omega$ is the activation function, and $\hat{L} = (\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_\Omega)^T \in \mathbb{R}^\Omega$ is the external input vector.

The response system of FONN is taken as

$$D^q \sigma_\hbar(t) = -d_\hbar \sigma_\hbar(t) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(\sigma_\hbar(t)) + \hat{R}_\hbar + u_\hbar(t), \tag{6.15}$$

or, in matrix form as

$$D^q \sigma(t) = -D\sigma(t) + Bg(\sigma(t)) + \hat{R} + u(t), \tag{6.16}$$

where $u(t)$ is the control input to be designed and $\sigma(t) \in \mathbb{R}^\Omega$ is the state vector. $D = diag(d_1, d_2, \ldots, d_\Omega)$, with $d_\hbar > 0$ is a diagonal self-connection weight matrix, and $B = (b_{\hbar\mu})$ is the interconnection weight matrix. $g(\sigma) = (g_1(\sigma_1), g_2(\sigma_2), \ldots, g_\Omega(\sigma_\Omega))^T : \mathbb{R}^\Omega \longrightarrow \mathbb{R}^\Omega$ is a diagonal mapping, where $g_\hbar(\sigma_\hbar)$ represents the activation function of the $\hbar$-th neuron of the considered response system. $\hat{R} = (\hat{R}_1, \hat{R}_2, \ldots, \hat{R}_\Omega)^T \in \mathbb{R}^\Omega$ is the external input.

Assumption 6.2.1. Let $F_\mu^-, F_\mu^+, G_\mu^-, G_\mu^+$ are constants for all $\mu = 1, 2, \ldots, n$ such that for any $x \neq y \in \mathbb{R}$, the following Lipschitz condition holds

$$F_\mu^- \leq \frac{F_\mu(x) - F_\mu(y)}{x - y} \leq F_\mu^+, \quad G_\mu^- \leq \frac{G_\mu(x) - G_\mu(y)}{x - y} \leq G_\mu^+, \tag{6.17}$$

where F_μ and G_μ are the neuron activation functions.

Now define the synchronization error function as

$$\xi(t) = \sigma(t) - k(t)\chi(t), \quad \text{where} \quad k(t) = diag(k_1(t), k_2(t), \ldots, k_\Omega(t)).$$

By applying Caputo derivatives to the error function and utilising equation (6.15) and Property 1, the error system is obtained as

$$
\begin{aligned}
D^q \xi_\hbar(t) =& D^q \sigma_\hbar(t) - D^q (k_\hbar(t)\chi_\hbar(t)),\\
=& - d_\hbar \sigma_\hbar(t) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(\sigma_\hbar(t)) + \hat{R}_\hbar + u_\hbar(t) - D^q (k_\hbar(t)\chi_\hbar(t))
\end{aligned} \tag{6.18}
$$

Assumption 6.2.2. The elements of the function projective coefficient $k_\hbar(t)$ are continuously differentiable and bounded.

Definition 6.2.1. [120] The zero solution of the error system (6.18) is considered as GMLS if there exist positive constants $\mathcal{A}$ and λ s.t., for any solutions $\xi(t)$ of the error system (6.18) with different initial conditions denoted by $\xi(0)$, satisfy the following conditions

$$
\|\xi(t)\| \leq \mathcal{A}\|\xi(0)\|\mathbf{E}_q(-\lambda t^q), \quad t \geq 0, \tag{6.19}
$$

where $\mathbf{E}_q(-\lambda t^q)$ represents the Mittag-Leffler function.

Definition 6.2.2. [120] The master-response FONNs (6.13) and (6.15) are said to attain FPMLS if there exist $\mathcal{P} \geq 0$ and $\lambda > 0$ such that

$$
\|\xi(t)\| = \|\sigma(t) - k(t)\chi(t)\| \leq \mathcal{P}\|\sigma(t_0) - k(t)\chi(t_0)\|\mathbf{E}_q(-\lambda t^q), \text{for all} \quad t \geq 0,
$$

where $\chi(t)$ and $\sigma(t)$ are the solution of systems (6.13) and (6.15) having initial conditions $\chi(t_0) = (\chi_1(t_0), \chi_2(t_0), \ldots, \chi_\Omega(t_0))^T$ and $\sigma(t_0) = (\sigma_1(t_0), \sigma_2(t_0)), \ldots, \sigma_\Omega(t_0)))^T$, respectively.

Remark 6.2.5. The synchronization issue will be reduced to the PS and AS if the projective coefficient is $k(t) = k \in \mathbb{R}$ and $k(t) = -1$, respectively. The synchronization problem will be transformed into a complete synchronization if $k(t) = 1$.

Lemma 6.2.1. [121] Assume $\hat{\phi}(t) \in \mathbb{R}$ is a continuous and differentiable function, and D^q is Caputo fractional derivative of order q, then for every $t \geq t_0$,

$$\frac{1}{2} D^q \hat{\phi}^2(t) \leq \hat{\phi}(t) D^q \hat{\phi}(t), \forall\ q \in (0,1). \tag{6.20}$$

Furthermore, when $\hat{\phi}(t) \in \mathbb{R}^n$, the equation (6.20) is still valid. Specifically, it asserts that $\forall\ q \in (0,1)$ and $t \geq t_0$,

$$\frac{1}{2} D^q \hat{\phi}^T(t)\hat{\phi}(t) \leq \hat{\phi}^T(t) D^q \hat{\phi}(t).$$

Remark 6.2.6. When the function $\hat{\phi}(t)$ remains constant or when $q = 1$, we can expect an equality in equation (6.20). When $q = 1$, this corresponds to the product rule for integer-order derivatives. This rule is expressed as $\frac{1}{2}\frac{d\hat{\phi}^2(t)}{dt} = \hat{\phi}(t)\frac{d\hat{\phi}(t)}{dt}$, making it a specific instance of Lemma 6.2.1.

Lemma 6.2.2. [120] Consider the function $L(t)$ which is continuous on the interval $[0, +\infty)$ and there exists a constant α meets the following conditions

$$D^q L(t) \leq \alpha L(t), \quad q \in (0,1),$$

then

$$L(t) \leq L(0)\mathbf{E}_q(\alpha t^q), \quad t \geq 0.$$

Based on the error system (6.18), the controller is constructed as

$$\begin{aligned} u_\hbar(t) =& \hat{u}_\hbar(t) + \tilde{u}_\hbar(t), && (6.21)\\ \text{where} \quad \hat{u}_\hbar(t) =& k_\hbar(t) d_\hbar \chi_\hbar(t) - \sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(k_\hbar(t)\chi_\hbar(t)) - \hat{R}_\hbar(t) + D^q(k_\hbar(t)\chi_\hbar(t)), \end{aligned}$$

$$\text{and} \quad \tilde{u}_\hbar(t) = -\alpha_\hbar \xi_\hbar(t).$$

The control function contains both linear and nonlinear terms. $\tilde{u}_\hbar(t)$ is a linear term, with $\alpha_\hbar$ as the control gain, providing a stabilizing influence based on the error $\xi_\hbar(t)$. $\hat{u}_\hbar(t)$ is nonlinear due to the activation function g_μ which introduces a nonlinear dependency on the state.

6.3 Main results

Theorem 6.3.1. The considered class of FONNs (6.13) and (6.15) will be FPMLS under the Assumption 6.2.1 if control functions are chosen as (6.21) and the following conditions hold.

$$\alpha_\hbar < \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu |b_{\hbar\mu}| + \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\hbar |b_{\mu\hbar}| - d_\hbar, \quad \hbar = 1, 2, \ldots, \Omega.$$

Proof. Let us define Lyapunov function as

$$L(t) = \frac{1}{2}\sum_{\hbar=1}^{\Omega} \xi_\hbar^T(t)\xi_\hbar(t). \tag{6.22}$$

Taking the fractional order derivative of $L(t)$ w.r.to t, using Property 1 and Lemma 6.2.1, we get

$$\begin{aligned} D^q L(t) &\leq \sum_{\hbar=1}^{\Omega} \xi_\hbar^T(t) D^q \xi_\hbar(t) \\ &= \sum_{\hbar=1}^{\Omega} \xi_\hbar^T(t)\Big(- d_\hbar \sigma_\hbar(t) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(\sigma_\hbar(t)) + \hat{R}_\hbar + u_\hbar(t) - D^q(k_\hbar(t)\chi_\hbar(t))\Big). \end{aligned} \tag{6.23}$$

Putting the values of $u_\hbar(t)$ in equation (6.23) from equation (6.21), we obtain

$$D^q L(t) \le \sum_{\hbar=1}^{\Omega} \xi_\hbar^T(t)\Big(- d_\hbar\big(\sigma_\hbar(t) - k_\hbar(t)\chi_\hbar(t)\big) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu}\big(g_\mu(\sigma_\hbar(t)) - g_\mu(k_\hbar(t)\chi_\hbar(t))\big) \\ - \alpha_\hbar \xi_\hbar(t)\Big).$$

From definition of error function $\sigma_\hbar(t) - k_\hbar(t)\chi_\hbar(t) = \xi_\hbar(t)$, we get

$$\begin{aligned} D^q L(t) &\le \sum_{\hbar=1}^{\Omega} \xi_\hbar^T(t)\Big(- d_\hbar \xi_\hbar(t) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(\xi_\mu(t)) - \alpha_\hbar \xi_\hbar(t)\Big) \\ &\le - \sum_{\hbar=1}^{\Omega} d_\hbar \xi_\hbar^2(t) + \sum_{\hbar=1}^{\Omega}\sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(\xi_\mu(t))\xi_\hbar(t) - \sum_{\hbar=1}^{\Omega} \alpha_\hbar \xi_\hbar^2(t). \end{aligned}$$

From Assumption 6.2.1, $|g_\mu(\xi_\mu(t))| \le F_\mu|\xi_\mu(t)|$, where $F_\mu = max\{|F_\mu^-|, |F_\mu^+|\}$.

$$\begin{aligned} \text{Now,}\quad D^q L(t) &\le - \sum_{\hbar=1}^{\Omega} d_\hbar d_\hbar \xi_\hbar^2(t) + \sum_{\hbar=1}^{\Omega}\sum_{\mu=1}^{\Omega} d_\hbar |b_{\hbar\mu}||g_\mu(\xi_\mu(t))||\xi_\hbar(t)| - \sum_{\hbar=1}^{\Omega} \alpha_\hbar \xi_\hbar^2(t) \\ &\le - \sum_{\hbar=1}^{\Omega} d_\hbar \xi_\hbar^2(t) + \sum_{\hbar=1}^{\Omega}\sum_{\mu=1}^{\Omega} |b_{\hbar\mu}|F_\mu|\xi_\mu(t)||\xi_\hbar(t)| - \sum_{\hbar=1}^{\Omega} \alpha_\hbar \xi_\hbar^2(t) \\ &\le - \sum_{\hbar=1}^{\Omega} d_\hbar \xi_\hbar^2(t) + \frac{1}{2}\sum_{\hbar=1}^{\Omega}\sum_{\mu=1}^{\Omega} F_\mu|b_{\hbar\mu}|\xi_\mu^2(t) + \frac{1}{2}\sum_{\hbar=1}^{\Omega}\sum_{\mu=1}^{\Omega} F_\mu|b_{\mu\hbar}|\xi_\mu^2(t) \\ &\quad - \sum_{\hbar=1}^{\Omega} \alpha_\hbar \xi_\hbar^2(t) \\ &\le - \sum_{\hbar=1}^{\Omega} \Big\{ d_\hbar - \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\hbar\mu}| - \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\mu\hbar}| + \alpha_\hbar \Big\}\xi_\hbar^2(t). \end{aligned} \tag{6.24}$$

By choosing $\alpha_\hbar < \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\hbar\mu}| + \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\mu\hbar}| - d_\hbar$, we have

$$D^q L(t) \le - \lambda \sum_{\hbar=1}^{\Omega} \xi_\hbar^2(t) \le 0, \tag{6.25}$$

where $\lambda = \min_{1\leq\hbar\leq n} \{d_\hbar - \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\hbar\mu}| - \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\mu\hbar}| + \alpha_\hbar\}$.
From Lemma 6.2.2, we have

$$L(t) \leq L(0)\mathbf{E}_q(-\lambda t^q), \quad t \geq 0.$$

That is,

$$\|\xi_\hbar(t)\| \leq \|\xi_\hbar(0)\|\mathbf{E}_q(-\lambda t^q), \quad t \geq 0. \tag{6.26}$$

In the view of Definition 6.2.1, the considered class of FONNs (6.13) and (6.15) will be globally FPMLS. □

Corollary 6.3.1. If we take $k_\hbar(t) = \beta_\hbar$, $\hbar = 1, 2, \ldots, \Omega$, where $\beta_\hbar \in \mathbb{R}$, then the considered FONN error function is reduced into the following form

$$D^q\xi_\hbar(t) = D^q\sigma_\hbar(t) - \beta_\hbar D^q\chi_\hbar(t).$$

Now, from the equations (6.13) and (6.15), we get

$$\begin{aligned} D^q\xi_\hbar(t) = &- d_\hbar\xi_\hbar(t) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu}g_\mu(\xi_\mu(t)) + \beta_\hbar(c_\hbar - d_\hbar)\chi_\hbar(t) - \beta_\hbar\sum_{\mu=1}^{\Omega} a_{\hbar\mu} \\ &\times f_\mu(\chi_\mu(t)) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu}g_\mu(\beta_\hbar\chi_\mu(t)) + \hat{R}_\hbar - \beta_\hbar\hat{L}_\hbar + u_\hbar(t). \end{aligned}$$

Then (6.13) and (6.15) of FONNs will be global Mittag-Leffler PS under the controller

$$\begin{aligned} u_\hbar(t) = &- \alpha_\hbar\xi_\hbar(t) - \beta_\hbar(c_\hbar - d_\hbar)\chi_\hbar(t) + \beta_\hbar\sum_{\mu=1}^{\Omega} a_{\hbar\mu}f_\mu(\chi_\mu(t)) - \sum_{\mu=1}^{\Omega} b_{\hbar\mu} \\ &\times g_\mu(\beta_\hbar\chi_\mu(t)) - \hat{R}_\hbar + \beta_\hbar\hat{L}_\hbar. \end{aligned} \tag{6.27}$$

Corollary 6.3.2. When $k_\hbar(t) = 1$, then the control function will be

$$u_\hbar(t) = -\alpha_\hbar \xi_\hbar(t) - (c_\hbar - d_\hbar)\chi_\hbar(t) + \sum_{\mu=1}^{\Omega} a_{\hbar\mu} f_\mu(\chi_\mu(t)) - \sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(\chi_\mu(t)) - \hat{R}_\hbar + \hat{L}_\hbar,$$

with the condition $\alpha_\hbar < \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu |b_{\hbar\mu}| + \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu |b_{\mu\hbar}| - d_\hbar - \frac{\varrho}{2}$, $\hbar = 1, 2, \ldots, \Omega$, and considered fractional order master and response systems (6.13) and (6.15) of NNs will be simplified to GMLS.

Corollary 6.3.3. When $k_\hbar(t) = -1$, the control function is reduced in the following form

$$u_\hbar(t) = -\alpha_\hbar \xi_\hbar(t) + (c_\hbar - d_\hbar)\chi_\hbar(t) - \sum_{\mu=1}^{\Omega} a_{\hbar\mu} f_\mu(\chi_\mu(t)) + \sum_{\mu=1}^{\Omega} b_{\hbar\mu} g_\mu(\chi_\mu(t)) - \hat{R}_\hbar - \hat{L}_\hbar,$$

with the same conditions and the fractional order master and response systems (6.13) and (6.15) of NNs will be changed into the global Mittag-Leffler AS.

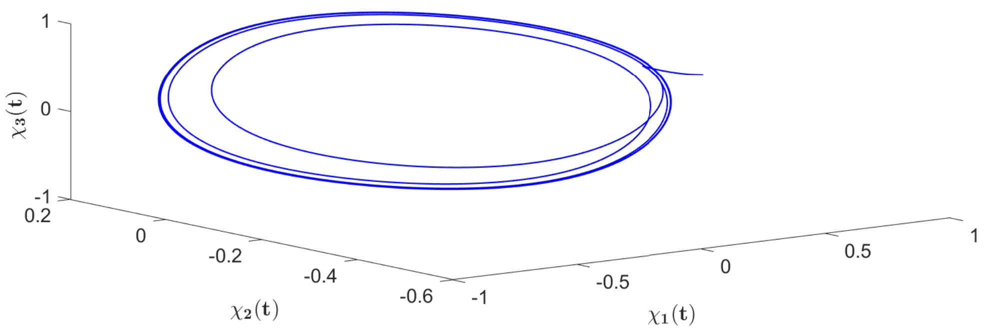
Remark 6.3.1. Global exponential synchronization can be regarded as a specific instance of GMLS when the fractional order q approaches 1. In simpler terms, as q tends to 1, the Mittag-Leffler function converges to the exponential function, and the GMLS expression gradually resembles the characteristic form of global exponential synchronization. FONNs inherently encompass memory effects, signifying that the current state relies on the entire historical trajectory of the system. The Mittag-Leffler function is particularly well-suited for representing fractional-order dynamics, offering a more precise depiction of memory effects in contrast to conventional exponential functions.

6.4 Numerical example

In this section, we will verify the efficiency and reliability of the obtained results given in section 6.2.1.

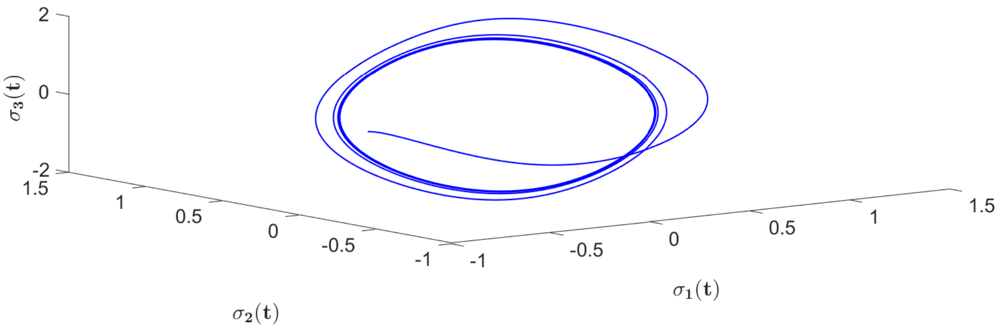
Consider the three dimensional FONNs as a master system defined as

$$D^q\chi_1(t) = -6\chi_1(t) + 8f_1(\chi_1(t)) + 5.2f_2(\chi_2(t)) + 6f_3(\chi_3(t)) + \hat{L}_1,$$
$$D^q\chi_2(t) = -6\chi_1(t) - 1.2f_1(\chi_1(t)) + 2f_2(\chi_2(t)) + 1.15f_3(\chi_3(t)) + \hat{L}_2,$$
$$D^q\chi_3(t) = -6\chi_1(t) - 4.25f_1(\chi_1(t)) + 3.1f_3(\chi_3(t)) + \hat{L}_3, \tag{6.28}$$



(a)

where $t \geq 0$ and $f_\hbar(\chi_\hbar(t)) = tanh(\chi_\hbar(t)),\ \hbar = 1,2,3$, in the view of equation (6.14), $C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, $A = \begin{bmatrix} 8 & 5.2 & 6 \\ -1.2 & 2 & 1.15 \\ -4.25 & 0 & 3.1 \end{bmatrix}$, and $\hat{L} = [0,0,0]^T$. The following FONN



(b)

FIGURE 6.1: Phase portraits of FONNs with master (6.28) and response (6.29) systems

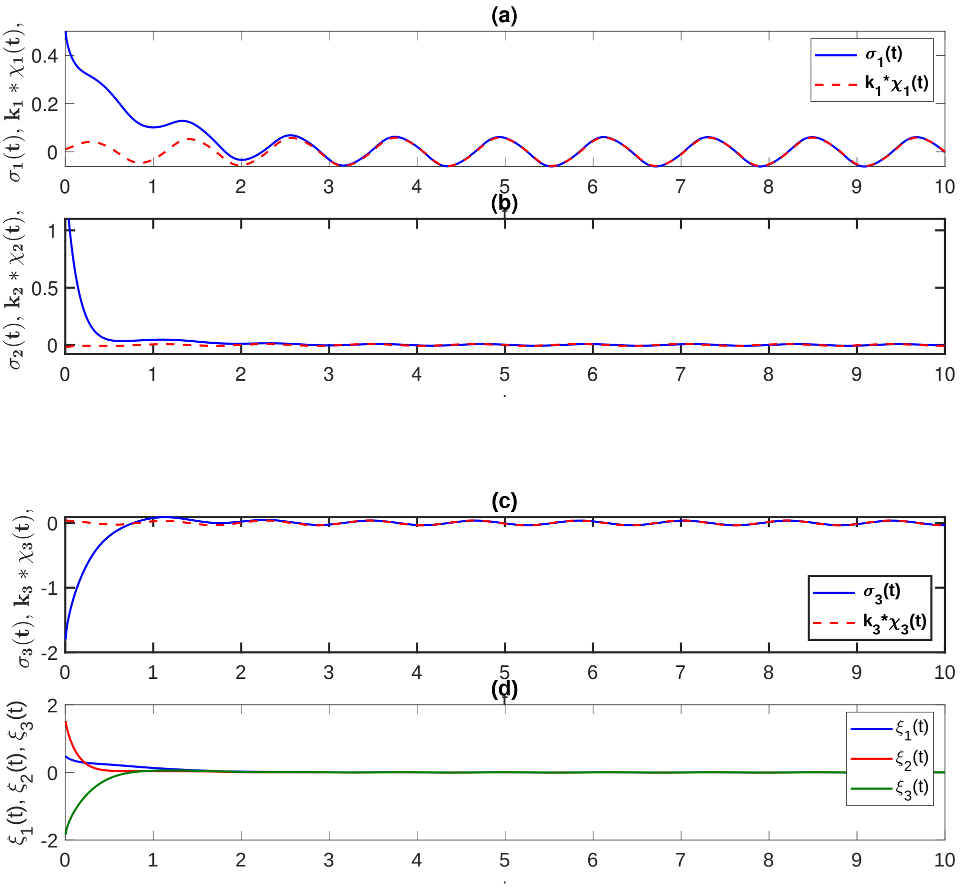


FIGURE 6.2: Plots of the state trajectories of master and response systems (6.28) and (6.29) with $k_1(t) = 0.05cos(0.02\chi_1(t)){+}0.01$, $k_2(t) = 0.02cos(0.01\chi_2(t)){+}0.02$, and $k_3(t) = 0.01cos(0.03\chi_3(t)) + 0.04$ at $q = 0.97$.

is considered as the corresponding response system as

$$\begin{aligned}
D^q\sigma_1(t) &= -0.8\sigma_1(t) + 1.5g_1(\sigma_1(t)) - 1.2g_2(\sigma_2(t)) - 0.5g_3(\sigma_3(t)) + \hat{R}_1 + u_1(t),\\
D^q\sigma_2(t) &= -1.2\sigma_1(t) + 1.3g_1(\sigma_1(t)) - 1.5g_2(\sigma_2(t)) + 1.15g_3(\sigma_3(t)) + \hat{R}_2 + u_2(t),\\
D^q\sigma_3(t) &= -0.5\sigma_1(t) - 2.75g_1(\sigma_1(t)) - 2g_2(\sigma_2(t)) + 1.15g_3(\sigma_3(t)) + \hat{R}_3 + u_3(t),
\end{aligned} \tag{6.29}$$

where $u_\hbar(t), \hbar = 1,2,3$ are the control functions, $g_\hbar(\sigma_\hbar(t)) = sin(\sigma_\hbar(t)),\ \hbar = 1,2,3$, and $\hat{R} = [0,0,0]^T$. From the equations (6.29) and (6.16), we get

$$D = \begin{bmatrix} 0.8 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 1.5 & -1.2 & -0.5\\ 1.3 & 1.5 & 1.15\\ 2.75 & -2 & 1.15 \end{bmatrix}.$$

Figure 6.1 depicts the phase portrait of the systems (6.28)-(6.29) with initial conditions $\chi(0) = (0.2, -0.5, 0.8)^T$, $\sigma(0) = (0.5, 1.5, -1.8)^T$ and $q = 0.97$. Let $F_\mu = 1$, for $\mu = 1,2,3$, then we can obtain $\alpha_1 = 1.2, \alpha_2 = 0.3, \alpha_3 = 0.7$, and $\lambda = \min_{1\leq\hbar\leq 3}\big\{d_\hbar - \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\hbar\mu}| - \frac{1}{2}\sum_{\mu=1}^{\Omega} F_\mu|b_{\mu\hbar}| + \alpha_\hbar\big\} = \min\{1.675, 0.625, 0.65\} = 0.625$. It is straightforward to ascertain that the criteria outlined in Theorem 6.3.1 are met. For the FPMLS, the projective functions are taken as $k_1(t) = 0.05cos(0.02\chi_1(t)) + 0.01$, $k_2(t) = 0.02cos(0.01\chi_2(t)) + 0.02$, and $k_3(t) = 0.01cos(0.03\chi_3(t)) + 0.04$. The initial value of the error system is considered with the help of error functions as $(0.059, 0.039, 0.049)$. The fractional derivative is taken for both global Mittag-Leffler PS and FPS cases. Figs. 6.2(a)-(c) depict the state trajectories of FONN systems (6.28) and (6.29) are FPMLS. The state variables of synchronization error of systems (6.28) and (6.29) are depicted in Figure 6.2(d), which clearly shows that the FPMLS is achieved.

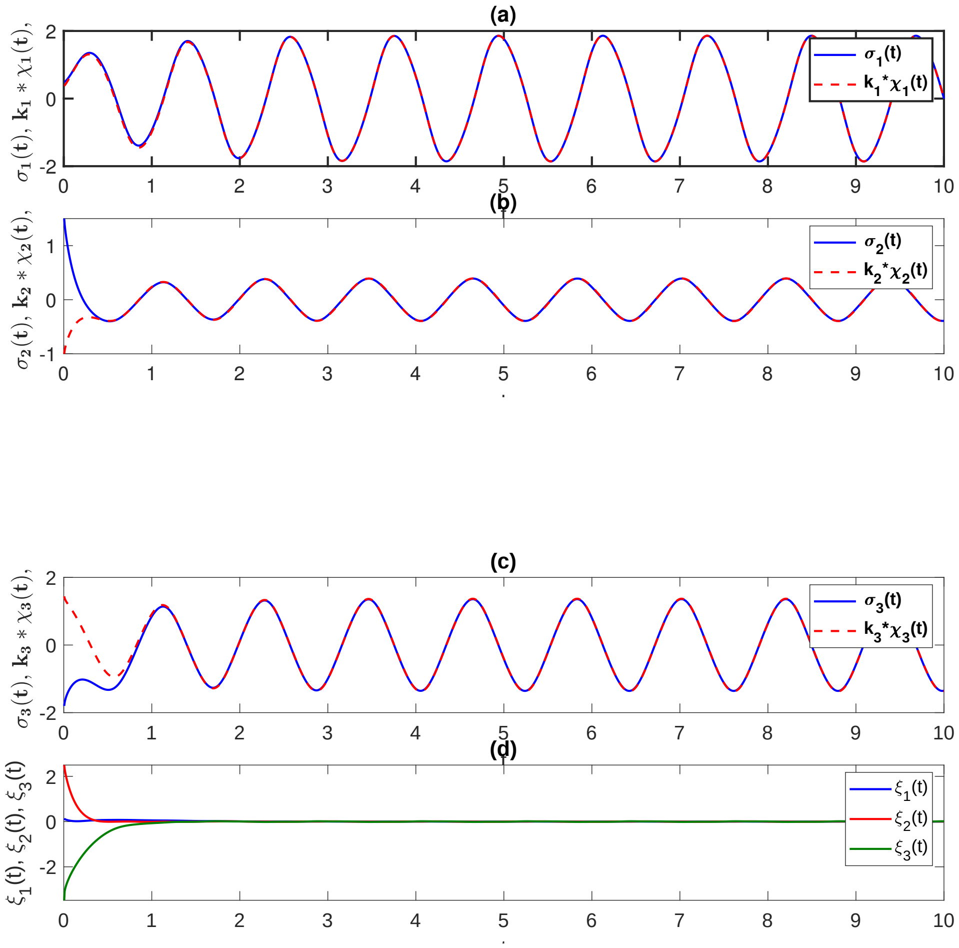
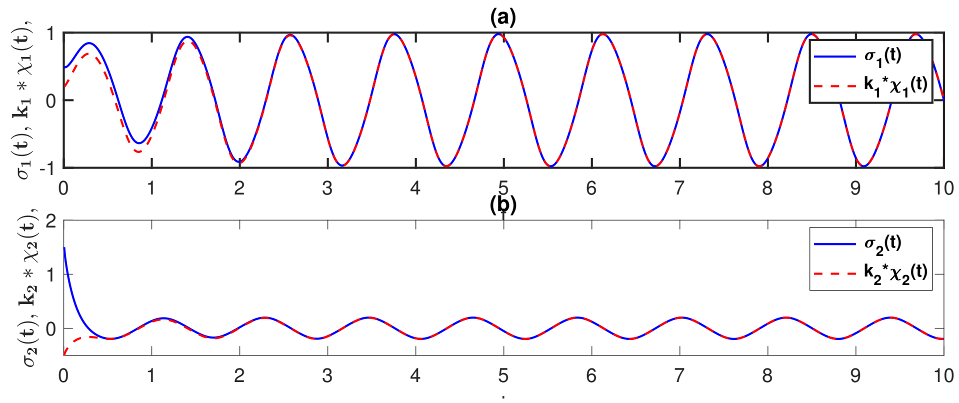


FIGURE 6.3: Plots of the state trajectories of master and response systems (6.28) and (6.29) with $k_1 = 1.9$, $k_2 = 2$, $k_1 = 1.8$ at $q = 0.97$.



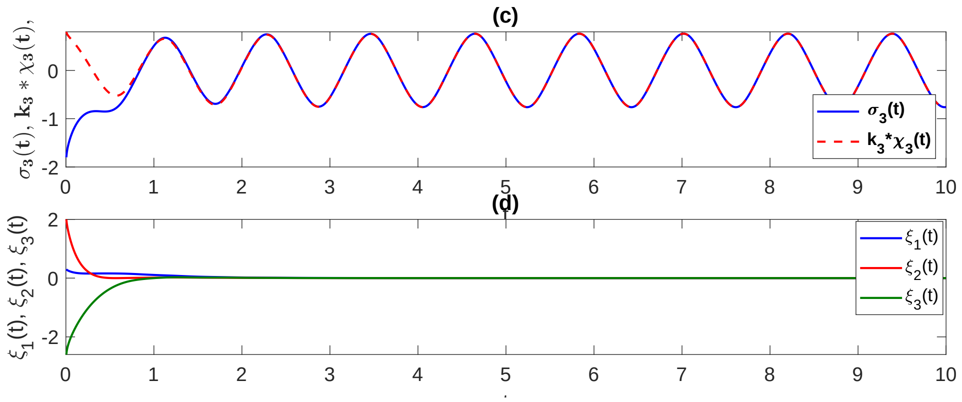


FIGURE 6.4: Plots of the state trajectories of the systems (6.28) and (6.29) with $k_\hbar = 1$, $\hbar = 1, 2, 3$ at $q = 0.97$.

Case 1 From Corollary 6.3.1, for the global Mittag-Leffler PS, the values of projective constants k_1, k_2, k_3 are the taken as 1.9, 2 and 1.8, respectively. During synchronization, the initial condition of the error system is taken as $(0.14, 2.5, -3.48)$. Figure 6.3(d) depicts the state variables of synchronization error between considered systems (6.28) and (6.29), which clears that the global Mittag-Leffler PS is also achieved after a small time duration. Figs. 6.3(a)-(c) show the time evolution of state variables of master and response systems. The PS criteria enables faster communication due to its advantageous properties, as demonstrated in the literature [122], the application of PS in secure communication can be utilized to extend binary digital to M-nary digital communication for achieving quick communication.

Case 2 In view of Corollary 6.3.2, for GMLS, we consider $k_\hbar(t) = 1$. The initial condition of the considered systems (6.28) and (6.29) are taken as $(0.2, -0.5, 0.8)$ and $(0.5, 1.5, -1.8)$ and hence the initial condition of the error system will be $(0.3, 2, -2.6)$. The values of $\alpha_\hbar$, $\hbar = 1, 2, 3$ are taken as 1.5, 3, 4, respectively. The state trajectories of the systems (6.28) and (6.29) are depicted in Figs. 6.4(a)-(c), taking the order of fractional derivative as $q = 0.97$. In a brief time, it is seen that the systems under consideration are GMLS. The error functions also exponentially converge to zero after a short time, as seen in Figure 6.4(d).

Case 3 According to the concept of the error functions for Corollary 6.3.2, the initial conditions of the systems (6.28) and (6.29) are assumed to be the same as the case 6.4 in light of Corollary 6.3.2, so that the initial condition of the error system will be $(0.7, 1, -1)$. The state trajectories of the systems (6.28) and (6.29) are AS after a short time, as shown in Figs. 6.5(a)-(c), and the plots of error functions confirm that the AS is achieved after a short time depicted in in Figure 6.5(d).

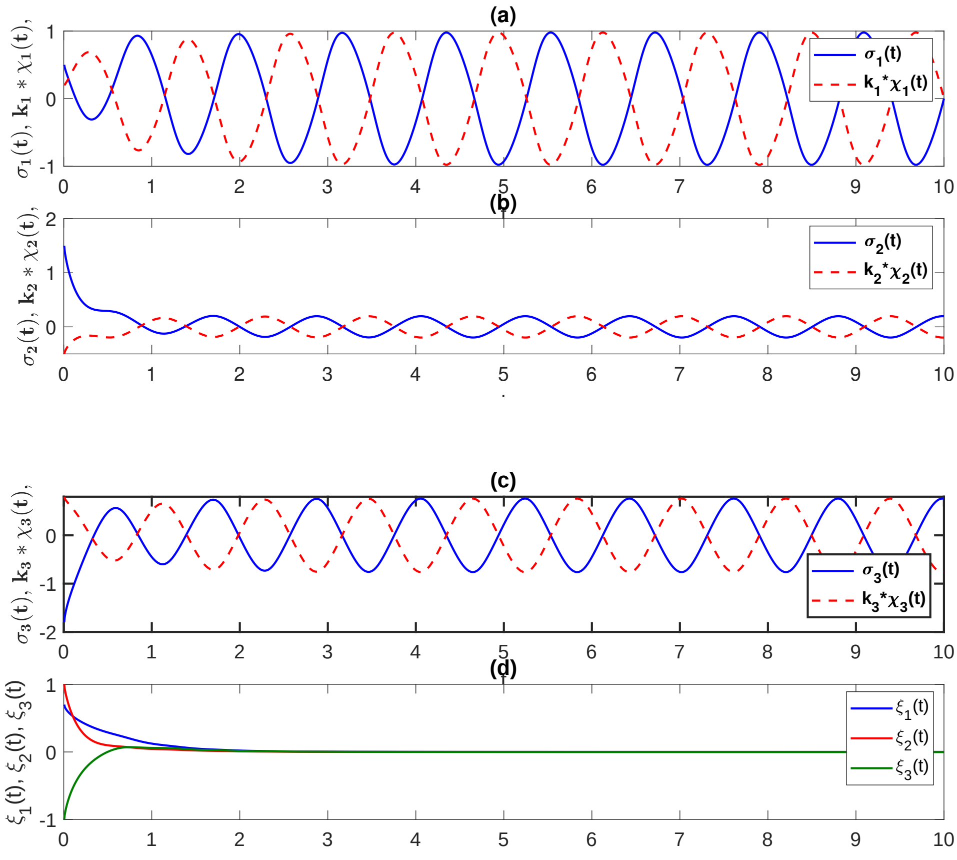


FIGURE 6.5: Plots of the state trajectories of the systems (6.28) and (6.29) with $k_\hbar = -1$, $\hbar = 1, 2, 3$ at $q = 0.97$.

Remark 6.4.1. Fractional-order systems are more complex than integer-order systems, and their analyses often require specialized mathematical tools. The study of FPS for FONNs involves fractional calculus, stability analysis, and control theory techniques tailored to fractional-order systems. In [123], PS of FONNs with delays has been investigated in the Mittag-Leffler sense. The authors have used identical master and response systems to achieve the PS [123]. In PS, the projective coefficient is a scaling factor, but in FPS, the projective coefficient is a function. So, when we take the Caputo fractional derivative, the error system will not follow the product rule, similar to the integer order system. Therefore, the study of FPS in FONNs is more complicated than that of integer-order neural networks. In this chapter, by designing an appropriate controller based on the error system, the FPS of FONN in the Mittage-Leffler sense is achieved for non-identical systems. Also, in the Mittag-Leffler sense, special cases are provided for PS, AS, and CS for non-identical FONN systems. Hence, our results are more general here.

6.5 Conclusion

The present chapter investigates global FPMLS for non-identical FONNs, employing global Mittag-Leffler stability theory, Lyapunov stability theory, and various assumptions and lemmas. Additionally, the chapter delves into global Mittag-Leffler PS, AS, and CS on non-identical FONNs, elucidating the definitions of error functions. Remark 6.2.3 indicates that exponential synchronization is achieved more swiftly in fractional order scenarios compared to integer order cases. A numerical example in Section 6.4 is provided to underscore the effectiveness and reliability of the scheme, demonstrating global Mittag-Leffler AS, PS, and CS schemes for non-identical FONNs through Cases 1–3. It is anticipated that the synchronization scheme for non-identical cases will offer practical guidelines for engineering applications of FONNs. Moreover, according to this research, adjusting the function scaling factor for FPMLS can significantly enhance communication security and confidentiality due to its complex and unpredictable nature. This flexibility empowers the users to tailor the function scaling factor to specific applications, making it more challenging for hackers to discern the correct path and fortifying the system's defense against attacks. The advent of the significant data era necessitates heightened technical standards for communication security and confidentiality.

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