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# Abbreviations

<b>FC</b>	<b>F</b> ractional <b>C</b> alculus
<b>SLP(s)</b>	<b>S</b> turm- <b>L</b> iouville <b>P</b> roblem(s)
<b>FSLP(s)</b>	<b>F</b> ractional <b>S</b> turm- <b>L</b> iouville <b>P</b> roblem(s)
<b>GFSLP(s)</b>	<b>G</b> eneralized <b>F</b> ractional <b>S</b> turm- <b>L</b> iouville <b>P</b> roblem(s)
<b>TFSLP(s)</b>	<b>T</b> empered <b>F</b> ractional <b>S</b> turm- <b>L</b> iouville <b>P</b> roblem(s)
<b>FSLO</b>	<b>F</b> ractional <b>S</b> turm- <b>L</b> iouville <b>O</b> perator
<b>R-L</b>	<b>R</b> iemann- <b>L</b> iouville
<b>CFD(s)</b>	<b>C</b> aputo <b>F</b> ractional <b>D</b> erivative(s)
<b>TFI(s)</b>	<b>T</b> empered <b>F</b> ractional <b>I</b> ntegral(s)
<b>TFD(s)</b>	<b>T</b> empered <b>F</b> ractional <b>D</b> erivative(s)
<b>CGFD(s)</b>	<b>C</b> aputo <b>G</b> eneralized <b>F</b> ractional <b>D</b> erivative(s)



# Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$AC[a, b]$	Absolutely continuous on $[a, b]$
$L^p(a, b)$	$L^p$ -space
$\Omega$	$[a, b]$
$L_w^2(a, b)$	Weighted $L^2$ -space
$C^2(\Omega)$	$2^{nd}$ derivative continuous on $\Omega$
$\oplus$	Direct sum