# Chapter 6

# Conclusion and future directions

An overview of the main conclusions that can be made from this thesis is given in this chapter. Further, this chapter also includes a summary of the directions for future research scopes.

## 6.1 General conclusion

The key conclusions of this thesis are summarized as follows:

- We have given the notion of weak sharp minima for convex IVFs and its primal and dual characterizations.

- A more generalized form of $gH$-subdifferentiability, called $gH_{\epsilon}$-subdifferentiability, for convex IVFs has been proposed.

- Some fundamental characteristics of $gH_{\epsilon}$-subdifferential set, such as closeness, convexity, nonemptiness, etc., have been proved.

- The definition of $\epsilon$-solution for IOPs is provided. Subsequently, two necessary and sufficient optimality conditions to find an $\epsilon$-solution of IOPs with the help of $gH_{\epsilon}$-subdifferentiability have been given.

- A nonlinear conjugate gradient method without imposing a restriction on conjugate parameter has been proposed for set optimization problems whose objective function has finite cardinality. Further, two variants, Fletcher-Reeves and conjugate descent, have also been given for the considered set optimization problem.
- Two projected gradient methods for constrained set optimization problems having objective function finitely many vector-valued functions have been developed.

Chapter-wise detailed contributions in this thesis are given below.

## 6.2 Contributions of the thesis

This thesis deals initially with the theories on the analysis of IVFs and characterization of the solutions of IOPs with the help of the concepts of weak sharp minima, $gH$-subdifferentiability, $gH_\epsilon$-subdifferentiability of IVFs. Next, to solve unconstrained and constrained set optimization problems, nonlinear conjugate and projected gradient methods, respectively, are developed in this thesis. Contributions of each chapter of this thesis are highlighted below.

In Chapter 2, the conventional concepts of support function and subdifferentiability have been extended to IVFs. Additionally, several important properties of the $gH$-subdifferential set, such as nonemptiness, boundedness, convexity, and closedness, have been established. It has been shown that the $gH$-subdifferential set of a $gH$-differentiable convex IVF is a singleton set containing the $gH$-gradient. Furthermore, a relationship between the $gH$-directional derivative and the support function of the $gH$-subdifferential set for convex IVFs has been demonstrated. Next, we have introduced the concept of WSM for convex IVFs. Using the newly proposed notions of $gH$-subdifferentiability and support function, both a primal characterization and several dual characterizations of WSM have been developed. Two applications of the proposed theoretical framework are provided: the first focuses on the sets of WSM for portfolio programming with interval uncertainty, while the second establishes a connection between WSM and weak efficient solutions for both linear and nonlinear interval optimization problems.

In Chapter 3, we introduced the concepts of the $gH_\epsilon$-subgradient and $gH_\epsilon$-directional derivative for convex IVFs along with several key properties. It has been demonstrated that the $gH_\epsilon$-subdifferential set for convex IVFs is nonempty, closed, and convex. Additionally, the boundedness of the $gH_\epsilon$-subdifferential set has been established for the interior of the effective domain. A connection between the $gH$-subdifferential set and the $gH_\epsilon$-subdifferential set has also been provided. Moreover, it has been observed that a convex IVF is $gH$-Lipschitz continuous if it is $gH_\epsilon$-subdifferentiable at every point in its domain. Furthermore, the notion of an $\epsilon$-solution was introduced, along with two optimality conditions. Finally, a theorem has been proved to address interval minimax optimization problems using $gH_\epsilon$-subdifferentiability.

In Chapter 4, we have proposed nonlinear conjugate gradient methods for solving unconstrained set-valued optimization problems, where the objective function consists of a finite collection of continuously differentiable vector-valued functions. A general algorithm has been presented using the conjugate parameter $\beta_k$. Two specific variants—Fletcher-Reeves and conjugate descent—have been then analyzed based on different choices of $\beta_k$. The standard and strong Wolfe line search conditions for the considered set-valued optimization problems have been introduced. Subsequently, a Zoutendijk type condition has also been proved. Furthermore, we demonstrated the existence of a step length $\alpha_k$ that satisfies both the standard and strong Wolfe conditions. The global convergence of the general nonlinear conjugate gradient method has been proved with and without regularity assumption. The convergence of the FR method has also been established, with and without regularity assumptions, provided that $\beta_k$ is nonnegative and bounded above by a fraction of the FR choice $\beta_k^{FR}$. We further demonstrated that $d_k$ satisfies the sufficient descent condition with a constant

$(1-\sigma)$ when $\beta_k$ is nonnegative and bounded by the conjugate descent choice $\beta_k^{CD}$ under regularity assumptions. This result allowed us to establish the convergence of the CD method under regularity assumptions when $\beta k$ is nonnegative and bounded by an appropriate fraction of $\beta_k^{CD}$. We have given some numerical example also to test the proposed methods on some existing and freshly introduced set optimization problems. A comparison of the performance of FR, CD, and the steepest descent method [32] on these set optimization problems has been shown. From comparison, we have noticed that the CD method delivered the best performance, followed by the FR method as the second best among the three approaches.

In Chapter 5, we have proposed a projected gradient method for addressing constrained set-valued optimization problems, where the objective function consists of finitely many continuously differentiable vector-valued functions. We examine two variations of this method: one utilizing a constant projection parameter and the other employing a variable projection parameter. For both variants, we have proved well-definedness. The global convergence of both methods also established. Importantly, the global convergence of these methods have been proved without imposing any convexity assumptions on the objective function of the considered constrained set optimization problems. To construct these methods, we have derived necessary optimality conditions. The feasibility of the points generated by the algorithms has also been shown. Furthermore, we demonstrated that the sequence of projected gradient descent directions is bounded. Finally, several numerical examples have been presented to illustrate the performance and effectiveness of the proposed methods.

## 6.3 Future scope of studies

There are several interesting and challenging research problems that can be explored in the future. This section outlines potential directions for future work that could further build on and consolidate the findings presented in this thesis.

- Although, we have studied many properties and characterizations of the $gH$-subdifferential and $gH_\epsilon$-subdifferential set, we could not provide subdifferential sum rule for convex IVFs. One can attempt to prove the generalization of the conventional subdifferential or $\epsilon$-subdifferential sum rule for convex IVFs. This is important because with the sum rule, we can use the proposed study of $gH$-subdifferential and $gH_\epsilon$-subdifferential set and optimality conditions to find solution of constrained IOPs.

- One can attempt to develop $gH$-subgradient and $gH_\epsilon$-subgradient methods to solve IOPs.

- Similar to conventional optimization problems, weak sharp minima for IVFs can be used to provide necessary and sufficient condition under which global error bound may exist for a convex inequality system as follows:

$$\mathbf{H}_\lambda(x) \preceq \mathbf{0},\ \ \lambda \in \Lambda \text{ and } x \in C, \tag{6.1}$$

where $\Lambda$ is an index set, and for each $\lambda \in \Lambda$, $\mathbf{H}_\lambda : X \subseteq \mathbb{R}^n \to I(\mathbb{R})$ is $gH$-lsc, convex, proper, and the set $C$ is closed convex subset of $X$. By a global error bound for the inequality system (6.1), we mean the existence of a constant $\beta > 0$ such that

$$\beta\, \text{dist}\,(x, \Omega) \preceq \text{dist}(x, C) \oplus \mathbf{H}_{\lambda+}(x) \text{ for each } \lambda \in \Lambda \text{ and } x \in X, \tag{6.2}$$

where $\Omega = \{x : x \in C \text{ and } \mathbf{H}_\lambda(x) \preceq \mathbf{0}\}$ and $\mathbf{H}_{\lambda+}(x) = \max\{\mathbf{0}, \mathbf{H}_\lambda(x)\}$.

- In this thesis, we have considered set optimization problems whose objective function has finite vector-valued functions. Therefore, a potential future direction could be to extend the proposed methods for set optimization problems whose objective function may have infinitely many vector-valued functions.

- Different variants of the conjugate method and projected method can be developed for considered unconstrained and constrained set optimization problems. One may attempt to propose these methods with the nonmonotone line search as, to our best knowledge, nonmonotone line search has not been applied for set optimization problems yet.

- We have used a lower set less relation in this paper. For future research, one may extend these results and methods with the other set relations, such as upper set less, set less, minmax less, etc.

- Other conventional smooth and nonsmooth optimization algorithms for considered unconstrained and constrained set optimization problems can be developed as well.