Chapter 1

Introduction

In this chapter, we provide the background theories, preliminaries, notations, and definitions that are used in this thesis. We also offer the rationale behind the research and a comprehensive summary of all the chapters included in this thesis.

1.1 What is Fractal?

"Fractal geometry will make you see everything differently. You risk the loss of your childhood vision of clouds, forests, galaxies, leaves, feathers, flowers, rocks, mountains, and much else. Never again your interpretations of these things be quite the same" -Michael Barnsley.

In our universe, we come across various structures that cannot be explained using traditional Euclidean geometry. Examples include the intricate network of bronchial branches in the respiratory tract, the extensive arteries in the heart, and the massive network of tubules in the renal framework. These structures possess a unique mathematical complexity that requires mathematical modeling and further investigation for accurate understanding. The limitations of Euclidean geometry prevent its application in solving such issues. However, the fractal structures, which have existed since the beginning of the universe, it was coined only in 1982 as *fractal* by Mandelbrot [55].

The publication of Mandelbrot's book *The Fractal Geometry of Nature* sparked significant interest in researching fractal structures. In fact, fractals deeply reveal mathematical concepts with numerous practical applications. However, the existing literature lacks clear and explicit mathematical definitions for fractals. Mandelbrot initially defined a fractal as a set with a *Hausdorff dimension* greater than its *topological dimension*. However, this definition was later proved to be inadequate as it excluded certain sets that should be considered fractals. Therefore, alternative perspectives are needed to define fractals.

In [46], Hutchinson conducted a study on fractals as a self-similar set. This perspective, along with its connection to the Hausdorff dimension, is found to be the most effective approach. In this thesis, we have also adopted this view of fractals, specifically referring to Hutchinson's definition. Many theories have been developed based on the idea of fractals as self-similar sets, meaning that they possess a geometric shape that exhibits intricate structures even at extremely small scales. In [46], Hutchinson utilized the concept of an *iterated function system* to define the self-similarity of a set in his paper. In what follows, we will provide an overview of Hutchinson's theory on fractals.

Let (X, d) be a complete metric space, then it is known that the space of all nonempty compact subsets $\mathcal{K}(X)$ of X, endowed with the Hausdorff metric

$$H_d(B, C) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(b, a)\right\}$$

is also complete.

Definition 1.1. Let (X, d) be a complete metric space and $m \in \mathbb{N}$ be a natural number. Let for each $i \in \Sigma_m$, $v_i : X \to X$ is a contraction map, i.e., for each

$i \in \Sigma_m$, there exists $0 \le \kappa_i < 1$ such that

$$d(\upsilon_i(x), \upsilon_i(y)) \le \kappa_i d(x, y) \text{ for all } x, y \in X,$$

then, the system $\{X;\ \upsilon_1, \ldots, \upsilon_m\}$ is called a (hyperbolic) iterated function system (IFS).

Moreover, if for each $i \in \Sigma_m$, we have $d(\upsilon_i(x), \upsilon_i(y)) = \kappa_i d(x, y)$ for all $x, y \in X$ and $x \ne y$, then υ_i is known as *simlitude* and IFS $\{X;\ \upsilon_1, \ldots, \upsilon_m\}$ is known as *self-similar* IFS.

Theorem 1.2. [9, 46] *Let $\{X;\ \upsilon_1, \ldots, \upsilon_m\}$ be an IFS. Further, let $\mathcal{F}_\upsilon : \mathcal{K}(X) \to \mathcal{K}(X)$ such that $\mathcal{F}_\upsilon(B) = \bigcup_{i=1}^m \upsilon_i(B)$ for $B \in \mathcal{K}(X)$ be the Hutchinson operator associated to this IFS. Then, $\mathcal{F}_\upsilon$ is a contraction map with respect to Hausdorff metric $\mathcal{H}_d$ with contraction ratio $\kappa = \max\{\kappa_1, \ldots, \kappa_m\}$. Hence, using the Banach contraction principle, there exists a unique nonempty subset $\mathcal{A} \in \mathcal{K}(X)$ such that*

$$\mathcal{A} = \mathcal{F}_\upsilon(\mathcal{A}) = \bigcup_{i=1}^m \upsilon_i(\mathcal{A}).$$

This $\mathcal{A}$ is known as an attractor or fractal corresponding to the given IFS.

Definition 1.3. [9] An IFS $\{X;\ \upsilon_1, \ldots, \upsilon_m\}$ is said to satisfy *open set condition* (OSC) if there exists a nonempty open set $V \subset \mathbb{R}$ such that

$$\bigcup_{i=1}^m \upsilon_i(V) \subset V \quad \text{such that} \quad \upsilon_i(V) \cap \upsilon_j(V) = \emptyset \quad \text{for} \quad i \ne j.$$

Moreover, if $\mathcal{A}$ is the attractor of this IFS such that $V \cap \mathcal{A} \ne \emptyset$, then the IFS is said to satisfy the *strong open set condition* (SOSC).

Definition 1.4. [63] Let $\{X;\ \upsilon_1, \ldots, \upsilon_m\}$ be an IFS and $\mathcal{A}$ be the attractor of this IFS, then the set defined as

$$\Theta_\upsilon = \bigcup_{i \neq j} (\upsilon_i(\mathcal{A}) \cap \upsilon_j(\mathcal{A})) \text{ for } i, j \in \Sigma_m \tag{1.1}$$

is known as the *overlapping* set of the IFS $\{X;\ \upsilon_1, \ldots, \upsilon_m\}$.

Definition 1.5. [7] Let (X, d) be a complete metric space and $\{X;\ \upsilon_1, \ldots, \upsilon_m\}$ be an IFS and $\mathcal{A}$ be its attractor. Let $\mathcal{F}_\upsilon : \mathcal{K}(X) \to \mathcal{K}(X)$ such that $\mathcal{F}_\upsilon(B) = \bigcup_{i=1}^m \upsilon_i(B)$ for $B \in \mathcal{K}(X)$ be the Hutchinson operator associated to this IFS. Then, the *dynamical boundary* of $\mathcal{A}$ with respect to this IFS is defined as

$$\partial\mathcal{A} = \overline{\bigcup_{k=1}^{\infty} \mathcal{F}_\upsilon^{-k}(\Theta_\upsilon) \cap \mathcal{A}}$$

such that $\mathcal{F}_\upsilon^{-1}(U) = \bigcup_{i=1}^m \upsilon_i^{-1}(U)$ and Θ_υ is the overlapping set of this IFS. Further, if $\mathcal{A} \neq \partial\mathcal{A}$, then $\mathcal{A}$ is said to be *nonoverlapping* set.

Moreover, the set $\widehat{\Theta}_\upsilon$ defined as

$$\widehat{\Theta}_\upsilon = \bigcup_{k=0}^{\infty} \mathcal{F}_\upsilon^{k}(\Theta_\upsilon)$$

is known as *inner boundary* of the attractor $\mathcal{A}$ with respect to this given IFS.

1.2 Fractal Dimension

The fractal dimension is one of the most engrossing areas of fractal theory. It provides the statistical ratio of the complexity of the fractal patterns with the details of how it changes with the measured scale. Several notions of fractal dimension have been introduced so far in the literature. For example, Hausdorff dimension, box dimension, packing dimension [11, 36, 58], etc. In disparate to the topological or algebraic dimensions, fractal dimensions permit the noninteger values as a dimension of a set. In the course of this thesis, we give two notions of fractal dimension: the Hausdorff dimension and the box (Minkowski) dimension. For more details on these two and other types of fractal dimensions of the sets, please refer [11, 36].

For a nonempty subset U of Euclidean space $\mathbb{R}^n$, the diameter is defined as

$$|U| = \sup \left\{ \|(x_1, \dots, x_n) - (y_1, \dots, y_n)\| : (x_1, \dots, x_n), (y_1, \dots, y_n) \in U \right\}.$$

A countable (or finite) collection of sets, say $\{U_i\}$, will be pronounced as δ-cover of a subset $E \subseteq \mathbb{R}^n$ if all the sets of the collection have a diameter of at most δ. Let s be a nonnegative real number and $\delta > 0$. Define

$$H^s_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}.$$

Definition 1.6. [36] The s-dimensional Hausdorff measure of E is defined by

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E).$$

Definition 1.7. [36] Let $E \subseteq \mathbb{R}^n$ and $s \geq 0$. The Hausdorff dimension of E is defined as

$$\dim_H(E) = \inf \left\{ s : H^s(E) = 0 \right\} = \sup \left\{ s : H^s(E) = \infty \right\}.$$

Remark 1.8. If $s = \dim_H(E)$, then $H^s(E)$ may be zero or infinite or may satisfy $0 < H^s(E) < \infty$.

Definition 1.9. [36] Let $E \neq \emptyset$ be a bounded subset of $\mathbb{R}^n$ and let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which can cover E. The lower box dimension and upper box dimension of E are defined as follows:

$$\underline{\dim}_B(E) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \text{ and } \overline{\dim}_B(E) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}, \text{ respectively.}$$

If the above two are equal, then the common value is known as the box dimension of E. That is,

$$\dim_B(E) = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

The following relations are established between these fractal dimensions (see [36]):

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E).$$

1.3 Fractal Interpolation Functions

Interpolation is a method used to estimate data points within a known set of data points. It is commonly used in various fields to determine the value of a function at a specific point based on nearby data points. Different interpolation techniques, such as polynomial, trigonometric, and spline functions, have been explored. These methods typically produce smooth interpolants that are differentiable in most of the domains.

However, in real-world scenarios, some functions and signals do not exhibit smooth behavior. They can be complex and lack smoothness. Therefore, traditional interpolants with smoothness may not accurately represent the geometric structure of these signals. This creates a need for a new interpolation approach to generate interpolants that are nowhere differentiable and have a more intricate structure.

To address this issue, Barnsley [9] introduced the concept of fractal interpolation functions (FIFs) using the idea of an iterated function system (IFS). This interpolation technique exhibits some level of self-similarity. Classical interpolation methods described in existing literature may not capture the self-similarity and nonsmoothness in the data set. Barnsley's contribution sparked significant interest in fractal interpolation, leading to a plethora of publications on theory and applications by various researchers, see, for instance, [8, 13, 14, 19, 25, 26, 67] and references therein.

Numerous types of FIFs have been constructed, and their properties have been studied in the literature. For instance, in [12], Barnsley et al. studied the concept of Hidden variable fractal interpolation functions. In [68], Navascues et al. have introduced a new type of fractal interpolation function associated with a continuous real-valued univariate function, later known as α-fractal function. In [58], Massopust has introduced the concept of fractal functions and fractal surfaces. In [80], Ruan et al. have explored the study of fractal interpolation function on a rectangular grid. Recently in [90], Verma et al. have established the idea of a fractal operator related to bivariate fractal interpolation functions on a rectangular grid.

The study of fractal functions is receiving significant attention due to its practical applications and relevance in various fields. In addition to creating different types of FIFs, researchers are also focusing on analyzing their properties. Several important characteristics, such as calculus, dimension, stability, perturbation error, and smoothness, have been extensively investigated. For instance, in [15], Barnsley examined the box dimension of the bilinear FIF, while in [2], the box dimension of the α-fractal function has been explored. Recently in [28], Chandra et al. have conducted a study on the calculus of bivariate FIF. Further research on FIF and its properties can be found in [25, 41, 57, 58, 65, 93] and references therein.

Definition 1.10. Let $f : X \to Y$ be a function from a topological space X to a topological space Y, then graph of function f is denoted by $\mathcal{G}(f)$ and it is defined as follows:

$$\mathcal{G}(f) = \{(x, f(x)) : \ x \in X\}.$$

1.3.1 Univariate Fractal Interpolation Functions

Let us recall the construction of the univariate fractal interpolation function introduced by Barnsley [9].

Consider a data set $\Delta = \{(x_i, y_i) \in I \times \mathbb{R} : i \in \Sigma_{N,0}\}$ with $0 = x_0 < \cdots < x_N = 1$. Denote $I_i = [x_{i-1}, x_i]$ for $i \in \Sigma_N$. Let $u_i : I \to I_i$ be homeomorphisms for all $i \in \Sigma_N$ such that for all $x, x^* \in I$ and for some $0 \leq \eta_i < 1$,

$$|u_i(x) - u_i(x^*)| \leq \eta_i |x - x^*|$$
$$u_i(x_0) = x_{i-1} \text{ and } u_i(x_N) = x_i.$$

For each $i \in \Sigma_N$, let $F_i : I \times \mathbb{R} \to \mathbb{R}$ be a continuous map such that for all $x \in I, \ y, y^* \in \mathbb{R}$ and $0 \leq \rho_i < 1$ it satisfies

$$\begin{aligned} &|F_i(x, y) - F_i(x, y^*)| \leq \rho_i |y - y^*| \\ &F_i(x_0, y_0) = y_{i-1} \ \text{ and } \ F_i(x_N, y_N) = y_i. \end{aligned} \tag{1.2}$$

For all $i \in \Sigma_N$, let us take $u_i(x) = a_i x + b_i$ and $F_i(x, y) = \alpha_i(x) y + q_i(x)$ throughout. Then, using the conditions given on u_i and F_i, we have $a_i = x_i - x_{i-1}$, $b_i = x_{i-1}$ and $q_i : I \to \mathbb{R}$ such that

$$q_i(x_0) = y_{i-1} - \alpha_i(x_0) y_0 \text{ and } q_i(x_N) = y_i - \alpha_i(x_N) y_N. \tag{1.3}$$

And $\alpha_i : I \to \mathbb{R}$ such that $\|\alpha_i\|_\infty \leq 1$ is known as scaling factor.

Now for each $i \in \Sigma_N$, define a map $\Omega_i : I \times \mathbb{R} \to I \times \mathbb{R}$ such that

$$\Omega_i(x, y) = (u_i(x), F_i(x, y)). \tag{1.4}$$

Then, the IFS $\{I \times \mathbb{R} : \ \Omega_1, \dots, \Omega_N\}$ has a unique attractor [9] which is the graph of a continuous function h^* such that it satisfies the following self-referential equation

$$h^*(x) = \alpha_i(u_i^{-1}(x))h^*(u_i^{-1}(x)) + q_i(u_i^{-1}(x)). \tag{1.5}$$

Function h^* is known as FIF.

Remark 1.11. For the sake of simplicity, we deal with a constant scaling function $\alpha_i(x) = \alpha_i$ for each $i \in \Sigma_N$. Therefore, in this case $\|\alpha_i\|_\infty = |\alpha_i| \leq 1$.

Remark 1.12. Notice that the construction of interpolants using IFS provides a self-similarity nature of the graph of FIF. In general, the FIF constructed using the set of IFS (1.4) is nondifferentiable in nature. However, if the problem is of a differential type, the parameters of the generating IFS may be chosen such that a smooth fractal interpolant can be constructed; see, for instance, [24, 56]. Moreover, the Hausdorff and the box dimension of the graph of FIF can be of noninteger order.

Remark 1.13. If we choose the mappings F_i as $F_i(x) = c_i x + \alpha_i y + d_i$, where the coefficients c_i and d_i can be determined by the condition (1.2), and $\alpha_i \in (-1, 1)$, then constructed FIF is known as *affine*.

Example 1.14. Consider a set of data of interpolation given as follows

$$\Delta = \{(0,0),(0.1, 0.09), (0.2, 0.16), (0.3, 0.21), (0.4, 0.24), (0.5, 0.25), (0.6, 0.24),$$
$$(0.7, 0.21), (0.8, 0.16), (0.9, 0.09), (1, 0)\}.$$

Let $\Omega_i(x,y) = (u_i(x), F_i(x,y))$ such that $u_i(x) = 0.1x + 0.(i-1)$ and $F_i(x,y) = \alpha_i y + u_i(x)(1-u_i(x)) - \alpha_i x^2 u_i(x)(1-u_i(x))$.

The graph of FIF obtained by choosing $\boldsymbol{\alpha} = (0.9, 0.9, 0.8, 0.8, 0.9, 0.9, 0.8, 0.8, 0.9, 0.9)$ is plotted in Figure 1.1. Figure 1.2 represents the graph of FIF obtained by choosing $\boldsymbol{\alpha} = (0.2, 0.3, 0.5, 0.1, 0.5, 0.1, 0.8, 0.7, 0.5, 0.6)$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{10})$.

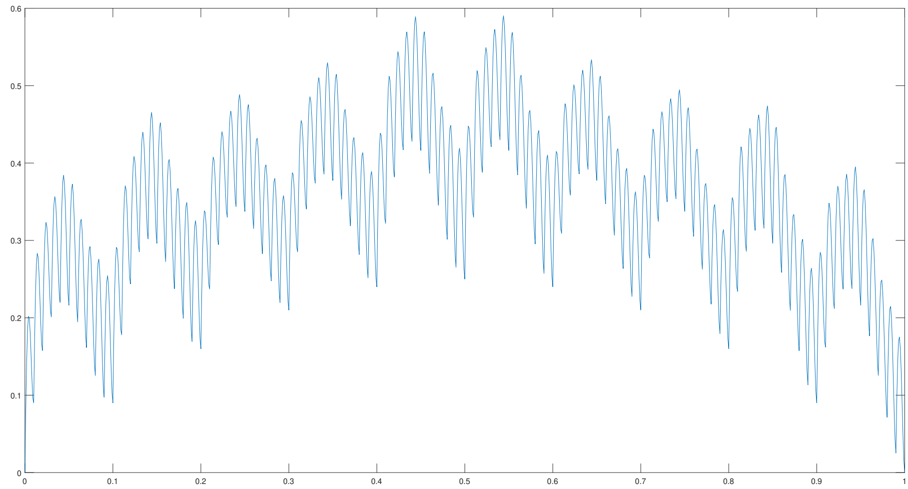

FIGURE 1.1: when $\boldsymbol{\alpha} = (0.9, 0.9, 0.8, 0.8, 0.9, 0.9, 0.8, 0.8, 0.9, 0.9)$

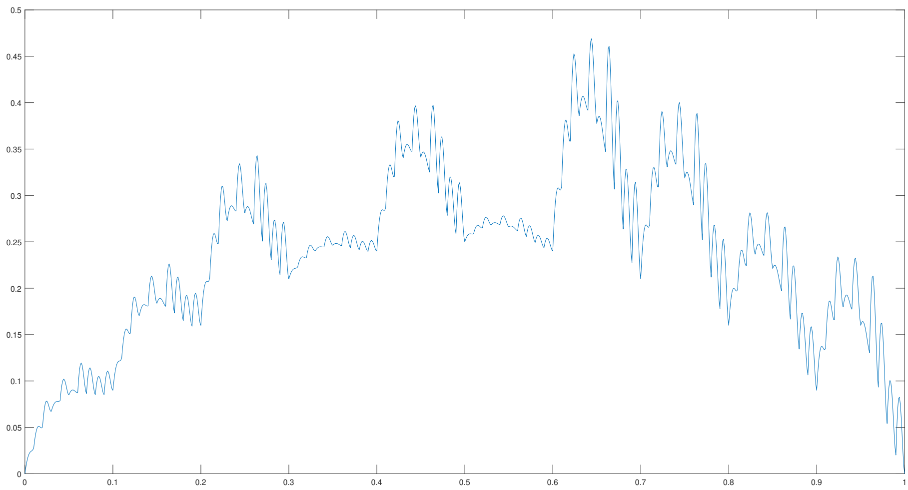

FIGURE 1.2: when $\boldsymbol{\alpha} = (0.2, 0.3, 0.5, 0.1, 0.5, 0.1, 0.8, 0.7, 0.5, 0.6)$

1.3.2 Univariate α-Fractal Functions

The theory of interpolation is interlaced with approximation theory, and it is like two sides of the same coin. This fact is endorsed in the context of FIF with α-fractal function, which is introduced by Navascués. In 2005, after the pioneering work of Barnsley [9], Navascués [65, 68] has studied a parameterized class of fractal interpolation function, known as α-fractal function, associated with the continuous function defined on a real compact interval.

Let $I = [0,1]$ be the unit interval of $\mathbb{R}$ and $\mathcal{C}(I,\mathbb{R})$ be endowed with the uniform norm. Let $f \in \mathcal{C}(I,\mathbb{R})$ be the prescribed function, known as *germ* function.

Let us consider $\Delta = \{x_0, \ldots, x_N\}$ to be a partition of I such that $0 = x_0 < \cdots < x_N = 1$. For each $i \in \Sigma_N$, let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a *scaling* vector, where $\alpha_i : I \to \mathbb{R}$ are continuous functions with $\|\alpha_i\|_\infty < 1$. Let $s : I \to \mathbb{R}$ be such that $s(x_0) = f(x_0),\ s(x_N) = f(x_N)$ and $s \neq f$, known as *base* functions. Now, in this case, the data set is $\{(x_i, f(x_i)) : i \in \Sigma_{N,0}\}$. Let us define maps

$$\begin{aligned} u_i(x) =& a_i x + b_i, \\ F_i(x,y) =& \alpha_i(y - s(x)) + f(u_i(x)),\ i \in \Sigma_N. \end{aligned} \tag{1.6}$$

Hence, corresponding IFS $\{I \times \mathbb{R} :\ \Omega_1, \ldots, \Omega_N\}$ has a unique attractor, which is the graph of a continuous function denoted by $f^\alpha_{\Delta,s}$ such that

$$f^\alpha_{\Delta,s}(x_i) = f(x_i) \ \text{ for } i \in \Sigma_{N,0}.$$

Moreover, $f^\alpha_{\Delta,s}$ satisfies the self-referential equation

$$f^\alpha_{\Delta,s}(x) = f(x) + \alpha_i(f^\alpha_{\Delta,s} - s)(u_i^{-1}(x)) \text{ for all } x \in I_i,\ i \in \Sigma_N. \tag{1.7}$$

The function $f^{\alpha}_{\Delta,s}$ is known as *α-fractal function*, and it is usually nondifferentiable in nature, and its graph can have noninteger Hausdorff and box dimension.

Note 1.15. If there is no ambiguity, then for notational convenience, we write "f^{α}" instead of "$f^{\alpha}_{\Delta,s}$". It is also known as *fractal perturbation* of f and using (1.7), it is easy to notice that ([65, 68])

$$\|f^{\alpha} - f\|_{\infty} \leq \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} \|f - s\|_{\infty}. \tag{1.8}$$

1.3.3 Multivariate Fractal Interpolation Functions

After the concept of FIF pioneered by Barnsley [9] using the idea of IFS, several works have been given to construct FIF. For instance, in [39], Feng introduced a construction of bivariate FIF on a rectangular domain. However, the condition of continuity for this construction is not easy to check. In [30], Dalla constructed bivariate FIF under the condition that interpolation points are collinear on each rectangle edge. Xie and Sun improved this construction in [96]. In [54], Malysz gave a construction of bivariate FIF with arbitrary interpolation points and equal vertical scaling on a rectangular domain. Ruan and Xu have recently given a general construction of bivariate FIF in [80]. Some more research can be found here [15, 18, 19, 23] in the context of bivariate FIF.

In contrast to the theory of univariate and bivariate FIF, higher dimensional analogous is inadequate in the literature. For instance, in [44, 58], a study of fractal interpolation surface (FIS) as the graph of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ for $m, n \in \mathbb{N}$ is presented and in [18], the general construction of FIFs on arbitrary interpolation points situated on a rectangular grid of $\mathbb{R}^n$ has been studied. These constructions are based on recurrent IFSs. Here, we will discuss the construction of multivariate FIF based on the framework of Ruan and Xu [80] in the bivariate case.

The data of interpolation are given as

$$\left\{\left(x_{1,i_1},\dots,x_{N,i_N},z_{(1,i_1),\dots,(N,i_N)}\right)\in\mathbb{R}^{N+1}: i_1\in\Sigma_{M_1,0},\cdots,i_N\in\Sigma_{M_N,0}\right\},$$

such that $0=x_{k,0}<\cdots<x_{k,M_k}=1$ for each I_k, where I_k denotes the k^{th} unit interval in I^N. Denote a net Δ on I^N defined as follows:

$$\Delta:=\left\{(x_{1,i_1},\dots,x_{N,i_N})\in I^N: i_k\in\Sigma_{M_k,0},\ 0=x_{k,0}<\cdots<x_{k,M_k}=1,\ k\in\Sigma_N\right\}.$$

For each $i_k\in\Sigma_{M_k}$, let us consider $I_{k,i_k}=[x_{k,i_k-1},x_{k,i_k}]$ and define contraction functions $u_{k,i_k}:I_k\to I_{k,i_k}$ such that

$$\begin{aligned}
u_{k,i_k}(x_{k,0}) &= x_{k,i_k-1}, \qquad u_{k,i_k}(x_{k,M_k})=x_{k,i_k},\ \text{if } i_k \text{ is odd},\\
u_{k,i_k}(x_{k,0}) &= x_{k,i_k}, \qquad u_{k,i_k}(x_{k,M_k})=x_{k,i_k-1},\ \text{if } i_k \text{ is even, and}\\
|u_{k,i_k}(x)-u_{k,i_k}(y)| &\le \mu_{k,i_k}|x-y| \text{ for all } x,y\in I_k,\ \text{and } 0\le\mu_{k,i_k}<1.
\end{aligned} \tag{1.9}$$

It is easy to observe from the definition of u_{k,i_k} that

$$u^{-1}_{k,i_k}(x_{k,i_k})=u^{-1}_{k,i_k+1}(x_{k,i_k}) \text{ for all } i_k\in\Sigma_{M_k},\ \text{where } k\in\Sigma_N.$$

Define a function $\eta:\mathbb{Z}\times\{0,M_1,\dots,M_N\}\to\mathbb{Z}$ by

$$\begin{cases}
\eta(i,0)=i-1, \quad \eta(i,M_1)=\cdots=\eta(i,M_N)=i, & \text{when } i \text{ is odd},\\
\eta(i,0)=i, \quad \eta(i,M_1)=\cdots=\eta(i,M_N)=i-1, & \text{when } i \text{ is even}.
\end{cases}$$

Set $K = I^N \times \mathbb{R}$. For each $(i_1, \ldots, i_N) \in \prod_{k=1}^N \Sigma_{M_k}$, let $F_{i_1,\ldots,i_N} : K \to \mathbb{R}$ be a continuous function satisfying

$$F_{i_1\cdots i_N}\left(x_{1,k_1}, \ldots, x_{N,k_N}, z_{(1,k_1)\cdots(N,k_N)}\right) = z_{(1,\eta(i_1,k_1))\cdots(N,\eta(i_N,k_N))}$$

for all $(k_1, \ldots, k_N) \in \prod_{j=1}^N \partial\Sigma_{M_j,0}$, and

$$|F_{i_1\cdots i_N}(x_1, \ldots, x_N, z^*) - F(x_1, \ldots, x_N, z^{**})| \leq \gamma_{i_1\cdots i_N}|z^* - z^{**}|$$

for all $(x_1, \ldots, x_N) \in I^N$, $z^*, z^{**} \in \mathbb{R}$, and $0 < \gamma_{i_1\cdots i_N} < 1$ is any given constant. For each $(i_1, \ldots, i_N) \in \prod_{k=1}^N \Sigma_{M_k}$, define a function $\Omega_{i_1\cdots i_N} : K \to \prod_{k=1}^N I_{k,i_k} \times \mathbb{R}$ such that

$$\Omega_{i_1\cdots i_N}(x_1, \ldots, x_N, z) = (u_{1,i_1}(x_1), \ldots, u_{N,i_N}(x_N), F_{i_1\cdots i_N}(x_1, \ldots, x_N, z)). \tag{1.10}$$

Then, with the assumptions on $F_{i_1\cdots i_N}$ and u_{k,i_k}, for each $k \in \Sigma_N$, one can prove that $\Omega_{i_1\cdots i_N}$ is a contraction function, hence $\{K,\ \Omega_{i_1\cdots i_N} : (i_1, \ldots, i_N) \in \prod_{k=1}^N \Sigma_{M_k}\}$ is an IFS.

Theorem 1.16. *Let $\{K, \Omega_{i_1\cdots i_N} : (i_1, \ldots, i_N) \in \prod_{k=1}^N \Sigma_{M_k}\}$ be the IFS as defined in (1.10). For every $k \in \Sigma_N$, suppose $F_{i_1\cdots i_N}$ satisfies the following matching conditions: for all $i_k \in int\Sigma_{M_k,0}$, and $j \neq k$, $i_j \in \Sigma_{M_j}$ and $x_k^* = u_{k,i_k}^{-1}(x_{k,i_k}) = u_{k,i_k+1}^{-1}(x_{k,i_k})$,*

$$F_{i_1\cdots i_N}(x_1, \ldots, x_k^*, x_{k+1}, \ldots, x_N, z) = F_{i_1\cdots i_{k-1}, i_k+1 i_{k+1}\cdots i_N}(x_1, \ldots, x_k^*, x_{k+1}, \ldots, x_N, z) \tag{1.11}$$

for all $x_j \in I_j$, $z \in \mathbb{R}$. Then, there is a unique continuous function $g^* : I^N \to \mathbb{R}$ such that

$$g^*(x_{1,i_1}, \ldots, x_{N,i_N}) = z_{(1,i_1)\cdots(N,i_N)} \text{ for all } (i_1, \ldots, i_N) \in \prod_{k=1}^N \Sigma_{M_k,0},$$

and $H = \cup\{\Omega_{i_1\cdots i_N}(H) : (i_1, \ldots, i_N) \in \prod_{k=1}^{N} \Sigma_{M_k}\}$*, where* $H = \mathcal{G}(g^*)$*. We call* H *the FIS and* g^* *the FIF concerning the IFS defined in* (1.10)*. Moreover, for all* $(i_1, \ldots, i_N) \in \prod_{k=1}^{N} \Sigma_{M_k}$*, take Read-Bajraktarević (RB) operator,* $T : \mathcal{C}(I^N) \to \mathcal{C}(I^N)$ *such that*

$$Tg(x_1, \ldots, x_N) = F_{i_1\cdots i_N}\left(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N), g\left(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\right)\right).$$

Then, the multivariate FIF, g^**, will be the unique fixed point of* T*.*

The proof of the above theorem can be accomplished by using a technique similar to Ruan and Xu [80]; hence it is omitted here.

Throughout this chapter, for $(x_1, \ldots, x_N) \in \prod_{k=1}^{N} I_{k,i_k}$, where $i_k \in \Sigma_{M_k}$, let us define $F_{i_1\cdots i_N} : K \to \mathbb{R}$ such that

$$F_{i_1\cdots i_N}(x_1, \ldots, x_N, z) = \delta z + \mathcal{B}_{i_1\cdots i_N}(x_1, \ldots, x_N), \text{ where}$$

$$\mathcal{B}_{i_1\cdots i_N}(x_{1,k_1}, \ldots x_{N,k_N}) = z_{(1,\eta(i_1,k_1))\cdots(N,\eta(i_N,k_N))} - \delta z_{(1,k_1)\cdots(N,k_N)}$$

for all $(k_1, \ldots, k_N) \in \prod_{k=1}^{N} \partial\Sigma_{M_k,0}$ and $|\delta| < 1$. Then, observe that $F_{i_1\cdots i_N}$ satisfies the conditions of Theorem 1.16. Now for all $(x_1, \ldots, x_N) \in \prod_{k=1}^{N} I_{k,i_k}$, where $i_k \in \Sigma_{M_k}$, define Read-Bajraktarević (RB) operator, $T : \mathcal{C}(I^N) \to \mathcal{C}(I^N)$ such that

$$Tg(x_1, \ldots, x_N) = F_{i_1\cdots i_N}\left(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N), g\left(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\right)\right).$$

Then, the multivariate FIF, g^* will be the unique fixed point of T and for all $(x_1, \ldots, x_N) \in \prod_{k=1}^{N} I_{k,i_k}$, where $i_k \in \Sigma_{M_k}$, g^* will satisfy the following self-referential equation:

$$g^*(x_1,\ldots,x_N) = F_{i_1\cdots i_N}\left(u_{1,i_1}^{-1}(x_1),\ldots,u_{N,i_N}^{-1}(x_N), g^*\left(u_{1,i_1}^{-1}(x_1),\ldots,u_{N,i_N}^{-1}(x_N)\right)\right).$$

Therefore, we have

$$g^*(x_1,\ldots,x_N) = \delta g^*\left(u_{1,i_1}^{-1}(x_1),\ldots,u_{N,i_N}^{-1}(x_N)\right) + \mathcal{B}_{i_1\cdots i_N}\left(u_{1,i_1}^{-1}(x_1),\ldots,u_{N,i_N}^{-1}(x_N)\right). \tag{1.12}$$

Note 1.17. For every $x \in I_{k,i_k}$, define $u_{k,i_k}(x) = a_{k,i_k}(x) + b_{k,i_k}$ such that

$$\begin{aligned} a_{k,i_k} &= \frac{x_{k,i_k} - x_{k,i_{k-1}}}{x_{k,M_k} - x_{k,0}} \quad \text{and} \quad b_{k,i_k} = \frac{x_{k,i_k}x_{k,M_k} - x_{k,i_{k-1}}x_{k,0}}{x_{k,M_k} - x_{k,0}}, \text{ if } i_k \text{ is odd} \\ a_{k,i_k} &= \frac{x_{k,i_{k-1}} - x_{k,i_k}}{x_{k,M_k} - x_{k,0}} \quad \text{and} \quad b_{k,i_k} = \frac{x_{k,i_{k-1}}x_{k,M_k} - x_{k,i_k}x_{k,0}}{x_{k,M_k} - x_{k,0}}, \text{ if } i_k \text{ is even.} \end{aligned}$$

Then, obviously u_{k,i_k} satisfies (1.9).

1.3.4 Multivariate α-Fractal Functions

Following the work of Ruan and Xu, [80], Verma and Viswanathan [90] have recently introduced a bivariate version of the α-fractal function and studied their applications in approximation theory. Here, we give the concept of α-fractal function corresponding to a multivariate continuous function.

Let $f \in \mathcal{C}(I^N)$, and $\alpha \in \mathcal{C}(I^N)$ be such that $\|\alpha\|_\infty < 1$. Further, a net Δ on I^N is defined as follows:

$$\Delta := \left\{(x_{1,i_1},\ldots,x_{N,i_N}) \in I^N : i_k \in \Sigma_{M_k,0},\ 0 = x_{k,0} < \cdots < x_{k,M_k} = 1,\ k \in \Sigma_N\right\}.$$

Assume further that $s \in \mathcal{C}(I^N)$ satisfies

$$s(x_{1,i_1}, \dots, x_{N,i_N}) = f(x_{1,i_1}, \dots, x_{N,i_N}) \text{ for all } (i_1, \dots, i_N) \in \prod_{k=1}^{N} \partial\Sigma_{M_k,0}.$$

For each $(x_1, \dots, x_N) \in \prod_{k=1}^N I_{k,i_k}$, where $i_k \in \Sigma_{M_k}$, define $F_{i_1 \dots i_N} : K \to \mathbb{R}$ as follows

$$\begin{aligned} &F_{i_1 \dots i_N}(x_1, \dots, x_N, z) \\ =&f\left(u_{1,i_1}(x_1), \dots, u_{N,i_N}(x_N)\right) + \alpha(x_1, \dots, x_N)\left(z - s(x_1, \dots, x_N)\right). \end{aligned} \tag{1.13}$$

Then, $W_{i_1 \dots i_N} : K \to \prod_{k=1}^N I_{k,i_k} \times \mathbb{R}$ defined by

$$W_{i_1 \dots i_N}(x_1, \dots, x_N, z) = \left(u_{1,i_1}(x_1), \dots, u_{N,i_N}(x_N), F_{i_1 \dots i_N}(x_1, \dots, x_N, z)\right)$$

is an IFS on K and notice that $F_{i_1 \dots i_N}$ satisfies the conditions of Theorem 1.16, therefore we have the following:

Theorem 1.18. *Let* $\left\{K, W_{i_1 \dots i_N} : (i_1, \dots, i_N) \in \prod_{k=1}^N \Sigma_{M_k}\right\}$ *be the IFS. Then, there exists a unique continuous function* $f^\alpha_{\Delta,s} : I^N \to \mathbb{R}$ *such that* $f^\alpha_{\Delta,s}\left(x_{1,i_1}, \dots, x_{N,i_N}\right) = f\left(x_{1,i_1}, \dots, x_{N,i_N}\right)$ *for all* $(i_1, \dots, i_N) \in \prod_{k=1}^N \Sigma_{M_k,0}$ *and*

$$G = \bigcup \left\{ W_{i_1 \cdots i_N}(G) : (i_1, \dots, i_N) \in \prod_{k=1}^{N} \Sigma_{M_k,0} \right\},$$

where $\mathcal{G}(f^\alpha_{\Delta,s}) = G$*.*

Now define a RB Operator, $T : \mathcal{C}(I^N) \to \mathcal{C}(I^N)$ such that

$$T(h)(x_1, \dots, x_N) = F_{i_1 \dots i_N}\left(u^{-1}_{1,i_1}(x_1), \dots, u^{-1}_{N,i_N}(x_N), h\left(u^{-1}_{1,i_1}(x_1), \dots, u^{-1}_{N,i_N}(x_N)\right)\right),$$

then there exists a fixed point of this operator, $f^{\alpha}_{\Delta,s}$, known as α-fractal function and it satisfies the following self-referential equation,

$$f^{\alpha}_{\Delta,s}(x_1,\ldots,x_N)=F_{i_1\ldots i_N}\left(u^{-1}_{1,i_1}(x_1),\ldots,u^{-1}_{N,i_N}(x_N),f^{\alpha}_{\Delta,s}\left(u^{-1}_{1,i_1}(x_1),\ldots,u^{-1}_{N,i_N}(x_N)\right)\right)$$

for all $(x_1,\ldots,x_N)\in\prod_{k=1}^{N}I_{k,i_k}$. That is,

$$\begin{aligned}&f^{\alpha}_{\Delta,s}(x_1,\ldots,x_N)\\ =&f(x_1,\ldots,x_N)+\alpha\left(u^{-1}_{1,i_1}(x_1),\ldots,u^{-1}_{N,i_N}(x_N)\right)\left[f^{\alpha}_{\Delta,s}\left(u^{-1}_{1,i_1}(x_1),\ldots,u^{-1}_{N,i_N}(x_N)\right)\right.\\ &\left.\qquad\qquad\qquad\qquad -s\left(u^{-1}_{1,i_1}(x_1),\ldots,u^{-1}_{N,i_N}(x_N)\right)\right]\end{aligned} \tag{1.14}$$

for $(x_1,\ldots,x_N)\in\prod_{k=1}^{N}I_{k,i_k}$.

• Denote $f^{\alpha}_{\Delta,s}$ as f^{α} if there is no ambiguity.

1.4 Fractional Integral

Integration and differentiation are valuable tools not only in mathematics but in various other fields as well. It is known that integration and differentiation are interconnected. For instance, consider the function $g(x)=\int_a^x f(x)dx$, where $f:[a,b]\to\mathbb{R}$ is a continuous function. In this case, f is referred to as the derivative of g, and g is recognized as the integral of f. This definition is commonly known as classical integration and differentiation. It is important to note that these functions are of integer order. However, the study of differentiation and integration with noninteger orders falls under the field of Fractional Calculus. The study of fractional calculus, which has been around nearly for the last 300 years, focuses on the theory of derivatives and integrals with noninteger orders. Several formulations have been developed to define fractal integrals, such as the Riemann-Liouville fractional integral, Hadamard fractional integral, Katugampola fractional integral, etc. There have been numerous publications dedicated to the study of fractional calculus. For a more comprehensive study of fractional calculus and additional perspectives on its applications in science, engineering, and its interpretations from a physical and geometrical standpoint, one may consult [28, 64, 69, 71, 73, 74, 79, 83] and references therein.

1.4.1 Riemann-Liouville Fractional Integral

Definition 1.19. Let g be an integral map on a closed and bounded interval $[a, b]$ of $\mathbb{R}$, then Riemann-Liouville fractional integral of g is expressed as

$$ {}_a\mathfrak{J}^{\mu} g(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} g(t)\ \mathrm{d}t, $$

where $\mu > 0$.

1.4.2 Katugampola Fractional Integral

Definition 1.20. Let g be an integrable map on a closed interval $[a, b]$. Then, the Katugampola fractional integral of g is defined as

$$ {}_a^{\rho}\mathfrak{J}^{\mu} g(x) = \frac{(\rho+1)^{1-\mu}}{\Gamma(\mu)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} g(t)\ \mathrm{d}t, $$

where $\mu > 0$, and $\rho \neq -1$ are real numbers.

Remark 1.21. Here, we note that the Katugampola fractional integral generalizes the Riemann-Liouville and Hadamard fractional integral as follows.

- Choosing $\rho = 0$, we get

$$ {}^{\rho}_{a}\mathfrak{J}^{\mu} g(x) = \frac{1^{1-\mu}}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} g(t)\ \mathrm{d}t = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} g(t)\ \mathrm{d}t, $$

this is the Riemann-Liouville fractional integral of $g : [a, b] \to \mathbb{R}$.

- Now, taking $\rho \to -1^{+}$, we have

$$ \begin{aligned} \lim_{\rho \to -1^{+}} {}^{\rho}_{a}\mathfrak{J}^{\mu} g(x) = & \lim_{\rho \to -1^{+}} \frac{(\rho+1)^{1-\mu}}{\Gamma(\mu)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} g(t)\ \mathrm{d}t \\ = & \frac{1}{\Gamma(\mu)} \int_a^x \lim_{\rho \to -1^{+}} \left(\frac{x^{\rho+1} - t^{\rho+1}}{\rho+1} \right)^{\mu-1} t^{\rho} g(t) \mathrm{d}t \\ = & \frac{1}{\Gamma(\mu)} \int_a^x \left(\log \frac{x}{t} \right)^{\mu-1} \frac{g(t)}{t}\ \mathrm{d}t, \end{aligned} $$

using the L'hospital rule. This is the Hadamard fractional integral.

1.5 Set-Valued Maps

Definition 1.22. [4] Let X and Y be metric spaces and $F : X \rightrightarrows Y$ be a set-valued map from X to Y, then the graph of F is defined as,

$$ G_F = \{(x, y) \in X \times Y : y \in F(x)\}. \tag{1.15} $$

Here $F(x)$ is the image (or) the value of F at x. If there is at least one element $x \in X$ such that $F(x)$ is nonempty, then F is considered nontrivial. If $F(x)$ is nonempty for each $x \in X$, then F is known to be strict. The domain and range of F are defined as

$$ \mathrm{Dom}(F) := \{x \in X : F(x) \neq \emptyset\} \text{ and } \mathrm{Im}(F) := \bigcup_{x \in X} F(x), \text{respectively.} $$

Definition 1.23. Let $F, G : I \rightrightarrows \mathbb{R}$ be set-valued maps. Then, $F \leq G$ if and only if $F(x) \subseteq G(x)$ for all $x \in I$.

Remark 1.24. If $F(x)$ is closed (convex, compact, bounded), then F is said to be closed (convex, compact, bounded).

Definition 1.25. [4] Let $F : X \rightrightarrows Y$ be a set-valued mapping and $x \in \mathrm{Dom}(F)$ such that for every neighborhood $\mathfrak{U}$ of $F(x)$,

$$\text{there exists } \eta > 0 \text{ such that } F(x') \subset \mathfrak{U} \text{ for all } x' \in B_X(x, \eta), \tag{1.16}$$

where $B_X(x, \eta)$ is the η-neighbourhood of x in X. Then, F is characterized as upper semicontinuous at x. If it satisfies (1.16) for each $x \in \mathrm{Dom}(F)$, then F is known as an upper semicontinuous function.

If for every $y \in F(x)$ and every sequence $\{x_n\} \subset \mathrm{Dom}(F)$ converging to x, there exists a sequence of elements $y_n \in F(x_n)$ which converges to y, then F is characterized as lower semicontinuous at x. If it is lower semicontinuous at each $x \in \mathrm{Dom}(F)$, then F is said to be the lower semicontinuous function.

Lemma 1.26. [4] *A set-valued map F is said to be convex if and only if for all $x_1, x_2 \in$ $Dom(F)$, $\lambda \in [0, 1]$, we have*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F\left(\lambda x_1 + (1 - \lambda)x_2\right).$$

1.6 Invariant Measures and Fractal Transformations

1.6.1 Invariant Measures

Definition 1.27. [11] Let (X, d) be a compact metric space and ν be a Borel measure on X; if $\nu(X) = 1$, then ν is said to be Borel probability measure.

Definition 1.28. [11] Let (X, d) be a complete metric space. Let $\mathfrak{M}(X)$ denote the Borel probability measures on X. Define the metric $d_{\mathcal{H}} : \mathfrak{M}(X) \times \mathfrak{M}(X) \to \mathbb{R}$ as

$$d_{\mathcal{H}}(\mu, \nu) = \sup \left\{ \int_X f d\mu - \int_X f d\nu : \ f \in \mathcal{L}ip(1) \right\} \text{ for all } \mu, \nu \in \mathfrak{M}(X),$$

where $\mathcal{L}ip(1)$ denotes the collection of Lipschitz functions with Lipschitz constant 1, then the set $\mathfrak{M}(X)$ equipped with this metric $d_{\mathcal{H}}$ is a complete metric space.

Definition 1.29. [11] Let (X, d) be a complete metric space and $\mathfrak{M}(X)$ is the space of Borel measures on X. Let $\{X; \upsilon_1, \ldots, \upsilon_m\}$ be an IFS and $\boldsymbol{r} = (r_1, \ldots, r_m)$ be a probability vector. Then, the Markov operator $\mathcal{M} : \mathfrak{M}(X) \to \mathfrak{M}(X)$ associated with the IFS is defined as

$$\mathcal{M}(\mu) = \sum_{i=1}^{m} r_i \mu \circ \upsilon_i^{-1} \text{ for all } \mu \in \mathfrak{M}(X). \tag{1.17}$$

Lemma 1.30. [11] *Let $\{X; \upsilon_1, \ldots, \upsilon_m\}$ be an IFS and $\boldsymbol{r} = (r_1, \ldots, r_m)$ be a probability vector and $\mathcal{M}$ be a Markov operator as defined in (1.17) associated with the IFS. Let $f : X \to \mathbb{R}$ be either a simple function or a continuous function, then*

$$\int_X g \ d(\mathcal{M}(\mu)) = \sum_{i=1}^{m} r_i \int_X g \circ \upsilon_i \ d\mu \quad \textit{for all } \mu \in \mathfrak{M}(X).$$

Theorem 1.31. [11] *Let $\{X; \upsilon_1, \dots \upsilon_m\}$ be an IFS on a complete metric space (X, d) and $\boldsymbol{r} = (r_1, \dots, r_m)$ be a probability vector. Then, there exists a unique Borel probability measure $\mu_r \in \mathfrak{M}(X)$ such that*

$$\mu_r = \sum_{i=1}^{m} p_i \mu_r \circ \upsilon_i^{-1}.$$

Moreover, the support of μ_r is the attractor of the IFS.

Definition 1.32. For a measure μ on $\mathbb{R}$, Hausdorff dimension is defined as

$$\begin{aligned} \dim_H(\mu) &= \inf\{\dim_H(A) : A \text{ is Borel}, \mu(\mathbb{R} \backslash A) = 0\} \\ &= \inf\{\dim_H(A) : A \text{ is Borel}, \mu(A) > 0\}. \end{aligned} \tag{1.18}$$

Moreover, if $\{X; \upsilon_1, \dots, \upsilon_m\}$ is a self-similar IFS and satisfies the OSC and if $\boldsymbol{r} = (r_1, \dots, r_m)$ is a probability vector such that μ_r is the invariant measure corresponding to $\boldsymbol{r}$. Then, it has been proved in [31], that the Hausdorff dimension of μ_r is given by

$$\dim_H(\mu_r) = \frac{\sum_{i=1}^{m} r_i \log(r_i)}{\sum_{i=1}^{m} r_i \log(\kappa_i)}, \tag{1.19}$$

where κ_i is the similarity ratio of υ_i for each $i \in \Sigma_m$.

1.6.2 Fractal Transformations

Recall that the space $\Lambda^\infty = \{1, \dots, m\}^\infty$ referred as *code space*, is the set of all infinite sequences $\zeta = \zeta_1\zeta_2\cdots$ with elements from Σ_m. The operator $\mathcal{S}(\zeta_1\zeta_2\cdots) = \zeta_2\zeta_3\cdots$ is known as *shift operator*. The space $(\Lambda^\infty, d_{\Lambda^\infty})$ is a compact space with respect to the metric d_{Λ^∞}, where $d_{\Lambda^\infty} : \Lambda^\infty \times \Lambda^\infty \to \mathbb{R}$ such that for $\zeta, \theta \in \Lambda^\infty$ with

$\zeta \neq \theta$, we have

$$d_{\Lambda^\infty}(\zeta, \theta) = 2^{-n}, \text{where } n \text{ is the least integer such that } \zeta_n \neq \theta_n.$$

Definition 1.33. [7] Let the IFS $\{\Lambda^\infty; \varpi_1, \dots, \varpi_m\}$, where $\varpi_i : \Lambda^\infty \to \Lambda^\infty$ be defined as $\varpi_i(\theta) = i\theta$. The attractor of this IFS is Λ^∞. Each ϖ_i is a contraction map with contraction ratio $\frac{1}{2}$. This IFS is known as *code space IFS*.

Definition 1.34. [10] Let (X, d) be a metric space and $\mathcal{A}$ be the attractor of IFS $\{X; \upsilon_1, \dots, \upsilon_m\}$, then *coding map* $\pi_\upsilon : \Lambda^\infty \to \mathcal{A}$ is defined as

$$\pi_\upsilon(\theta) = \lim_{k \to \infty} \upsilon_{\theta_1} \circ \cdots \circ \upsilon_{\theta_k}(b) \text{ for a fixed } b \in \mathcal{A} \text{ and for all } \theta = \theta_1\theta_2 \cdots \in \Lambda^\infty.$$

Since the IFS is contractive, this limit will be a single point independent of the choice of b, implying that the convergence is uniform and π_υ is continuous and onto.

If $\pi_\upsilon(\theta) = x$ for $\theta \in \Lambda^\infty$, then θ is known as *address* of x.

Moreover, the map π_υ satisfies $\pi_\upsilon(\varpi_i(\theta)) = \upsilon_i(\pi_\upsilon(\theta))$.

Definition 1.35. [10] Let Λ^∞ be a code space. For $\zeta, \theta \in \Lambda^\infty$, we have $\zeta > \theta$ if $\zeta_n < \theta_n$, where n is the smallest number such that $\zeta_n \neq \theta_n$. Then, this ordering is known as *lexicographical* ordering.

Definition 1.36. [7] Let $\pi_\upsilon : \Lambda^\infty \to \mathcal{A}$ be a coding map and there is map $\tau_\upsilon : \mathcal{A} \to \Lambda^\infty$ such that $\pi_\upsilon \circ \tau_\upsilon$ is the identity. Then, τ_υ is said to be *section* map of π_υ. Further, if $\tau_\upsilon : \mathcal{A} \to \Lambda^\infty$ satisfies the condition $\tau_\upsilon(x) = \max \pi_\upsilon^{-1}(x)$ for all $x \in \mathcal{A}$, where maximum is taken with respect to the lexicographic ordering. Then, τ_υ is known as *top section* of π_υ

Definition 1.37. [7] Let $\mathcal{A}_\upsilon$ and $\mathcal{A}_{\upsilon'}$ be the attractors of the IFSs $\{X; \upsilon_1, \dots, \upsilon_m\}$ and $\{X; \upsilon'_1, \dots, \upsilon'_m\}$, respectively. Then, the *fractal transformations* $T_{\upsilon\upsilon'} : \mathcal{A}_\upsilon \to \mathcal{A}_{\upsilon'}$

and $T_{\upsilon'\upsilon} : \mathcal{A}_{\upsilon'} \to \mathcal{A}_{\upsilon}$ are defined as

$$T_{\upsilon\upsilon'} = \pi_{\upsilon'} \circ \tau_{\upsilon} \text{ and } T_{\upsilon'\upsilon} = \pi_{\upsilon} \circ \tau_{\upsilon'}.$$

1.7 Motivation and Objective of this Thesis

A significant amount of research is available on the fractal dimension of univariate and bivariate real-valued functions. Additionally, a wealth of research focuses on fractal interpolation functions and the fractional integral of functions with a single value (see, for instance, [12, 15, 19, 23, 25, 27])

Despite the significant interest and practical applications observed in real-valued fractal interpolation and fractal dimension, there have been limited investigations into its extension to *vector-valued*, *multivariate*, and *set-valued maps*. In fact, we have made the initial effort to introduce the concept of α-fractal function for *set-valued maps* and have obtained results regarding the fractal dimension of *vector-valued maps*. In the present work, we attempt to advance the concept of fractal dimension and fractal interpolation function in higher dimensional functions such as the fractal dimension of vector-valued maps, fractal interpolation, and fractal dimension of set-valued maps. Further, we have studied some properties of invariant measures associated with the fractal interpolation function.

Calculating the fractal dimension of a function's graph is a challenging task, even for functions with real values. In Chapter 2, we have tried to estimate the fractal dimension of the graph of the Katugampola fractional integral of a vector-valued continuous function with bounded variation on a closed bounded interval in $\mathbb{R}$. We are able to prove that the dimension of the graph of a continuous vector-valued function with bounded variation is 1, and the same applies to the graph of its

Katugampola fractional integral. Additionally, for an Hölder continuous function, we have provided an upper bound for the upper box dimension of the graph of each coordinate function of the Katugampola fractional integral.

In Chapter 5, we have expanded the classical approximation of set-valued maps to the fractal approximation of set-valued maps. We have introduced the concept of an α-fractal function for set-valued maps. However, unlike single-valued maps, we have observed that the set-valued α-fractal function is generally not interpolatory in nature. Furthermore, we have focused on estimating the fractal dimension of the graph of certain special classes of set-valued maps.

1.8 Delineation of this Thesis

This thesis consists of seven chapters; the organization of these chapters is as follows.

Chapter 1 is reserved for the introduction and basic definitions of this thesis that are being used throughout this thesis. It consists of the literature survey and recent developments in the field of fractal interpolation and fractal dimension. It also includes the motivation and mentions the contributions made in this thesis.

Chapter 2 presents the study of the fractional integral of univariate vector-valued maps. This chapter presents the initial theory for the dimension of the graphs of vector-valued maps and, in particular, the study of the fractal dimension of the graph of Katugampola fractional integral of a vector-valued continuous function of bounded variation defined on a closed and bounded interval of $\mathbb{R}$.

Chapter 3 proposes the study of multivariate fractal interpolation functions for a given set of data points. Further, this chapter includes the study of the restriction of the α-fractal function on the coordinate axis. In addition, the box dimension and the Hausdorff dimension of the graph of the multivariate α-fractal function and its restriction are investigated.

Chapter 4 consists of the study of approximation of continuous multivariate functions defined on the domain $[0,1]^N$. Further, a few well-known multivariate constrained approximation results are established. Also, the existence of a one-sided approximation of multivariate functions using fractal functions is proved. Moreover, the existence of the Schauder basis consisting of multivariate fractal functions for the space of all real-valued continuous functions defined on $[0,1]^N$ is established, and the existence of multivariate fractal polynomials for the approximation is proved.

Chapter 5 is devoted to the study of set-valued α-fractal functions. This chapter introduces the concept of the α-fractal function and fractal approximation for a set-valued continuous map defined on a closed and bounded interval of real numbers, and a few properties of such fractal functions are given. Further, the study of the fractal dimension of the graph of some special class of functions is carried out in this chapter.

Chapter 6 explores the measure theoretic aspects of FIFs using the method developed by Barnsley [9]. This chapter discusses the characteristics of the invariant measures that are supported on the graph of FIF. Additionally, it investigates the fractal transformation between two FIFs that share the same interpolation data. The latter part of the chapter defines certain function spaces and outlines the conditions under which the FIF can be considered a member of these spaces.

Chapter 7 wraps up the thesis and provides suggestions for potential future research directions.

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