Chapter 4

The Application of Differential Constraint Method for the Solution of Non-homogeneous Generalized Riemann problem

4.1 Introduction

It is always interesting and complicated to compute an exact solution for the generalized Riemann problem (GRP) that is prominent in quasi-linear hyperbolic systems. The Riemann problem (RP) is described as an initial value problem given by two constant states having a jump discontinuity at a fixed point. When the piecewise constant data are perturbed in a problem, it is called GRP. The solutions of GRP are the basic building blocks for the treatment of complex hyperbolic systems. Because of tremendous applications of the GRP, researchers, engineers, and mathematicians have shown great interest in last few decades. From the starting work of Riemann, a wide significant addition has been provided on this burning topic by Smoller [21], Dafermos [116, 117] and in the references quoted therein. For a homogeneous hyperbolic system, the general theory of the RP was developed by Lax in his celebrated chapter [3]. He proved the existence and uniqueness of the solution of the RP by assuming very small jumps in initial states. Also, the solutions are given by constant states divided by elementary waves, namely, rarefaction, shock, and/or contact discontinuities. Further, in the case of a homogeneous hyperbolic system, solutions are characterized by simple waves, but unfortunately, this fails for the nonhomogeneous case. But, within the frame of reference of the differential constraints, it is possible to generalize the simple waves in nonhomogeneous systems. There is a very small amount of literature available in this area, and it primarily concerns the existence and uniqueness, together with asymptotic solutions.

Solving a GRP is complex, and getting an exact solution is more challenging. Lefloch and Raviart [118] found an asymptotic expansion approximate result in solving GRP, Bourgeade and the team [119] explained this with an application in gas dynamics. In recent years, distinct computational methods have been used to capture an exact solution of a quasi-linear hyperbolic system. There are no general ways or techniques to solve a GRP to date. Examples of techniques that can produce exact or approximate solutions are the similarity transformation approach, asymptotic expansion, perturbation, differential constraint method, and so on. Among them, the reduction method (DCM) plays a decisive part in finding exact solutions for a class of nonlinear PDEs. In this chapter, a novel approach to studying the application of DCM for the solution of non-homogeneous GRP is analyzed. The system of non-linear PDEs and its exact solution play a significant role in developing a better understanding and qualitative characterization of several physical events. Also, it has a variety of applications in the field of physical and natural sciences. Many researchers have recently used the differential constraint approach to achieve the precise solution of the hyperbolic system [120] connected by elementary waves (see [55, 121]). The differential constraint method is one of the best reduction methods for obtaining particular exact solutions of quasi-linear PDEs. Janenko [59] first proposed the DCM in 1964, and Sidorov, Shapeev et al. gave a full, detailed review of this method in their book [122]. This method involves a group of differential equations added to the governing system of interest. We assume that a solution to the governing system is also a solution to the additional differential equations. These appended equations are known as differential constraints. We can understand the importance of this method as we know that in isentropic flow, the 1D gas-dynamic equation is reducible to the homogeneous hyperbolic equations drafted in Riemann invariants. In this case, we get simple waves, but we can not find such waves in non-isentropic flows because no Riemann invariants exist for such a kind of flow. We can remove this obstacle by DCM. By this approach, we obtained solutions that can substitute for the simple waves. These solutions are described as a non-isentropic rarefaction wave. The decay of arbitrary discontinuity, pulling piston problems, and many more are the applications of rarefaction waves.

In quasi-linear hyperbolic systems of balance laws, it is generally accepted that the solution to the RP plays a significant role [20, 22]. A generalized Chaplygin gas (GCG) dynamics model combined with a frictional term was analyzed by [123, 124]

$$\begin{aligned} \rho_t + \rho u_x + u\rho_x &= 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x &= \alpha\rho, \end{aligned} \tag{4.1.1}$$

with the equation of state as

$$p(\rho) = -\frac{A}{\rho^a}. \tag{4.1.2}$$

The GCG generally consider a barotropic perfect fluid. Where p and ρ denote the pressure and energy density, respectively. Also, α, a, and $A > 0$, and the range of a is in between 0 to 1. The system (4.1.1) with the equation of state (4.1.2) is called the GCG dynamics equation. This gas dynamics model has been advertised as a possible quintessence prototype for unifying dark matter and dark energy. When $a = 1$ and $A = 1$, Chaplygin [8] introduced it as isentropic Chaplygin gas in a particular case; Mathematically, it is employed as an approximation for calculating the lifting force on an aeroplane wing in aerodynamics [104, 125, 126]. In solving $1D$ RP, Y. Brenier [127] found the Riemann solutions with concentration, and for the Relativistic Chaplygin gas dynamics model, Cheng and Yang [128] investigated the RP. For $\alpha = 0$, the system (4.1.1) reduced to Euler's equation for GCG, which was studied by [124]. Additionally, it served as a forerunner to the unified model [9, 129]. In which a combination of dark energies and dark matter is interconnected and treated as an unmixed, unique fluid. Further, Chaplygin cosmology provides an excellent method to interpret recent discoveries about the universe's expansion.

Our interest is particularly concentrated on finding solutions that enable us to examine GRP when a friction term is present in the governing model. The initial data in the GRP are discontinuous at $x = 0$. Existence and uniqueness theorems have received a lot of attention in the latter topic [130].

The content of this chapter is structured as follows: In section 4.3, in order to explain the current method in investigating GRP, we take into consideration the governing model to transform the homogeneous model via a transformation. We have solved the Cauchy problem by using the differential constraint method for GCG equations, including the frictional term. This method is described in Section 4.2. In sections 4.4 and 4.5, a generalized piecewise linear GRP is also taken into consideration for the Cauchy problem under governing equations. In order to handle GRP in closed form, we demonstrate how the corresponding constraint equation produces a generalized simple wave (Rarefaction wave) and shock wave precisely as solutions. We used rarefaction wave solutions to examine a nonlinear GRP in section 4.6. The results and analysis of the problem under consideration are included in section 4.7.

4.2 The Differential Constraint Method

This section describes the Differential Constraint Method (DCM) briefly, given in [131]. We look at the nonhomogeneous partial differential system that has the following representation:

$$\mathbf{U}_t + B(\mathbf{U})\mathbf{U}_x = D(\mathbf{U}), \tag{4.2.1}$$

where the dependent field variables and the friction term are denoted by the column vector $\mathbf{U} \in \mathbb{R}^n$ and $D(\mathbf{U})$, respectively, and $B(\mathbf{U})$ denotes the matrix coefficient. Hereafter, the subscript denotes a partial derivative corresponding to the given variable. System (4.2.1) is considered strictly hyperbolic, assuming that the coefficient matrix $B(\mathbf{U})$ has distinct and real eigenvalues. Without loss of generality, we have

$$L^{(E^{(\lambda)})} \cdot R^{(E^{(\mu)})} = \begin{cases} 1, & E^{(\lambda)} = E^{(\mu)}, \\ 0, & E^{(\lambda)} \neq E^{(\mu)}, \end{cases} \tag{4.2.2}$$

where $R^{(E^{(\mu)})}$ and $L^{(E^{(\lambda)})}$ denote the right and left eigenvectors of the matrix $B(\mathbf{U})$ corresponding to an eigenvalue $E^{(\mu)}$ and $E^{(\lambda)}$. Denote $L^{(E^{(\lambda)})}$ as $L^{(\lambda)}$ and $R^{(E^{(\mu)})}$ as $R^{(\mu)}$. We consider a group of differential constraints as suggested in [55], of having order one to solve system (4.2.1) is given by

$$L^{(\lambda)}(\mathbf{U}) \cdot \mathbf{U}_x = C^{(\lambda)}(\mathbf{U}), \quad 1 \leq \lambda \leq N - 1, \tag{4.2.3}$$

where $C^{(\lambda)}(\mathbf{U})$ is an arbitrary function that will be examined using the compatibility conditions listed in [55, 56]. From equation (4.2.3), we have

$$\begin{aligned}
&C_t^{(\lambda)} + E^{(\lambda)}C_x^{(\lambda)} + \nabla C^{(\lambda)}\left(D - \sum_{\mu=1}^{N-1} C^{(\mu)}\left(E^{(\mu)} - E^{(\lambda)}\right)R^{(\mu)}\right)\\
&+ \sum_{\mu=1}^{N-1}\sum_{r=1}^{N-1} C^{(\mu)}C^{(r)}(E^{(\mu)} - E^{(r)})L^{(\mu)}\nabla R^{(\mu)}R^{(r)}\\
&+ \sum_{r=1}^{N-1} C^{(r)}\left(L^{(r)}\left(\nabla R^{(r)}D - \nabla D R^{(r)}\right) + p^{(\lambda)}\nabla E^{(\lambda)}.R^{(r)}\right)\\
&= 0,
\end{aligned} \tag{4.2.4}$$

$$\begin{aligned}
&\left(E^{(\lambda)} - E^{(N)}\right)\nabla C^{(\lambda)}.R^{(N)} + \left(\sum_{r=1}^{N-1} C^{(r)}\left(E^{(r)} - E^{(N)}\right)L^{(\lambda)}\right)\\
&\times\left(\nabla R^{(r)}.R^{(N)} - \nabla R^{(N)}.R^{(r)}\right)\\
&+ L^{(\lambda)}\left(\nabla R^{(N)}.D - \nabla D.R^{(N)}\right) + C^{(\lambda)}\nabla E^{(\lambda)}.R^{(N)} = 0,
\end{aligned} \tag{4.2.5}$$

where $1 \leq \lambda \leq N-1$ and $\nabla = \partial/\partial\mathbf{U}$.

Substituting the value (4.2.4) and (4.2.5), and using Equation (4.2.3), equation (4.2.1) can be written as

$$\mathbf{U}_t + E^{(N)}\mathbf{U}_x = D + \sum_{\lambda=1}^{N-1} C^{(\lambda)}\left(E^{(N)} - E^{(\lambda)}\right)R^{(\lambda)}. \tag{4.2.6}$$

The equations (4.2.6) may be integrated along the characteriztic curves provided by $\frac{dx}{dt} = E^{(N)}$ with the required initial condition by determining $C^{(\lambda)}$, which satisfies equations (4.2.4), and (4.2.5).

4.3 Governing equation and solution procedures

Here, we analyzed the Cauchy problem for conservation laws of the system (4.1.1)-(4.1.2). By introducing a new velocity variable $\nu = u - \alpha t$, the system (4.1.1) will be transformed in a conservative homogeneous form as

$$\begin{aligned} \rho_t + (\rho(\nu + \alpha t))_x &= 0, \\ (\rho\nu)_t + (\rho\nu(\nu + \alpha t) - \frac{A}{\rho^a})_x &= 0. \end{aligned} \tag{4.3.1}$$

From the system (4.3.1), it is simple to find the solution of the systems (4.1.1)-(4.1.2). So, firstly, we solve the system (4.3.1). Considering shallow water equation with friction terms, authors in [132] investigated the transformation of state variables to analyze the shock and rarefaction waves solutions of the RP .

The system of equations (4.3.1) can be rewritten in the form of (4.2.1) as

$$\mathbf{U} = \begin{pmatrix} \rho \\ \nu \end{pmatrix}, B(\mathbf{U}) = \begin{pmatrix} \nu + \alpha t & \rho \\ \frac{aA}{\rho^{(a+2)}} & \nu + \alpha t \end{pmatrix}, \text{and} \quad D(\mathbf{U}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.3.2}$$

The eigenvalues for the coefficient matrix $B(\mathbf{U})$ are determined as

$$\begin{aligned} &E^{(1)} = \nu + \alpha t - \sqrt{\frac{aA}{\rho^{a+1}}}, \quad E^{(2)} = \nu + \alpha t + \sqrt{\frac{aA}{\rho^{a+1}}}, \\ &i.e., \quad E^{(i)} = \nu + \alpha t + E_{(0)}\sqrt{\frac{aA}{\rho^{a+1}}}, \end{aligned} \tag{4.3.3}$$

where $i = 1$ to 2 and $E_{(0)} = \mp 1$.

Accordingly, the left and right eigenvectors may be given as

$$\begin{aligned} &L^{(E^{(1)})} = \left[\frac{1}{2}\sqrt{\frac{aA}{\rho^{a+1}}}, \frac{-\rho}{2}\right], \qquad L^{(E^{(2)})} = \left[\frac{1}{2}\sqrt{\frac{aA}{\rho^{a+1}}}, \frac{\rho}{2}\right] \\ &\text{and} \\ &R^{(E^{(1)})} = \begin{bmatrix} \rho \\ -\sqrt{\frac{aA}{\rho^{a+1}}} \end{bmatrix}, \qquad R^{(E^{(2)})} = \begin{bmatrix} \rho \\ \sqrt{\frac{aA}{\rho^{a+1}}} \end{bmatrix}. \end{aligned} \tag{4.3.4}$$

We investigated a non-trivial solution of the system (4.1.1)-(4.1.2) for the given initial conditions

$$\rho(x,0) = \rho_0(x), \qquad \nu(x,0) = \nu_0(x). \tag{4.3.5}$$

Let the differential constraint

$$\rho\nu_x + E_{(0)}\sqrt{\frac{aA}{\rho^{a+1}}}\rho_x = q(x,t,\rho,\nu). \tag{4.3.6}$$

Consequently, the compatibility conditions determined in (4.2.4) and (4.2.5) are as follows:

$$C_t + E^{(1)}C_x + C^2\nabla E^{(1)}R^{(E^{(1)})} = 0, \tag{4.3.7}$$

$$\begin{aligned} &(E^{(1)} - E^{(2)})\nabla C R^{(E^{(2)})} + C\nabla E^{(1)}R^{(E^{(2)})} + C(E^{(1)} - E^{(2)})L^{(E^{(1)})} \\ &\times \left(\nabla R^{(E^{(1)})}R^{(E^{(2)})} - \nabla R^{(E^{(2)})}R^{(E^{(1)})}\right) = 0. \end{aligned} \tag{4.3.8}$$

In view of (4.3.3) and (4.3.4), we rewrite the compatibility condition (4.3.7) and (4.3.8) as

$$C_t + E^{(i)} C_x + C^2 \left(\frac{a-1}{2}\right) \sqrt{\frac{aA}{\rho^{a+1}}} = 0, \tag{4.3.9}$$

$$\rho C_\rho - E_{(0)} \sqrt{\frac{aA}{\rho^{a+1}}} C_\nu = C\left(\frac{a+3}{4} + \rho \sqrt{\frac{aA}{\rho^{a+1}}}\right). \tag{4.3.10}$$

Which contains a solution of the form

$$C = c_0 \rho^{\frac{a+3}{4}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho^{a-1}}}}, \tag{4.3.11}$$

where c_0 is an arbitrary constant. Consequently, in consideration of Equation (4.3.6) and (4.3.11), the system of equations (4.3.1) can be stated as follows:

$$\rho_t + \left(\nu + \alpha t - E_{(0)} \sqrt{\frac{aA}{\rho^{a+1}}}\right) \rho_x + C = 0, \tag{4.3.12}$$

$$\nu_t + \left(\nu + \alpha t - E_{(0)} \sqrt{\frac{aA}{\rho^{a+1}}}\right) \nu_x + E_{(0)} \frac{C}{\rho} \sqrt{\frac{aA}{\rho^{a+1}}} = 0. \tag{4.3.13}$$

Given the initial conditions (4.3.5), this can be solved as

$$\rho(x,t) = \frac{\rho_0(\zeta)}{\left(1 + \frac{c_0}{4}(a-1)\rho_0^{\frac{a-1}{4}} t\right)^{4/(a-1)}}, \tag{4.3.14}$$

$$\nu(x,t) = \nu_0(\zeta) - E_{(0)} c_0 \sqrt{\frac{aA}{\rho_0^{(a+3)/2}}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_0^{a-1}}}} t, \tag{4.3.15}$$

$$x = \zeta + \left(\nu_0(\zeta) - E_{(0)} \sqrt{\frac{aA}{\rho_0^{a+1}}}\right) t - E_{(0)} c_0 \sqrt{\frac{aA}{\rho_0^{(a+3)/2}}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_0^{a-1}}}} t^2 + \alpha t^2. \tag{4.3.16}$$

In this case, $\zeta(x,t)$ refers to the $x-axis$ point located on the characteriztic defined by (4.3.16) and passes through (x,t) at a speed of $\nu+\alpha t+E_{(0)}\sqrt{\frac{aA}{\rho^{a+1}}}$, provided by

$$\rho_0\nu_0'(\zeta)+E_{(0)}\sqrt{\frac{aA}{\rho_0^{a+1}}}\rho_0'(\zeta)=c_0\rho^{\frac{a+3}{4}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho^{a-1}}}}. \tag{4.3.17}$$

For a given x and t, the equations (4.3.14) and (4.3.15) declare unique values for ρ and ν, if there is a unique ζ satisfying (4.3.16); therefore, for each x in $(-\infty,\infty)$ and every $t>0$, a unique solution exists provided

$$1+\left(\rho_0\nu_0'(\zeta)+E_{(0)}\frac{(a+1)}{2\rho_0}\sqrt{\frac{aA}{\rho_0^{a+1}}}\rho_0'(\zeta)\right)t\neq 0. \tag{4.3.18}$$

Thus, the solution given in system (4.3.14)-(4.3.18) of equations (4.1.1),(4.1.2) with initial condition (4.3.5) are summarized as

$$\begin{aligned}
&\rho(x,t)=\frac{\rho_0(\zeta)}{\left(1+\frac{c_0}{4}(a-1)\rho_0^{\frac{a-1}{4}}t\right)^{4/(a-1)}},\\
&u(x,t)=u_0(\zeta)-E_{(0)}c_0\sqrt{\frac{aA}{\rho_0^{(a+3)/2}}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_0^{a-1}}}}t+\alpha t,\\
&x=\zeta+\left(u_0(\zeta)-E_{(0)}\sqrt{\frac{aA}{\rho_0^{a+1}}}\right)t\\
&\qquad-E_{(0)}c_0\sqrt{\frac{aA}{\rho_0^{(a+3)/2}}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_0^{a-1}}}}t^2+\alpha t^2,\\
&\rho_0u_0'(\zeta)+E_{(0)}\sqrt{\frac{aA}{\rho_0^{a+1}}}\rho_0'(\zeta)=c_0\rho^{\frac{a+3}{4}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho^{a-1}}}},\\
&1+\left(\rho_0u_0'(\zeta)+E_{(0)}\frac{(a+1)}{2\rho_0}\sqrt{\frac{aA}{\rho_0^{a+1}}}\rho_0'(\zeta)\right)t\neq 0,
\end{aligned} \tag{4.3.19}$$

where $\quad E_{(0)}=\mp 1.$

Since for $a\neq 1$, $\nabla E^{(i)}.R^{(E^{(i)})}=\beta\frac{(1-a)}{2}\sqrt{\frac{aA}{\rho^{a+1}}}\neq 0,(i=1,2)$, $\beta=\mp 1$, and $\nabla=$

$(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial u})$. Which shows that the characteriztics fields $E^{(i)}$ are genuinely nonlinear. Therefore in classical sense, the associated waves are either rarefaction or shock waves, symbolized by $\mathcal{R}$ and $\mathcal{S}$.

4.4 Rarefaction Wave

We assume the governing system (4.1.1),(4.1.2) with the given discontinuous initial data as

$$(\rho,\nu)(x,0) = \begin{cases} (\rho_+,\nu_+), & x > 0, \\ (\rho_-,\nu_-), & x \leq 0, \end{cases} \tag{4.4.1}$$

where $\rho_+, \nu_+, \rho_-,$ and ν_- are arbitrary functions such that $\nu_-(0) \neq \nu_+(0)$ and $\rho_-(0) \neq \rho_+(0)$.

Since we know the system (4.3.1) and the Riemann initial condition (4.4.1) are invariant under uniform stretching of coordinates: $(x,t) \to (bx, bt)$ (b-constant), we look for the self-similar solution

$$(\rho,\nu)(x,t) = (\rho,\nu)(\sigma), \;\; \sigma = \frac{x}{t}.$$

Then the Riemann problem (4.3.1) with the discontinuous initial data (4.4.1) is reduced to a boundary value problem of ordinary differential equations;

$$\begin{aligned} &-\sigma\rho_\sigma + (\rho(\nu+\alpha t))_\sigma = 0, \\ &-\sigma(\rho(\nu+\alpha t))_\sigma + \left(\rho\nu(\nu+\alpha t) - \frac{A}{\rho^a}\right)_\sigma = 0, \end{aligned} \tag{4.4.2}$$

with the boundary condition $(\rho,\nu)(\pm\infty) = (\rho_\pm, \nu_\pm)$.

System (4.4.2) can be expressed for any smooth solution as:

$$\begin{pmatrix} (\nu+\alpha t)-\sigma & \rho \\ -\sigma\nu+\nu(\nu+\alpha t)+\frac{aA}{\rho^{(a+1)}} & -\sigma\rho+\rho(2\nu+\alpha t) \end{pmatrix}\begin{pmatrix} \rho_\sigma \\ \nu_\sigma \end{pmatrix}=0. \tag{4.4.3}$$

It gives either the constant solution ($\rho > 0$) or it provides a singular solution, which is called a Rarefaction wave. For a given state (ρ_-, ν_-), the possible state (ρ, ν) that can be connected to the given state on the right by a centered rarefaction wave which is denoted by $\mathcal{R}_1$ and defined as

$$\mathcal{R}_1(\rho_-, \nu_-) : \begin{cases} \sigma = E^{(1)} = (\nu+\alpha t) - \sqrt{\frac{aA}{\rho^{a+1}}}, \\ \nu = \nu_- - \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{aA}{s^{a+1}}}}{s} ds, \\ \rho < \rho_-. \end{cases} \tag{4.4.4}$$

Similarly, for a given state (ρ_-, ν_-), the possible state (ρ, ν) that can be connected to the given state on the right by a centered rarefraction wave which is denoted by $\mathcal{R}_2$ and defined as

$$\mathcal{R}_2(\rho_-, \nu_-) : \begin{cases} \sigma = E^{(2)} = (\nu+\alpha t) + \sqrt{\frac{aA}{\rho^{a+1}}}, \\ \nu = \nu_- + \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{aA}{s^{a+1}}}}{s} ds, \\ \rho > \rho_-, \end{cases} \tag{4.4.5}$$

or general solutions (constant states)

$$(\rho, \nu)(\sigma) = \text{constant} \ \ (\rho > 0).$$

Differentiating $(4.4.4)_2$ with respect to ρ, we get $\nu_\rho = -\frac{\sqrt{\frac{aA}{\rho^{a+1}}}}{\rho} < 0$, and after that,

$$\nu_{\rho\rho} = \frac{a+3}{2\rho^2}\sqrt{\frac{aA}{\rho^{a+1}}} > 0,$$

which is clear that $\mathcal{R}_1(\rho_-, \nu_-)$ is monotonically decreasing and convex in (ρ, ν)-plane $(\rho > 0)$. Similarly, the equation $(4.4.5)_2$ yields $\nu_\rho > 0$ and $\nu_{\rho\rho} < 0$, indicating that $\mathcal{R}_2(\rho_-, \nu_-)$ is monotonically increasing and concave in (ρ, ν)- plane $(\rho > 0)$.

4.5 Shock Wave

In a genuinely non-linear field $E^{(i)}$, the two constant states W_L and W_R of a shock wave are connected by a single jump discontinuity, and the following circumstances apply.

The Rankine-Hugoniot conditions for a bounded discontinuity at $\sigma = V$

$$\begin{aligned} &-V(t)[\rho] + [\rho\nu(\nu + \alpha t)] = 0, \\ &-V(t)[\rho\nu] + [\rho\nu(\nu + \alpha t) - \frac{A}{\rho^a}] = 0, \end{aligned} \tag{4.5.1}$$

where $V(t)$ is the velocity of discontinuity and $[\rho] = \rho - \rho_-$. Solving the system (4.5.1), we have two shock S_1 and S_2

$$S_1(\rho_-, \nu_-) : \begin{cases} \nu = \nu_- - \rho\sqrt{\frac{A}{\rho\rho_-(\rho-\rho_-)}\left(\frac{1}{\rho_-^a} - \frac{1}{\rho^a}\right)}, \\ \nu = \nu_- - \sqrt{\frac{A}{\rho\rho_-(\rho-\rho_-)}\left(\frac{1}{\rho_-^a} - \frac{1}{\rho^a}\right)}(\rho - \rho_-), \\ \rho > \rho_-, \end{cases} \tag{4.5.2}$$

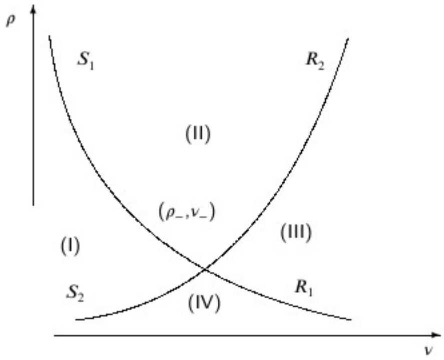

FIGURE 4.1: The (ρ, ν) phase plane for the model (4.1.1).

and

$$S_2(\rho_-, \nu_-): \begin{cases} \nu = \nu_- + \rho\sqrt{\frac{A}{\rho\rho_-(\rho-\rho_-)}\left(\frac{1}{\rho_-^a} - \frac{1}{\rho^a}\right)}, \\ \nu = \nu_- + \sqrt{\frac{A}{\rho\rho_-(\rho-\rho_-)}\left(\frac{1}{\rho_-^a} - \frac{1}{\rho^a}\right)}(\rho - \rho_-), \\ \rho < \rho_-. \end{cases} \tag{4.5.3}$$

Differentiating the equation $(4.5.2)_2$ with respect to ρ as $(\rho > \rho_-)$, we obtain that $\nu_\rho < 0$, which implies that 1-shock curve (S_1) is monotonically decreasing in (ρ, ν)-plane $(\rho > 0)$. Similarly, the equation $(4.5.3)_2$ yields $\nu_\rho > 0$ for $\rho < \rho_-$, indicating that the 2-shock wave curve (S_2) in monotonically increasing in (ρ, ν)-plane $(\rho > 0)$.

It is clear from the data evaluated in [133] for the homogeneous p-system connected to (4.4.4), (4.4.5) and (4.5.2), (4.5.3) and utilizing the conclusion from there that the situations depicted in Figure (4.1) are generated by the rarefaction curves $\mathcal{R}_1$, $\mathcal{R}_2$ and the shock curves $\mathcal{S}_1$, $\mathcal{S}_2$ in the (ρ, ν) plane.

4.6 Generalized Riemann problem (GRP)

We consider the initial discontinuous data of the generalized Riemann problem as follows:

$$(\rho, u)(x,0) = \begin{cases} (\rho_r(x), u_r(x)), & x > 0, \\ (\rho_l(x), u_l(x)), & x \leq 0, \end{cases} \tag{4.6.1}$$

where $\rho_r, u_r, \rho_l,$ and u_l are arbitrary functions that are smooth with the property that $u_l(0) \neq u_r(0)$ and $\rho_l(0) \neq \rho_r(0)$. The initial condition (4.6.1) when placed into (4.3.6) and by integrating further, we get

$$\begin{cases} u_l(x) = u_L + E_0\frac{2}{a+1}\left(\sqrt{\frac{aA}{\rho_l^{a+1}(x)}} - \sqrt{\frac{aA}{\rho_l^{a+1}(0)}}\right) + \ c_0 \int_0^x \rho_l^{\frac{a-1}{4}}(z).e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_l^{a-1}(z)}}}dz, \\ u_r(x) = u_R + E_0\frac{2}{a+1}\left(\sqrt{\frac{aA}{\rho_r^{a+1}(x)}} - \sqrt{\frac{aA}{\rho_r^{a+1}(0)}}\right) + c_0 \int_0^x \rho_r^{\frac{a-1}{4}}(z).e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_r^{a-1}(z)}}}dz, \end{cases} \tag{4.6.2}$$

with $u_L \neq u_R$. In (4.6.2), $u_L = \lim_{x\to 0^-} u_l$ and $u_R = \lim_{x\to 0^+} u_r$. Here, the characteriztic speed $E^{(2)} = u + \sqrt{\frac{aA}{\rho^{a+1}}}$ is referred to as such in the following. Now, using the exact solution (4.3.19) to fit the initial data (4.6.1) and (4.6.2) for $x < 0$ yields the solution

$$\begin{cases} \rho(x,t) = \frac{\rho_l(\zeta)}{\left(1+\frac{c_0}{4}(a-1)\rho_l^{\frac{a-1}{4}}t\right)^{4/(a-1)}}, \\ u(x,t) = u_l(\zeta) - E_{(0)}c_0\sqrt{\frac{aA}{\rho_l^{(a+3)/2}}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_l^{a-1}}}}t + \alpha t, \\ x = \zeta + \left(u_l(\zeta) - E_{(0)}\sqrt{\frac{aA}{\rho_l^{a+1}}}\right)t - E_{(0)}c_0\sqrt{\frac{aA}{\rho_l^{(a+3)/2}}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_l^{a-1}}}}t^2 + \alpha t^2, \end{cases} \tag{4.6.3}$$

with $\zeta < 0$.

While for $x > 0$, we obtain the solution

$$\begin{cases} \rho(x,t) = \frac{\rho_r(\zeta)}{\left(1+\frac{c_0}{4}(a-1)\rho_r^{\frac{a-1}{4}}t\right)^{4/(a-1)}}, \\ u(x,t) = u_r(\zeta) - E_{(0)}c_0\sqrt{\frac{aA}{\rho_r^{(a+3)/2}}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_r^{a-1}}}}t + \alpha t, \\ x = \zeta + \left(u_r(\zeta) - E_{(0)}\sqrt{\frac{aA}{\rho_r^{a+1}}}\right)t - E_{(0)}c_0\sqrt{\frac{aA}{\rho_r^{(a+3)/2}}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_r^{a-1}}}}t^2 + \alpha t^2, \end{cases} \tag{4.6.4}$$

with $\zeta > 0$.

Therefore, we integrate the system of PDEs (4.3.12) and (4.3.13) to connect the left state (4.6.3) to the right state (4.6.4) smoothly, subject to the initial conditions listed below,

$$\begin{cases} \rho = V(\alpha'); \;\; u = U(\alpha'); \;\; x(0) = 0 \;\text{ with }\; \alpha' \in [0,1], \\ V(0) = \rho_L, \;\; V(1) = \rho_R; \;\; U(0) = u_L, \;\; U(1) = u_R. \end{cases} \tag{4.6.5}$$

We obtain the resulting solution

$$\begin{cases} \rho(x,t) = \frac{V(\alpha')}{\left(1+\frac{c_0}{4}(a-1)V^{\frac{a-1}{4}}(\alpha')t\right)^{4/(a-1)}}, \\ u(x,t) = U(\alpha') - E_{(0)}c_0\sqrt{\frac{aA}{V^{(a+3)/2}(\alpha')}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{V\rho_r^{a-1}(\alpha')}}}t + \alpha t, \\ x = \left(U(\alpha') - E_{(0)}\sqrt{\frac{aA}{V^{a+1}(\alpha')}}\right)t - E_{(0)}c_0\sqrt{\frac{aA}{V^{(a+3)/2}(\alpha')}} \cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{V^{a-1}(\alpha')}}}t^2 + \alpha t^2, \end{cases} \tag{4.6.6}$$

along with

$$\frac{dU}{d\alpha'} + E_0\sqrt{\frac{aA}{V^{a+3}(\alpha')}}\frac{dV}{d\alpha'} = 0. \tag{4.6.7}$$

Moreover, the rarefaction curve is produced by the integration of (4.6.7) under the condition (4.6.5),

$$U = u_L + E_0 \frac{2}{a+1}\left(\sqrt{\frac{aA}{V^{a+1}(1)}} - \sqrt{\frac{aA}{\rho_l^{a+1}(0)}}\right). \tag{4.6.8}$$

Thus, we obtain the relation (4.6.8) satisfying the condition $(4.6.5)_2$ given by

$$u_R = u_L + E_0 \frac{2}{a+1}\left(\sqrt{\frac{aA}{\rho_r^{a+1}(0)}} - \sqrt{\frac{aA}{\rho_l^{a+1}(0)}}\right). \tag{4.6.9}$$

Rewriting the equation (4.6.9) we have

$$u_R - E_0 \frac{2}{a+1}\sqrt{\frac{aA}{\rho_R^{a+1}}} = u_L - E_0 \frac{2}{a+1}\sqrt{\frac{aA}{\rho_L^{a+1}}}. \tag{4.6.10}$$

Now, we need $\frac{dE^{(1)}}{d\alpha'} > 0$ to demonstrate the existence of the rarefaction wave solution (4.6.6). There is an intermediate state (4.6.6) that exists in the region $E^{(2)}(\rho_l(0)) \leq E^{(2)}(V(\alpha')) \leq E^{(2)}(\rho_r(0))$ where the left state (4.6.3) is connected to the right state (4.6.4) substituted by a smooth transitions.

The solution, which is represented by equation (4.6.6), is the required solution of the generalized Riemann problem for the generalized Chaplygin gas equation with friction term (4.1.1), which is characterized by the following state:

Left state:

$$\begin{cases} \rho(x,t) = \frac{\rho_l(\zeta)}{\left(1+\frac{c_0}{4}(a-1)\rho_l^{\frac{a-1}{4}}t\right)^{4/(a-1)}}, \\ u(x,t) = u_l(\zeta) - E_{(0)}c_0\sqrt{\frac{aA}{\rho_l^{(a+3)/2}}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_l^{a-1}}}}t + \alpha t, \\ x = \zeta + \left(u_l(\zeta) - E_{(0)}\sqrt{\frac{aA}{\rho_l^{a+1}}}\right)t - E_{(0)}c_0\sqrt{\frac{aA}{\rho_l^{(a+3)/2}}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_l^{a-1}}}}t^2 + \alpha t^2. \end{cases} \tag{4.6.11}$$

Right state:

$$\begin{cases} \rho(x,t) = \frac{\rho_r(\zeta)}{\left(1+\frac{c_0}{4}(a-1)\rho_r^{\frac{a-1}{4}}t\right)^{4/(a-1)}}, \\ u(x,t) = u_r(\zeta) - E_{(0)}c_0\sqrt{\frac{aA}{\rho_r^{(a+3)/2}}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_r^{a-1}}}}t + \alpha t, \\ x = \zeta + \left(u_r(\zeta) - E_{(0)}\sqrt{\frac{aA}{\rho_r^{a+1}}}\right)t - E_{(0)}c_0\sqrt{\frac{aA}{\rho_r^{(a+3)/2}}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{\rho_r^{a-1}}}}t^2 + \alpha t^2. \end{cases} \tag{4.6.12}$$

Intermediate state:

$$\begin{cases} \rho(x,t) = \frac{V(\alpha')}{\left(1+\frac{c_0}{4}(a-1)V^{\frac{a-1}{4}}(\alpha')t\right)^{4/(a-1)}}, \\ u(x,t) = U(\alpha') - E_{(0)}c_0\sqrt{\frac{aA}{V^{(a+3)/2}(\alpha')}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{V\rho_r^{a-1}(\alpha')}}}t + \alpha t, \\ x = \left(U(\alpha') - E_{(0)}\sqrt{\frac{aA}{V^{a+1}(\alpha')}}\right)t - E_{(0)}c_0\sqrt{\frac{aA}{V^{(a+3)/2}(\alpha')}}\cdot e^{-\frac{2}{a-1}\sqrt{\frac{aA}{V^{a-1}(\alpha')}}}t^2 + \alpha t^2. \end{cases} \tag{4.6.13}$$

4.7 Conclusion

By using the differential constraint method, closed-form solutions of the $1D$ generalized Riemann problem for a generalized Chaplygin gas dynamics system with a Coulomb-type friction term are constructed. This method obtained compatibility conditions and differential constraint equations for the governing model. The exact solution obtained is a generalisation of the classic simple wave solution, which only applies in source-free situations. We investigated the non-homogeneous quasi-linear hyperbolic system (4.1.1) with the set of first-order differential constraints (4.2.3) and obtained a non-trivial solution for the generalized Chaplygin gas dynamics equation. The solutions to the generalized Riemann problem are determined, and their complete characterization is demonstrated for piecewise-linear initial data. Further, we analyzed a nonlinear generalized Riemann problem and showed the closed-form solution for the generalized Chaplygin gas dynamics through a rarefaction wave. Some of the remarkable outcomes of the current mathematical study can be highlighted as follows:

- Employed the Differential Constraint Method to derive compatibility conditions and differential constraint equations for a 1D generalized Chaplygin gas system with a Coulomb-type friction term. Produced closed-form solutions that extend beyond source-free (classic simple wave) cases.
- characterized generalized Riemann problem solutions in full for piecewise-linear initial data, showing their explicit form. analyzed a nonlinear GRP scenario and provided a closed-form rarefaction wave solution for the generalized Chaplygin gas dynamics.

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