Chapter 5

Set-Valued Fractal Functions and Dimensions

In this section, we explore the estimation of set-valued maps by utilizing set-valued fractal maps. The literature provides various approaches for approximating single-valued maps using fractal functions. One notable method is the α-fractal function, which was introduced by Navascués [68]. In this chapter, we expand upon the theory of α-fractal function to encompass set-valued maps.

5.1 Introduction

The topic of approximating set-valued maps is widely discussed in the literature. Various theories have been proposed for the classical approximation of set-valued maps. For instance, in [50], some generalizations of the notion of univariate data interpolation are presented, inducing the concept of set-valued interpolation in a general metric space. Initially, the approximation theory was mainly focused on set-valued having convex images (also known as convex set-valued maps). For instance, Vitale explored the approximation of convex set-valued maps using set-valued Bernstein polynomials in [92]. More research on the approximation of convex set-valued maps can be found in [6, 21, 34], where the Minkowski sum of the two sets is considered. Arstein, in [3], studied the approximation of set-valued maps with compact images, known as compact set-valued maps. Instead of using the Minkowski sum of sets, he used the set of a sum of special pairs of elements, later referred to as "metric pairs.” Further research on the approximation of compact set-valued maps can be found in [16, 35].

Numerous theories related to the concept of fractal interpolation of single-valued maps are given in the literature, while no theory related to the fractal approximation of set-valued maps is explored. In this chapter, we have laid out the theory on the fractal version of a set-valued map. We have expanded the traditional method of approximating set-valued maps to fractal approximation. We have introduced the concept of an α-fractal function for set-valued maps. However, unlike single-valued maps, we have observed that the set-valued α-fractal function is not typically interpolatory. Additionally, we have focused on estimating the fractal dimension of the graph for certain specific types of set-valued maps.

5.1.1 Delineation

This chapter is structured as follows. The following section focuses on the development of fractal functions and the examination of their properties. In Section 5.3, we have investigated the fractal approximations and constrained approximations of the set-valued map. Additionally, in Section 5.4, we have introduced a new definition for the graph of set-valued maps and presented some dimensional results for this novel graph. We have also discussed the rationale behind defining this new graph. Furthermore, we have demonstrated that there exists an iterated function system whose attractor is this new graph of the set-valued α-fractal function. Finally, we have concluded this chapter in Section 5.5.

5.2 Fractal Functions in $C(I, \mathcal{K}(\mathbb{R}))$

We know that space $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ when endowed with metric $d_{\mathcal{C}}$ is a complete metric space, where

$$d_{\mathcal{C}}(F,G) = \|F - G\|_\infty = \sup_{x \in I} H_d(F(x), G(x)).$$

Note 5.1. [45, Proposition 1.17] Recall some properties of the Hausdorff metric as follows.

1. Let X be a normed space. Then, for $B, C, D, E \in \mathcal{K}(X)$, we have

$$H_d(B + C, D + E) \leq H_d(B, D) + H_d(C, E),$$

   where $A+B := \{a+b : a \in A \in \mathcal{K}(X), b \in B \in \mathcal{K}(X)\}$ is known as Minkowski sum of A and B.

2. For $\lambda \in \mathbb{R}$, $H_d(\lambda B, \lambda D) = |\lambda| H_d(B, D)$, where $\lambda A = \{\lambda a : a \in A \in \mathcal{K}(X)\}$.

Theorem 5.2. *Assume $F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$. Let $\Delta := \{(x_0, \ldots, x_N) : 0 = x_0 < \cdots < x_N = 1\}$ be a given set of data points such that it forms a partition of I, and for $i \in \Sigma_N$, let $I_i = [x_{i-1}, x_i]$. Let $U_i : I \to I_i$ be contractive homeomorphism such that $U_i(x_0) = x_{i-1}$ and $U_i(x_N) = x_i$ or $U_i(x_0) = x_i$ and $U_i(x_N) = x_{i-1}$. Further, assume that the base function $S \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ satisfies $S(x_0) - F(x_0) = S(x_N) - F(x_N)$, where set difference is defined as $A - B = \{a - b : a \in A \in \mathcal{K}(\mathbb{R})$ and $b \in B \in \mathcal{K}(\mathbb{R})\}$ and scaling factor $\alpha \in \mathbb{R}$. If $|\alpha| < 1$, then there exists a unique function $F^\alpha_{\Delta,S} \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ satisfying the following self-referential equation*

$$F^\alpha_{\Delta,S}(x) = F(x) + \alpha[F^\alpha_{\Delta,S}(U_i^{-1}(x)) - S(U_i^{-1}(x))] \quad \text{for every} \quad x \in I_i. \tag{5.1}$$

Proof. Let us define the set $\mathcal{C}_F(I,\mathcal{K}(\mathbb{R})) = \{G \in \mathcal{C}(I,\mathcal{K}(\mathbb{R})) : G(x_0) - S(x_0) = G(x_N) - S(x_N)\}$. It is elementary to observe that $\mathcal{C}_F(I,\mathcal{K}(\mathbb{R}))$ is a closed subset of $\mathcal{C}(I,\mathcal{K}(\mathbb{R}))$, hence $(\mathcal{C}_F(I,\mathcal{K}(\mathbb{R})), d_{\mathcal{C}})$ is a complete metric space. Define *Read-Bajraktarević* (RB) operator $\Phi : \mathcal{C}_F(I,\mathcal{K}(\mathbb{R})) \to \mathcal{C}_F(I,\mathcal{K}(\mathbb{R}))$ by

$$(\Phi G)(x) = F(x) + \alpha[G(U_i^{-1}(x)) - S(U_i^{-1}(x))]$$

for every $x \in I_i$ and $i \in \Sigma_N$. Well-definedness of Φ can be observed using the assumptions we have made for F, S, and α. With the reference to Note 5.1, we get

$$\begin{aligned}
&H_d((\Phi G)(x), (\Phi H)(x)) \\
=&H_d\Big(F(x) + \alpha[G(U_i^{-1}(x)) - S(U_i^{-1}(x))], F(x) + \alpha[H(U_i^{-1}(x)) - S(U_i^{-1}(x))]\Big) \\
\le&H_d\Big(\alpha G(U_i^{-1}(x)), \alpha H(U_i^{-1}(x))\Big) = |\alpha| H_d\Big(G(U_i^{-1}(x)), H(U_i^{-1}(x))\Big) \\
\le&|\alpha| \sup_{x\in I} H_d(G(x), H(x)) = |\alpha| \|G - H\|_\infty.
\end{aligned}$$

Since $|\alpha|\|G-H\|_\infty$ is independent of x, hence we have $\|\Phi G - \Phi H\|_\infty \le |\alpha|\|G-H\|_\infty$. Since $|\alpha| < 1$, Φ is a contraction map. Hence, Φ has a fixed point in $\mathcal{C}(I,\mathcal{K}(\mathbb{R}))$. Let $F^\alpha_{\Delta,S}$ be that fixed point, then for every $x \in I_i$, where $i \in \Sigma_N$, it satisfies the self-referential following equation,

$$F^\alpha_{\Delta,S}(x) = F(x) + \alpha(F^\alpha_{\Delta,S}(U_i^{-1}(x)) - S(U_i^{-1}(x))). \tag{5.2}$$

□

Note 5.3. Throughout this chapter,

- we denote $F^\alpha_{\Delta,S}$ by F^α if there is no ambiguity
- we take Δ and S as the same as it is in Theorem 5.2, unless specified.

Remark 5.4. In the context of (5.1), we get

$$F^{\alpha}(x_i) = F(x_i) + \alpha F^{\alpha}(x_0) - \alpha S(x_0) = F(x_i) + \alpha F^{\alpha}(x_N) - \alpha S(x_N) \text{ for every } x_i \in \Delta,$$

where $i \in \Sigma_N$. Further, if F^{α} and S are single-valued at the endpoints such that $F^{\alpha}(x_0) - S(x_0) = F^{\alpha}(x_N) - S(x_N) = \{0\}$, then $F^{\alpha}(x_i) = F(x_i)$ for each $i \in \Sigma_N$, this implies that F^{α} is a set-valued fractal interpolation function.

Note 5.5. The above remark hints at the following: in case F and S are single-valued at the endpoints such that $F(x_0) - S(x_0) = F(x_N) - S(x_N) = \{0\}$, then the set

$$\mathcal{C}_F(I, \mathcal{K}(\mathbb{R})) = \Big\{ G \in \mathcal{C}(I, \mathcal{K}(\mathbb{R})) : G(x_0) - S(x_0) = G(x_N) - S(x_N) = \{0\} \Big\}$$

is a complete metric space, and the RB operator $\Phi : \mathcal{C}_F(I, \mathcal{K}(\mathbb{R})) \to \mathcal{C}_F(I, \mathcal{K}(\mathbb{R}))$ as defined in Theorem 5.2 is well-defined and a contraction mapping. Therefore, we have a unique fixed point F^{α} of Φ satisfying $F^{\alpha}(x_i) = F(x_i)$ for all $i \in \Sigma_N$, this shows that F^{α} is a set-valued fractal interpolation function.

Here we give some examples of base functions $S \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ satisfying $S(x_0) - F(x_0) = S(x_N) - F(x_N)$:

(i) $S(x) = F(t(x)) + (x - x_0)(F(x_0) - F(x_0) + (x_N - x)(F(x_N) - F(x_N))$, where $t : I \to I$ is a continuous function which satisfies $t(x_0) = x_0,\ t(x_N) = x_N$.

(ii) $S(x) = t(x)F(x) + (x - x_0)(F(x_0) - F(x_0) + (x_N - u)(F(x_N) - F(x_N))$, where $t : I \to \mathbb{R}$ is a continuous function which satisfies $t(x_0) = 1$ and $t(x_N) = 1$.

The Hölder space is defined as follows:

$$\mathcal{HC}^{\sigma}(I, \mathcal{K}_c(\mathbb{R})) := \{G : I \to \mathcal{K}_c(\mathbb{R}) :\ G \in \sigma\text{-}\mathcal{HC}\}.$$

Let us recall [62], if we endow the space $\mathcal{HC}^\sigma(I, \mathcal{K}_c(\mathbb{R}))$ with metric

$$H_\sigma^{(1)}(G,H) = \sup_{x\in I} H_d(G(x),H(x)) + \sup_{\substack{x,y\in I\\ x\neq y}} \frac{H_d\big(G(x)+H(y),H(x)+G(y)\big)}{|x-y|^\sigma}.$$

Then, by [62, Proposition 1], it forms a complete metric space.

Note 5.6. Throughout this chapter, unless specified, take $U_i : I \to I_i$ as affine maps, such that $U_i(x) = a_i x + b_i$ for all $i \in \Sigma_N$, where $a_i = \frac{x_i - x_{i-1}}{x_N - x_0}$ and $b_i = \frac{x_{i-1}x_N - x_0 x_i}{x_N - x_0}$.

Theorem 5.7. *Consider $F, S \in \mathcal{HC}^\sigma(I, \mathcal{K}_c(\mathbb{R}))$ such that $S(x_0) - F(x_0) = S(x_N) - F(x_N)$, and let $\alpha \in (-1, 1)$. Then, $F^\alpha \in \sigma$-$\mathcal{HC}$ provided $\frac{N|\alpha|}{a^\sigma} < 1$, where $a := \min\{a_i : i \in \Sigma_N\}$.*

Proof. Consider $\mathcal{HC}_F^\sigma(I, \mathcal{K}_c(\mathbb{R})) = \{G \in \mathcal{HC}^\sigma(I, \mathcal{K}_c(\mathbb{R})) : G(x_0) - S(x_0) = G(x_N) - S(x_N)\}$. It is easy to notice that $\mathcal{HC}_F^\sigma(I, \mathcal{K}_c(\mathbb{R}))$ is a closed subset of $\mathcal{HC}^\sigma(I, \mathcal{K}_c(\mathbb{R}))$, and hence complete with respect to the metric $H_\sigma^{(1)}$. Define a map
$\Phi : \mathcal{HC}_F^\sigma(I, \mathcal{K}_c(\mathbb{R})) \to \mathcal{HC}_F^\sigma(I, \mathcal{K}_c(\mathbb{R}))$ as

$$(\Phi G)(x) = F(x) + \alpha\ (G - S)(U_i^{-1}(x))$$

for each $x \in I_i$, where $i \in \Sigma_N$. Clearly, Φ is well-defined. Now for $G, H \in \mathcal{HC}_F^\sigma(I, \mathcal{K}_c(\mathbb{R}))$, we have

$$\begin{aligned}
&H_\sigma^{(1)}(\Phi(G),\Phi(H))\\
\leq &\sup_{x\in I} H_d(\Phi(G)(x),\Phi(H)(x))\\
&+ N \max_{i\in\Sigma_N} \sup_{\substack{x,y\in I_i\\ x\neq y}} \frac{H_d\big(\Phi(G)(x)+\Phi(H)(y),\Phi(H)(x)+\Phi(G)(y)\big)}{|x-y|^\sigma}\\
\leq &\ |\alpha| \sup_{x\in I} H_d(G(x),H(x))\\
&+ N \max_{i\in\Sigma_N} \sup_{\substack{x,y\in I_i\\ x\neq y}} \frac{H_d\Big(\alpha G(U_i^{-1}(x)) + \alpha H(U_i^{-1}(y)), \alpha H(U_i^{-1}(x)) + \alpha G(U_i^{-1}(y))\Big)}{|x-y|^\sigma}
\end{aligned}$$

$$
\begin{aligned}
&\leq |\alpha| \sup_{x\in I} H_d(G(x),H(x)) \\
&\quad + N|\alpha| \max_{i\in\Sigma_N} \sup_{\substack{x,y\in I_i\\ x\neq y}} \frac{H_d\Big(G(U_i^{-1}(x))+H(U_i^{-1}(y)),H(U_i^{-1}(x))+G(U_i^{-1}(y))\Big)}{|a_i|^\sigma |U_i^{-1}(x)-U_i^{-1}(y)|^\sigma} \\
&\leq |\alpha| \sup_{x\in I} H_d(G(x),H(x)) \\
&\quad + \frac{N|\alpha|}{a^\sigma} \max_{i\in\Sigma_N} \sup_{\substack{x,y\in I_i\\ x\neq y}} \frac{H_d\Big(G(U_i^{-1}(x))+H(U_i^{-1}(y)),H(U_i^{-1}(x))+G(U_i^{-1}(y))\Big)}{|U_i^{-1}(x)-U_i^{-1}(y)|^\sigma} \\
&\leq \frac{N|\alpha|}{a^\sigma}\left[\sup_{x\in I} H_d(G(x),H(x)) + \sup_{\substack{x,y\in I_i\\ x\neq y}} \frac{H_d\Big(G(x)+H(y),H(x)+G(y)\Big)}{|x-y|^\sigma}\right] \\
&\leq \frac{N|\alpha|}{a^\sigma} H_\sigma^{(1)}(G,H).
\end{aligned}
$$

Since $\frac{N|\alpha|}{a^\sigma} < 1$, which implies Φ is a contraction map on $\mathcal{HC}_F^\sigma(I,\mathcal{K}_c(\mathbb{R}))$. Now, the Banach contraction principle ensures that a unique fixed point of Φ exists. This completes the proof. □

Definition 5.8. Assume $F : I \to \mathcal{K}(\mathbb{R})$ be a set-valued map. For every partition $P := \{(t_0,\dots,t_m) : t_0 < \cdots < t_m\}$ of I, define

$$V(F,I) = \sup_P \sum_{j=1}^m H_d(F(t_j),F(t_{j-1})),$$

where the supremum runs over all partitions P of I.

We set $\|F\|_{\mathcal{BV}} := \|F\|_\infty + V(F,I)$, where $\|F\|_\infty := \sup_{x\in I} \|F(x)\| = \sup_{x\in I} H_d(F(x),\{0\})$. Then, F will be characterized as a bounded variation function if $\|F\|_{\mathcal{BV}} < \infty$. $\mathcal{BV}(I,\mathcal{K}(\mathbb{R}))$ will denote the collection of all bounded variation functions on I.

Remark 5.9. It is interesting to write the following small observation : define functions $F,T : [0,1] \to \mathcal{K}(\mathbb{R})$ as follows $F(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{when } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$ and $T(x) = [-1,1]$. Here $F(x) \subset T(x)$ for each $x \in [0,1]$, such that $T \in \mathcal{BV}(I,\mathcal{K}(\mathbb{R}))$

while $F \notin \mathcal{BV}(I, \mathcal{K}(\mathbb{R}))$. This example shows that for set-valued mappings satisfying $F \leq T$ does not imply $\|F\|_{\mathcal{BV}} \leq \|T\|_{\mathcal{BV}}$.

As a prelude to our next result, we note the following lemma.

Lemma 5.10. *Consider $\{F_n\}$ to be a sequence of set-valued continuous maps that uniformly converge to $F : I \to \mathcal{K}(\mathbb{R})$. Then, for a given partition $P = \{(t_0, \ldots, t_m) : t_0 < \cdots < t_m\}$ of I, we have*

$$\sum_{j=1}^{m} H_d(F_n(t_j), F_n(t_{j-1})) \to \sum_{j=1}^{m} H_d(F(t_j), F(t_{j-1})).$$

Moreover,

$$\sup_P \sum_{j=1}^{m} H_d(F(t_j), F(t_{j-1})) \leq \liminf_{n\to\infty} \sup_P \sum_{j=1}^{m} H_d(F_n(t_j), F_n(t_{j-1})).$$

Proof. Let $P = \{(t_0, \ldots, t_m) : t_0 < \cdots < t_m\}$ be a partition of I. The uniform convergence of $\{F_n\}$ implies

$$\lim_{n\to\infty} \sum_{j=1}^{m} H_d(F_n(t_j), F_n(t_{j-1})) = \sum_{j=1}^{m} H_d(F(t_j), F(t_{j-1})).$$

Now for a given partition $P = \{(t_0, \ldots, t_m) : t_0 < \cdots < t_m\}$ of I, we get

$$\begin{aligned} \sum_{j=1}^{m} H_d(F(t_j), F(t_{j-1})) &= \sum_{j=1}^{m} H_d(\lim_{n\to\infty} F_n(t_j), \lim_{n\to\infty} F_n(t_{j-1})) \\ &= \lim_{n\to\infty} \sum_{j=1}^{m} H_d(F_n(t_j), F_n(t_{j-1})) \\ &\leq \liminf_{n\to\infty} \sup_P \sum_{j=1}^{m} H_d(F_n(t_j), F_n(t_{j-1})), \end{aligned}$$

completing the proof. □

Theorem 5.11. *The space* $\Big(\mathcal{BV}(I,\mathcal{K}_c(\mathbb{R})), H_{\mathcal{BV}}\Big)$ *is a complete metric space, where*

$$H_{\mathcal{BV}}(G,H) := \|G-H\|_\infty + \sup_P \sum_{j=1}^{m} H_d\Big(G(t_j)+H(t_{j-1}), H(t_j)+G(t_{j-1})\Big),$$

where sup *runs over all partitions* P *of* I.

Proof. Assume that $\{F_n\}$ is a Cauchy sequence in $\mathcal{BV}(I,\mathcal{K}_c(\mathbb{R}))$ with respect to $H_{\mathcal{BV}}$. Equivalently, for $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$H_{\mathcal{BV}}(F_n,F_k)<\epsilon \text{ for all } n,k\geq n_0.$$

Using the definition of $H_{\mathcal{BV}}$, we obtain $\|F_n-F_k\|_\infty<\epsilon$ for all $n,k\geq n_0$. Since $(\mathcal{C}(I,\mathcal{K}_c(\mathbb{R})),\|.\|_\infty)$ is a complete metric space, there exists a continuous function F with $\|F_n-F\|_\infty\to 0$ as $n\to\infty$. We claim that $F\in\mathcal{BV}(I,\mathcal{K}_c(\mathbb{R}))$ and $H_{\mathcal{BV}}(F_n,F)\to 0$ as $n\to\infty$. Let $P=\{(t_0,\ldots,t_m): t_0<\cdots<t_m\}$ be a partition of I and $n\geq n_0$. Using Lemma 5.10, we get

$$\begin{aligned}
&H_{\mathcal{BV}}(F_n,F)\\
=&\|F_n-F\|_\infty+\sum_{j=1}^{m} H_d\Big(F_n(t_j)+F(t_{j-1}),F(t_j)+F_n(t_{j-1})\Big)\\
=&\lim_{k\to\infty}\left(\|F_n-F_k\|_\infty+\sum_{j=1}^{m} H_d\Big(F_n(t_j)+F_k(t_{j-1}),F_k(t_j)+F_n(t_{j-1})\Big)\right)\\
\leq&\lim_{k\to\infty}\left(\|F_n-F_k\|_\infty+\sup_P\sum_{j=1}^{m} H_d\Big(F_n(t_j)+F_k(t_{j-1}),F_k(t_j)+F_n(t_{j-1})\Big)\right)\\
\leq&\sup_{k\geq n_0}\left(\|F_n-F_k\|_\infty+\sup_P\sum_{j=1}^{m} H_d\Big(F_n(t_j)+F_k(t_{j-1}),F_k(t_j)+F_n(t_{j-1})\Big)\right)\\
\leq&\sup_{k\geq n_0} H_{\mathcal{BV}}(F_n,F_k)<\epsilon.
\end{aligned}$$

Since P is arbitrary, therefore we have $H_{\mathcal{BV}}(F_n, F) < \epsilon$ for all $n \geq n_0$.
It remains to show that $F \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$. Using $H_d(B + D, C + D) = H_d(B, C)$ for every $B, C, D \in \mathcal{K}_c(\mathbb{R})$ (see, for instance, [45]), we have

$$
\begin{aligned}
&\sum_{j=1}^{m} H_d(F(t_j), F(t_{j-1}))\\
=&\sum_{j=1}^{m} H_d(F(t_j) + F_n(t_{j-1}), F(t_{j-1}) + F_n(t_{j-1}))\\
\leq&\sum_{j=1}^{m} H_d(F(t_j) + F_n(t_{j-1}), F(t_{j-1}) + F_n(t_j)) + \sum_{j=1}^{m} H_d(F_n(t_j) + F(t_{j-1}), F(t_{j-1}) + F_n(t_{j-1}))\\
\leq&\sum_{j=1}^{m} H_d(F(t_j) + F_n(t_{j-1}), F(t_{j-1}) + F_n(t_j)) + \sum_{j=1}^{m} H_d(F_n(t_j), F_n(t_{j-1})) \leq H_{\mathcal{BV}}(F_n, F) + \|F_n\|_{\mathcal{BV}}.
\end{aligned}
$$

Since $H_{\mathcal{BV}}(F_n, F) < \epsilon$ and $F_n \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$, the above inequality yields that $F \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$. This completes the proof. □

Theorem 5.12. *Consider* $F \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$, Δ *to be defined in Theorem 5.2,* $S \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$ *such that* $S(x_0) - F(x_0) = S(x_N) - F(x_N)$, *and* $\alpha \in (-1, 1)$ *with* $|\alpha| < \frac{1}{N}$. *Then,* α*-fractal function,* F^{α} *corresponding to* F *is of bounded variation on* I.

Proof. Consider $\mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R})) = \{G \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R})) : G(x_0) - S(x_0) = G(x_N) - S(x_N)\}$. It is easy to prove that $\mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R}))$ is a closed subset of $\mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$, hence complete with respect to metric $H_{\mathcal{BV}}$. Define RB operator $\Phi : \mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R})) \to \mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R}))$ by

$$(\Phi G)(x) = F(x) + \alpha\big[G\big(U_i^{-1}(x)\big) - S\big(U_i^{-1}(x)\big)\big]$$

for each $x \in I_i$ and $i \in \Sigma_N$. It is easy to observe the well-definedness of Φ. For $m \in \mathbb{N}$, assume $P^i = \{(t_0^i, \ldots, t_m^i) : t_0^i < \cdots < t_m^i\}$ is a partition of I_i and $i \in \Sigma_N$.

Now for $G, H \in \mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R}))$ and $j \in \Sigma_m$, we have

$$\begin{aligned}
&H_d\Big(\Phi(G)(t_j^i) + \Phi(H)(t_{j-1}^i), \Phi(H)(t_j^i) + \Phi(G)(t_{j-1}^i)\Big)\\
\le\, & H_d\Big(\alpha G\big(U_i^{-1}(t_j^i)\big) + \alpha H\big(U_i^{-1}(t_{j-1}^i)\big), \alpha H\big(U_i^{-1}(t_j^i)\big) + \alpha G\big(U_i^{-1}(t_{j-1}^i)\big)\Big)\\
\le\, & |\alpha| H_d\Big(G\big(U_i^{-1}(t_j^i)\big) + H\big(U_i^{-1}(t_{j-1}^i)\big), H\big(U_i^{-1}(t_j^i)\big) + G\big(U_i^{-1}(t_{j-1}^i)\big)\Big).
\end{aligned}$$

Summing over $j = 1$ to m, we have

$$\begin{aligned}
&\sum_{j=1}^{m} H_d\Big(\Phi(G)(t_j^i) + \Phi(H)(t_{j-1}^i), \Phi(H)(t_j^i) + \Phi(G)(t_{j-1}^i)\Big)\\
\le & |\alpha| \sum_{j=1}^{m} H_d\Big(G\big(U_i^{-1}(t_j^i)\big) + H\big(U_i^{-1}(t_{j-1}^i)\big), H\big(U_i^{-1}(t_j^i)\big) + G\big(U_i^{-1}(t_{j-1}^i)\big)\Big)\\
\le & |\alpha| \sup_{P} \sum_{j=1}^{m} H_d\Big(G(t_j) + H(t_{j-1}), H(t_j) + G(t_{j-1})\Big),
\end{aligned}$$

since $P := \{(U_i^{-1}(t_0^i), \dots, U_i^{-1}(t_m^i)) : U_i^{-1}(t_0^i) < \cdots < U_i^{-1}(t_m^i)\}$ is a partition of I (without loss of generality), and the supremum is taken over all partitions $P = \{(t_0, \dots, t_m) : t_0 < \cdots < t_m\}$ of I. The above inequality is true for any partition P^i of I_i. Hence, we get

$$\begin{aligned}
&H_{\mathcal{BV}}(\Phi(G), \Phi(H))\\
=& \sup_{x \in I} H_d(\Phi(G)(x), \Phi(H)(x)) + \sup_{P} \sum_{j=1}^{m} H_d\Big(\Phi G(t_j) + \Phi H(t_{j-1}), \Phi H(t_j) + \Phi G(t_{j-1})\Big)\\
\le & |\alpha| \sup_{x \in I} H_d(G(x), H(x)) + \max_{i \in \Sigma_N} \sup_{P^i} \sum_{j=1}^{m} H_d\Big(\Phi(G)(t_j^i) + \Phi(H)(t_{j-1}^i), \Phi(H)(t_j^i) + \Phi(G)(t_{j-1}^i)\Big)\\
\le & |\alpha| \sup_{x \in I} H_d(G(x), H(x)) + N|\alpha| \sup_{P} \sum_{j=1}^{m} H_d\Big(G(t_j) + H(t_{j-1}), H(t_j) + G(t_{j-1})\Big)\\
\le & N|\alpha| H_{\mathcal{BV}}(G, H).
\end{aligned}$$

As $|\alpha| < \frac{1}{N}$, Φ is a contraction map. Then, the Banach fixed point theorem ensures that Φ has a unique fixed point, say F^α. Further, this fixed point satisfies the following self-referential equation,

$$F^{\alpha}(x) = F(x) + \alpha\big[F^{\alpha}\big(U_i^{-1}(x)\big) - S\big(U_i^{-1}(x)\big)\big]$$

for each $x \in I_i$, where $i \in \Sigma_N$. □

Notice that function F^{α} is a parametric function depending on parameters, the base function S, scaling function α, partition Δ, and the function F itself. To observe collective behavior of F^{α} depending on some such parameters, we define a set-valued map, $\mathcal{F}_S^{\alpha} : \mathcal{C}(I, \mathcal{K}(\mathbb{R})) \to \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ such that

$$\mathcal{F}_S^{\alpha}(F) = F^{\alpha}, \text{ where } \alpha \in (-1, 1). \tag{5.3}$$

This map is known as a fractal operator.

Remark 5.13. The notion of the fractal operator has already been studied extensively for single-valued maps. See, for instance, in [65, 68] fractal operator has been defined for univariate single-valued maps. In [90], the fractal operator for bivariate single-valued maps has been studied. Here, we have given the notion of the set-valued fractal operator.

Theorem 5.14. *$\mathcal{F}_S^{\alpha}$ defined as (5.3) is a continuous map.*

Proof. Let $\{F_k\}$ be a sequence in $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ such that $F_k \to F$, then to prove that $\mathcal{F}_S^{\alpha}$ is a continuous function, it is sufficient to prove that $F_n^{\alpha} \to F_{\alpha}$. Since $F_n \to F$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that,

$$\|F_n - F\|_{\infty} < \epsilon(1 - |\alpha|) \text{ for all } n \geq n_0,$$

$$\text{equivalently, } \sup_{x \in I} H_d(F_n(x), F(x)) < \epsilon(1 - |\alpha|) \text{ for all } n \geq n_0.$$

Now, for $x \in I_i$, we have

$$
\begin{aligned}
&H_d(F_n^\alpha(x), F^\alpha(x)) \\
=&H_d\Big(F_n(x) + \alpha\left[F_n^\alpha(U_i^{-1}(x)) - S(U_i^{-1}(x))\right], F(x) + \alpha\left[F^\alpha(U_i^{-1}(x)) - S(U_i^{-1}(x))\right]\Big) \\
\leq &H_d(F_n(x), F(x)) + |\alpha| H_d(F_n^\alpha(U_i^{-1}(x)), F^\alpha(U_i^{-1}(x))).
\end{aligned}
$$

This implies,

$$
\sup_{x\in I} H_d(F_n^\alpha(x), F^\alpha(x)) \leq \frac{1}{1-|\alpha|} \sup_{x\in I} H_d(F_n(x), F(x)),
$$

that is $\|F_n^\alpha(x) - F^\alpha(x)\|_\infty < \epsilon$ for all $n \geq n_0$.

This completes the proof. □

Theorem 5.15. *For a fixed partition* Δ*, the mapping* $\mathcal{T}_S^\Delta : \mathcal{C}(I, \mathcal{K}(\mathbb{R})) \rightrightarrows \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ *defined as,*

$$
\mathcal{T}_S^\Delta(F) = \{F^\alpha : \alpha \in (-1, 1)\}
$$

is lower semi-continuous.

Proof. Let $F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ and let $F^\alpha \in \mathcal{T}_S^\Delta(F)$ and a sequence $F_k \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ such that $F_k \to F$. Using Theorem 5.14, we have $F_k^\alpha \to F^\alpha$, then clearly $F_k^\alpha \in \mathcal{T}_S^\Delta(F_k)$, establishing the result. □

5.3 Approximation of Set-Valued Functions

In Section 5.2, we observe that F^α satisfies the following self-referential equation:

$$
F^\alpha(x) = F(x) + \alpha\left[F^\alpha\left(U_i^{-1}(x)\right) - S\left(U_i^{-1}(x)\right)\right]
$$

for every $x \in I_i$, where $i \in \Sigma_N$.

The following proposition gives the perturbation error between the map F and its α-fractal function F^α. We shall use this proposition as a prelude to our next theorem.

Proposition 5.16. *The perturbation error between F and F^α, can be estimated as:*

$$\|F^\alpha - F\|_\infty \leq \frac{|\alpha|}{1-|\alpha|}\|F - S\|_\infty + \frac{2|\alpha|}{1-|\alpha|}\|F\|_\infty.$$

Proof. For each $x \in I$, there exists $i \in \Sigma_N$ such that $x \in I_i$, now using the self-referential equation (Equation 5.2) and Note 5.1, we get

$$\begin{aligned}
&H_d(F^\alpha(x), F(x)) \\
=&H_d\Big(F(x) + \alpha\left[F^\alpha\left(U_i^{-1}(x)\right) - S\left(U_i^{-1}(x)\right)\right],\ F(x)\Big) \\
\leq&H_d\Big(\alpha\left[F^\alpha\left(U_i^{-1}(x)\right) - S\left(U_i^{-1}(x)\right)\right],\ \{0\}\Big) \\
=&|\alpha|H_d\Big(F^\alpha\left(U_i^{-1}(x)\right) - S\left(U_i^{-1}(x)\right),\ \{0\}\Big) \\
\leq&|\alpha|H_d\Big(F^\alpha\left(U_i^{-1}(x)\right) - S\left(U_i^{-1}(x)\right),\ F\left(U_i^{-1}(x)\right) - F\left(U_i^{-1}(x)\right)\Big) \\
&+|\alpha|H_d\Big(F\left(U_i^{-1}(x)\right) - F\left(U_i^{-1}(x)\right), \{0\}\Big) \\
\leq&|\alpha|H_d\Big(F^\alpha\left(U_i^{-1}(x)\right), F\left(U_i^{-1}(x)\right)\Big) + |\alpha|H_d\Big(-S\left(U_i^{-1}(x)\right), -F\left(U_i^{-1}(x)\right)\Big) \\
&+2|\alpha|H_d\Big(F\left(U_i^{-1}(x)\right), \{0\}\Big) \\
\leq&|\alpha|\sup_{\substack{x\in I_i \\ i\in\Sigma_N}} H_d\Big(F^\alpha\left(U_i^{-1}(x)\right), F\left(U_i^{-1}(x)\right)\Big) + |\alpha|\sup_{\substack{x\in I_i \\ i\in\Sigma_N}} H_d\Big(S\left(U_i^{-1}(x)\right), F\left(U_i^{-1}(x)\right)\Big) \\
&+2|\alpha|\sup_{\substack{x\in I_i \\ i\in\Sigma_N}} H_d\left(F\left(U_i^{-1}(x)\right), \{0\}\right) \\
\leq&|\alpha|\|F^\alpha - F\|_\infty + |\alpha|\|F - S\|_\infty + 2|\alpha|\|F\|_\infty.
\end{aligned}$$

This in turn yields $\|F^\alpha - F\|_\infty \leq |\alpha|\|F^\alpha - F\|_\infty + |\alpha|\|F - S\|_\infty + 2|\alpha|\|F\|_\infty$. This establishes the proof. □

Remark 5.17. The literature has already examined the perturbation error between single-valued maps and their corresponding α-fractal functions. See, for instance, in [66, 90] perturbation error for a univariate and bivariate single-valued map and its corresponding α-fractal function is given, respectively. In this study, we have focused on the perturbation error between a set-valued map and its corresponding α-fractal function.

Definition 5.18. Let $\mathcal{P} \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ be a set-valued polynomial function, then α-fractal function $\mathcal{P}^\alpha$ corresponding to $\mathcal{P}$ is defined as set-valued fractal polynomial.

Theorem 5.19. *Consider $F \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$. For any $\epsilon > 0$, there is a set-valued fractal polynomial $\mathcal{P}^\alpha$ such that*

$$\|F - \mathcal{P}^\alpha\|_\infty < \epsilon.$$

Proof. For $\epsilon > 0$ using [92], there is a set-valued polynomial function $\mathcal{P}$ such that

$$\|F - \mathcal{P}\|_\infty < \frac{\epsilon}{3}.$$

Choose a partition $\Delta_{\mathcal{P}} = \{t_0, \dots, t_M\}$ of I and a continuous function $S_{\mathcal{P}}$ satisfying $S_{\mathcal{P}}(t_0) - \mathcal{P}(t_0) = S_{\mathcal{P}}(t_M) - \mathcal{P}(t_M)$, and $\alpha \in (-1, 1)$ such that

$$|\alpha| < \min\left\{\frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + \|\mathcal{P} - S_{\mathcal{P}}\|_\infty}, \frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + 2\|\mathcal{P}\|_\infty}\right\}.$$

Then, using Proposition 5.16, we get

$$\begin{aligned}\|F - \mathcal{P}^\alpha\|_\infty &\leq \|F - \mathcal{P}\|_\infty + \|\mathcal{P} - \mathcal{P}^\alpha\|_\infty \\ &\leq \|F - \mathcal{P}\|_\infty + \frac{|\alpha|}{1 - |\alpha|}\|\mathcal{P} - S_{\mathcal{P}}\|_\infty + \frac{2|\alpha|}{1 - |\alpha|}\|\mathcal{P}\|_\infty < \epsilon.\end{aligned}$$

□

Remark 5.20. We took $\alpha \in \mathbb{R}$ in the above proof, such that

$$|\alpha| < \min\left\{\frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + \|\mathcal{P} - S_{\mathcal{P}}\|_\infty}, \frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + 2\|\mathcal{P}\|_\infty}\right\}.$$

In this situation, α may be "close" to 0, $\mathcal{P}^\alpha$ may not be self-referential, and it may behave as a classical polynomial. In alter, if we fix $\alpha \in (-1,1)$ such that $|\alpha| < 1$, but otherwise arbitrary and choose a polynomial $\mathcal{P}$ and a function $S_{\mathcal{P}} \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ satisfying $S_{\mathcal{P}}(t_0) - \mathcal{P}(t_0) = S_{\mathcal{P}}(t_M) - \mathcal{P}(t_M)$ and

$$\|\mathcal{P} - S_{\mathcal{P}}\| < \frac{(1-|\alpha|)\epsilon}{3|\alpha|} \text{ and } \|\mathcal{P}\|_\infty < \frac{(1-|\alpha|)\epsilon}{6|\alpha|}.$$

This forces F to be a zero set function. Hence, the analog of [90, Remark 5.2] cannot be established in set-valued mappings. In particular, the recently developed notion of Bernstein fractal functions will not be useful in approximating set-valued functions.

With the reference to Theorem 5.19, we have

Theorem 5.21. *The set of set-valued fractal polynomials with a nonzero scale vector is dense in* $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$.

5.3.1 Constrained Approximation

Here we target to study some constrained approximation aspects of fractal functions. Before proving the next theorem, let us recall a result and prove a lemma as a prelude.

Result 5.22. Consider X, Y to be topological spaces, $f : X \to Y$ to be a continuous function and S to be a dense subset in X. If $F(x) \leq 0$ $(F(x) \geq 0)$ for each $x \in S$, then $F(x) \leq 0$ $(F(x) \geq 0)$ for each $x \in X$.

Lemma 5.23. *The set* $C = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{1\le i_1,\dots,i_n\le N} U_{i_1\dots i_n}\big(\{x_0,\dots,x_N\}\big)\right)$ *is dense in the interval* $I=[0,1]$, *where* $U_{i_1\dots i_n}(x) = U_{i_1}(U_{i_2}(\dots(U_{i_n}(x))))$ *and* $n\in\mathbb{N}$.

Proof. Let $x \in I$ be any point. Observe that for some $y \in \{x_0,\dots,x_N\}$, we have $|x-y| \le \max_{i\in\Sigma_N}\left\{\frac{x_i-x_{i-1}}{2}\right\}$. Since each U_i is a contraction mapping with contraction coefficient a_i. Choose $a = \max_{i\in\Sigma_N}\{a_i\}$, then for each $x\in I$ and for each $\epsilon>0$ we can choose $y \in U_{i_1\dots i_n}\big(\{x_0,\dots,x_N\}\big)$ for some $n\in\mathbb{N}$ such that,

$$|x-y| \le a^n \max_{i\in\Sigma_N}\left\{\frac{x_i-x_{i-1}}{2}\right\} < \epsilon.$$

This completes the proof. □

Theorem 5.24. *Let* $F, G \in \mathcal{C}(I,\mathcal{K}(\mathbb{R}))$ *and* Δ *be defined as in Theorem 5.2, and* $F(x_0), F(x_N), G(x_0), G(x_N)$ *be single-valued. If* $F \le G$, *then* $F^\alpha \le G^\alpha$ *provided* $S_F, S_G \in \mathcal{C}(I,\mathcal{K}(\mathbb{R}))$ *satisfying* $S_F \le S_G$ *and* $S_F(x_0) = F(x_0), S_F(x_N) = F(x_N)$, $S_G(x_0) = G(x_0), S_G(x_N) = G(x_N)$.

Proof. Let $S_F, S_G \in \mathcal{C}(I,\mathcal{K}(\mathbb{R}))$ such that $S_F \le S_G$ and $S_F(x_0) = F(x_0),\ S_F(x_N) = F(x_N),\ S_G(x_0) = G(x_0),\ S_G(x_N) = G(x_N)$. Using Note 5.5, we have

$$F^\alpha(x_i) = F(x_i),\ G^\alpha(x_i) = G(x_i) \text{ for each } i \in \Sigma_N.$$

From self-referential equation (Equation (5.2)),

$$F^\alpha(U_i(x)) = F(U_i(x)) + \alpha(F^\alpha(x) - S_F(x)) \text{ and } G^\alpha(U_i(x)) = G(U_i(x)) + \alpha(G^\alpha(x) - S_G(x))$$

for each $x \in I_i$, where $i\in\Sigma_N$. For $x\in\Delta$, we deduce

$$F^\alpha\big(U_i(x)\big) \subset G^\alpha\big(U_i(x)\big) \text{ for any } i \in \Sigma_N.$$

Applying the process repeatedly, we get

$$F^{\alpha}\left(U_{i_1\ldots i_n}(x)\right) \subset G^{\alpha}\left(U_{i_1\ldots i_n}(x)\right) \text{ for any } i_1,\ldots,i_n \in \Sigma_N,\ x \in \{x_0,\ldots,x_N\},$$

where $U_{i_1\ldots i_n}(x) = U_{i_1}(U_{i_2}(\ldots(U_{i_n}(x))))$ and $n \in \mathbb{N}$.
Hence, $F^{\alpha}(x) \subset G^{\alpha}(x)$ for each $x \in \bigcup_{n\in\mathbb{N}}\left(\bigcup_{1\le i_1,\ldots,i_n\le N} U_{i_1\ldots i_n}\big(\{x_0,\ldots,x_N\}\big)\right)$. Now using Lemma 5.23 and Result 5.22, we are done. □

5.4 Dimensional Results

To move further in this section, we shall first observe some examples to understand the motivation behind this section.

Example 5.25. Let $F_1 : [0,1] \rightrightarrows \mathbb{R}$ be a set-valued map defined as $F_1(x) = \{0\}$, then according to (1.15) graph of this function will be a line segment in $\mathbb{R}^2$, and hence $\dim_H(G_{F_1}) = 1$.

Example 5.26. Let $F_2 : [0,1] \rightrightarrows \mathbb{R}$ be a set-valued map defined as $F_2(x) = [-1,1]$, then by (1.15), we have $G_{F_2} = [0,1] \times [-1,1]$, and hence $\dim_H(G_{F_2}) = 2$.

Example 5.27. Let $F_3 : [0,1] \rightrightarrows \mathbb{R}$ be a set-valued map defined as $F_3(x) = C$, where C is Cantor set. Then, by (1.15) we have $G_{F_3} = [0,1] \times C$, and hence $\dim_H(G_{F_2}) = 1 + \frac{\log 2}{\log 3}$.

Note that F_1, F_2, and F_3 are constant functions. As a result, these functions are also Lipschitz with bounded variations. In contrast to the case of a single-valued function, we can see that the Hausdorff dimension of the graph of a set-valued Lipschitz function is not necessarily 1. The same observation applies to the graph of a set-valued function with bounded variation. It is always possible to find a set-valued Lipschitz function or a set-valued function with bounded variation whose graph has a dimension β for any $1 \leq \beta \leq 2$. We notice that the definition of the graph in equation (1.15) does not yield any interesting dimensional results. Therefore, we propose a new definition of the graph for a set-valued function and investigate some dimensional results using this new definition.

Definition 5.28. Let $F : [0,1] \to \mathcal{K}(\mathbb{R})$ be a set-valued map, then a graph of F is defined as;

$$\mathcal{G}(F) = \{(x, F(x)) : F(x) \in \mathcal{K}(\mathbb{R})\} \subset [0,1] \times \mathcal{K}(\mathbb{R}). \tag{5.4}$$

Define a metric on this graph as,

$$D_{\mathcal{G}}((x, F(x)), (y, F(y))) = |x - y| + H_d(F(x), F(y)).$$

Next, we prove that the graph of F^α defined in (5.4) is an attractor of an IFS defined on $I \times \mathcal{K}_c(\mathbb{R})$.

Let us note the following lemma as a prelude. The motivation for this lemma comes from [15, Proposition 1].

Lemma 5.29. *Let F be a set-valued continuous map and F^α be its corresponding α-fractal function. Define a function $\mathfrak{d} : I \times \mathcal{K}_c(\mathbb{R}) \to [0, \infty)$ as*

$$\mathfrak{d}\big((x, A), (y, B)\big) = |x - y| + H_d\big(A + F^\alpha(y), B + F^\alpha(x)\big).$$

Then, $I \times \mathcal{K}_c(\mathbb{R})$ with respect to $\mathfrak{d}$ is a complete metric space.

Proof. Clearly, $\mathfrak{d}\big((x,A),(y,B)\big)=\mathfrak{d}\big((y,B),(x,A)\big)\geq 0$. Suppose $\mathfrak{d}\big((x,A),(y,B)\big)=0$, then

$$\begin{aligned}
&|x-y|+H_d\big(A+F^\alpha(y),B+F^\alpha(x)\big)=0\\
&\text{i.e., } |x-y|=0 \text{ and } H_d\big(A+F^\alpha(y),B+F^\alpha(x)\big)=0\\
&\text{i.e., } x=y \text{ and } H_d\big(A+F^\alpha(y),B+F^\alpha(x)\big)=H_d(A,B)=0\\
&\text{i.e., } x=y \text{ and } A=B\\
&\text{i.e., } (x,A)=(y,B).
\end{aligned}$$

Now to prove that $\mathfrak{d}$ satisfies the triangle inequality. Take $(y_i,A_i)\in I\times\mathcal{K}_c(\mathbb{R})$ for $i=1,2,3$. Then, we have

$$\begin{aligned}
&\mathfrak{d}\big((y_1,A_1),(y_2,A_2)\big)\\
=&|y_1-y_2|+H_d\big(A_1+F^\alpha(y_2),A_2+F^\alpha(y_1)\big)\\
=&|y_1-y_2|+H_d\big(A_1+F^\alpha(y_2)+A_3+F^\alpha(y_3),A_2+F^\alpha(y_1)+A_3+F^\alpha(y_3)\big)\\
\leq&\big\{|y_1-y_3|+|y_3-y_2|\big\}+\big\{H_d\big(A_1+F^\alpha(y_3),A_3+F^\alpha(y_1)\big)+H_d\big(A_3+F^\alpha(y_2),A_2+F^\alpha(y_3)\big)\big\}.
\end{aligned}$$

Hence,

$$\mathfrak{d}\Big((y_1,A_1),(y_2,A_2)\Big)\leq\mathfrak{d}\Big((y_1,A_1),(y_3,A_3)\Big)+\mathfrak{d}\Big((y_3,A_3),(y_2,A_2)\Big).$$

To prove completeness, let $\{(y_n,A_n)\}$ be a Cauchy sequence in $I\times\mathcal{K}_c(\mathbb{R})$. For $\epsilon>0$ there is an integer $N(\epsilon)$ such that

$$|y_n-y_m|+H_d\big(A_n+F^\alpha(y_n),A_m+F^\alpha(y_m)\big)<\epsilon,\ \text{whenever } m,n\geq N(\epsilon).$$

This shows $\{y_n\}$ is a Cauchy sequence of I. Hence, it converges to, say, $y^*\in I$. Since F^α is a uniformly continuous map, consequently $\{F^\alpha(y_n)\}$ will also be a Cauchy sequence with respect to Hausdorff metric, and hence converges to $F^\alpha(y^*)\in\mathcal{K}_c(\mathbb{R})$.

Then,

$$\begin{aligned} H_d(A_n, A_m) =& H_d\big(A_n + F^\alpha(y_n), A_m + F^\alpha(y_n)\big) \\ \leq& H_d\big(A_n + F^\alpha(y_n), A_n + F^\alpha(y_m)\big) + H_d\big(A_n + F^\alpha(y_m) + A_m + F^\alpha(y_n)\big) \\ =& H_d\big(F^\alpha(y_n), F^\alpha(y_m)\big) + H_d\big(A_n + F^\alpha(y_m) + A_m + F^\alpha(y_n)\big) \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This implies, $\{A_n\}$ is a Cauchy sequence of $\mathcal{K}_c(\mathbb{R})$ and so it converges to, say $A^* \in \mathcal{K}_c(\mathbb{R})$. Hence, $\{(y_n, A_n)\}$ converges to $(y^*, A^*) \in I \times \mathcal{K}_c(\mathbb{R})$. This completes the proof. □

Proposition 5.30. *Let $F \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ be a set-valued continuous map and $S \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ be the base function. Define $W_i : I \times \mathcal{K}_c(\mathbb{R}) \to I \times \mathcal{K}_c(\mathbb{R})$ for each $i \in \Sigma_N$ such that*

$$W_i(x, A) = \left(U_i(x), \alpha A + F(U_i(x)) - \alpha S(x)\right).$$

Then, each W_i is a contraction map with respect to the metric defined in Lemma 5.29, provided $\max\{|\alpha|, a_i\} < 1$ for each $i \in \Sigma_N$.

Proof. Let $(x, A), (y, B) \in I \times \mathcal{K}_c(\mathbb{R})$, then for each $i \in \Sigma_N$, we have

$$\begin{aligned} &\mathfrak{d}\big(W_i(x, A), W_i(y, B)\big) \\ =& \mathfrak{d}\Big(\big(U_i(x), \alpha A + F(U_i(x)) - \alpha S(x)\big), \big(U_i(y), \alpha B + F(U_i(y)) + \alpha S(x)\big)\Big) \\ =& \big|U_i(x) - U_i(y)\big| + H_d\Big(\alpha A + F(U_i(x)) - \alpha S(x) + F^\alpha(U_i(y)), \\ & \qquad \alpha B + F(U_i(y)) - \alpha S(y) + F^\alpha(U_i(x))\Big) \\ =& \big|U_i(x) - U_i(y)\big| + H_d\Big(\alpha A + F(U_i(x)) - \alpha S(x) + F(U_i(y)) + \alpha F^\alpha(y) - \alpha S(y), \\ & \qquad \alpha B + F(U_i(y)) - \alpha S(y) + F(U_i(x)) + \alpha F^\alpha(x) - \alpha S(x)\Big) \\ =& a_i\big|x - y\big| + H_d\Big(\alpha A + \alpha F^\alpha(y), \alpha B + \alpha F^\alpha(x)\Big) \end{aligned}$$

$$
\begin{aligned}
&= a_i\big|x-y\big| + |\alpha| H_d\Big(A + F^{\alpha}(y), B + F^{\alpha}(x)\Big)\\
&\le \max\{|\alpha|, a_i\}\Big(\big|x-y\big| + H_d\Big(A + F^{\alpha}(y), B + F^{\alpha}(x)\Big)\Big)\\
&= \max\{|\alpha|, a_i\}\mathfrak{d}\big((x, A), (y, B)\big).
\end{aligned}
$$

Since $\max\{|\alpha|, a_i\} < 1$, each W_i is a contraction mapping. □

Now, to prove the next theorem, we first note the following basic results. Their proofs can be found in the literature, but we include them here for the sake of completeness.

Lemma 5.31. *The space* $(\mathcal{K}_c(\mathbb{R}), H_d)$ *is a complete metric space.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{K}_c(\mathbb{R})$. This implies that $\{A_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{K}(\mathbb{R})$. Then, by the completeness of the space $(\mathcal{K}(\mathbb{R}), H_d)$, there exists $A^* \in \mathcal{K}(\mathbb{R})$ such that $A_n \to A^*$ with respect to the Hausdorff metric H_d. It is well-known that for $z^* \in A^*$, there exists a sequence $\{z_n\}_n$, where $z_n \in A_n$ for each $n \in \mathbb{N}$, such that $z_n \to z^*$ as $n \to \infty$.

Now it remains to prove that A^* is a convex set. For this let $z, t \in A^*$, then there exist sequences $\{z_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}}$, where $z_n, t_n \in A_n$ for each $n \in \mathbb{N}$ such that $z_n \to z$ and $t_n \to t$. Since $z_n,\ t_n \in A_n$ and A_n is convex, therefore $\lambda z_n + (1-\lambda)t_n \in A_n$ for all $\lambda \in [0, 1]$. This implies that $\lim\limits_{n\to\infty} (\lambda z_n + (1-\lambda)t_n) = \lambda z + (1-\lambda)t \in A^*$. This completes the proof. □

Lemma 5.32. *The space* $(\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}), d_{\mathcal{C}})$ *is a complete metric space.*

Proof. To prove this, it is sufficient to show that $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ is a closed subset of $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$. For this, let F^* be a limit point of $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$. Then, there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ such that $F_n \to F^*$ with respect to the metric $d_{\mathcal{C}}$.

This implies that $F_n(z) \to F^*(z)$ for all $z \in I$ with respect to the Hausdorff metric. Since $F_n(z) \in \mathcal{K}_c(\mathbb{R})$ for each $z \in I$, hence using Lemma 5.31, $F^*(z) \in \mathcal{K}_c(\mathbb{R})$ for each $z \in I$. This completes the proof. □

Theorem 5.33. *For each $i \in \Sigma_N$, let $W_i : I \times \mathcal{K}_c(\mathbb{R}) \to I \times \mathcal{K}_c(\mathbb{R})$ be the map defined in Proposition 5.30. Then, by Definition 5.28, the graph of F^α will be an attractor of the IFS, $\{(I \times \mathcal{K}_c(\mathbb{R}), \mathfrak{d})\,; W_1, \dots, W_N\}$.*

Proof. First we establish that $F^\alpha \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$. But this can be observed by using Lemma 5.32 and Theorem 5.2. Now since $I = \bigcup_{i \in \Sigma_N} U_i(I)$. Then, from (5.1), we have

$$\begin{aligned}
\bigcup_{i\in\Sigma_N} W_i(\mathcal{G}(F^\alpha)) &= \bigcup_{i\in\Sigma_N} \big\{W_i(x, F^\alpha(x)) : x \in I\big\} \\
&= \bigcup_{i\in\Sigma_N} \big\{\big(U_i(x), \alpha F^\alpha(x) + F(U_i(x)) - \alpha S(x)\big) : x \in I\big\} \\
&= \bigcup_{i\in\Sigma_N} \big\{\,(U_i(x), F^\alpha(U_i(x))) : x \in I\big\} \\
&= \bigcup_{i\in\Sigma_N} \big\{(x, F^\alpha(x)) : u \in U_i(I)\big\} \\
&= \mathcal{G}(F^\alpha).
\end{aligned}$$

This completes the proof. □

Schief [84] noted that the dimensional results for Euclidean spaces do not have simple generalizations to complete metric spaces. Following his work, Nussbaum et al. [70] have proved a more general result in the setting of a complete metric space. Answering a question raised in [70], recently Verma [87] has shown the Hausdorff dimension of the invariant set under the SOSC. He explores several dimensional aspects of sets in complete metric space. In the book [36], Falconer studied the dimensional results of sets in Euclidean spaces. Given [87], we may assure the reader that some results, which we will use, also hold in a general complete metric space.

Theorem 5.34. *Let $\mathcal{I} = \{I \times \mathcal{K}_c(\mathbb{R}); W_1, \dots W_N\}$ be the IFS defined in Theorem 5.33 such that*

$$r_i D_{\mathcal{G}}\big((x, A), (y, B)\big) \leq D_{\mathcal{G}}\big(W_i(x, A), W_i(y, B)\big) \leq R_i D_{\mathcal{G}}\big((x, A), (y, B)\big)$$

for every $(x, A), (y, B) \in I \times \mathcal{K}_c(\mathbb{R})$, where $0 < r_i \leq R_i < 1$ for all $i \in \Sigma_N$. Then, $t_* \leq \dim_H(\mathcal{G}(F^\alpha)) \leq t^*$, where t_* and t^* are characterized by $\sum_{i=1}^{N} r_i^{t^*} = 1$ and $\sum_{i=1}^{N} R_i^{t^*} = 1$, respectively.

Proof. For the purposed upper bound one can refer [36, Proposition 9.6](see also, [87, Theorem 2.12]). For the lower bound of the Hausdorff dimension of $\mathcal{G}(F^\alpha)$, we proceed as follows.
Set $V = (x_0, x_N) \times \mathcal{K}_c(\mathbb{R})$, an open set in $I \times \mathcal{K}_c(\mathbb{R})$. Since for each $i, j \in \Sigma_N$ with $i \neq j$, we have

$$U_j\big((x_0, x_N)\big) = (x_j, x_{j+1}) \text{ and } U_i\big((x_0, x_N)\big) \cap U_j\big((x_0, x_N)\big) = \emptyset,$$

hence for each $i, j \in \Sigma_N$ and $i \neq j$, we have

$$W_j(V) = (x_j, x_{j+1}) \times \mathcal{K}_c(\mathbb{R}) \text{ and } W_i(V) \cap W_j(V) = \emptyset.$$

Therefore,

$$\bigcup_{i=1}^{N} W_i(V) = \bigcup_{i=1}^{N} \big\{(x_i, x_{i+1}) \times \mathcal{K}_c(\mathbb{R})\big\} \subseteq V \text{ and } W_i(V) \cap W_j(V) = \emptyset.$$

Then, IFS satisfies OSC. We have $V \cap \mathcal{G}(F^{\alpha}) \neq \emptyset$ this implies that IFS is satisfying SOSC. Since $V \cap \mathcal{G}(F^{\alpha}) \neq \emptyset$, we have an $i \in \Sigma_N^*$ such that $\mathcal{G}(F^{\alpha})_i \subset V$, where $\Sigma_N^* = \bigcup\limits_{m \in \mathbb{N}} \{1, \dots, N\}^m$, collection of all finite sequences whose terms are in Σ_N and

$$\mathcal{G}(F^{\alpha})_i = W_i(\mathcal{G}(F^{\alpha})) := W_{i_1} \circ W_{i_2} \circ \dots \circ W_{i_m}(\mathcal{G}(F^{\alpha}))$$

for $i \in \Sigma_N^m = \Sigma_N \times \dots \times \Sigma_N$ (m-times) and $m \in \mathbb{N}$. Observe that for any $j \in \Sigma_N^m$ and $k \in \mathbb{N}$, the sets, $\mathcal{G}(F^{\alpha})_{j_i}$, are disjoint. Further, the IFS $\{W_{j_i} : j \in \Sigma_N^k\}$ satisfies the hypothesis of [36, Proposition 9.7] (see also, [87, Theorem 2.35]). Therefore, with the notation $r_j = r_{j_1} r_{j_2} \cdots r_{j_k}$ we have $t_k \leq \dim_H(G^*)$, where G^* is an attractor of the IFS and $\sum\limits_{j \in \Sigma_N^k} r_{j_i}^{t_k} = 1$. Since $G^* \subset \mathcal{G}(F^{\alpha})$, $t_k \leq \dim_H(G^*) \leq \dim_H(\mathcal{G}(F^{\alpha}))$. If possible, assume that $\dim_H(\mathcal{G}(F^{\alpha})) < t_*$, where $\sum\limits_{i=1}^{N} r_i^{t_*} = 1$. Then, $t_k < t_*$. Now, we have

$$\begin{aligned} r_i^{-t_k} &= \sum_{j \in \Sigma_N^k} r_j^{t_k} \geq \sum_{j \in \Sigma_N^k} r_j^{\dim_H(\mathcal{G}(F^{\alpha}))} = \sum_{j \in \Sigma_N^k} r_j^{t_*} r_j^{\dim_H(\mathcal{G}(F^{\alpha})) - t_*} \\ &\geq \sum_{j \in \Sigma_N^k} r_j^{t_*} r_{max}^{k(\dim_H(\mathcal{G}(F^{\alpha})) - t_*)} \\ &= r_{max}^{k(\dim_H(\mathcal{G}(F^{\alpha})) - t_*)}, \end{aligned}$$

where $r_{\max} = \max\{r_1, r_2, \dots, r_N\}$. Since $r_{\max} < 1$, $r_{max}^{k(\dim_H(\mathcal{G}(F^{\alpha})) - t_*)}$ tends to infinity as k tends to infinity, and therefore $r_i^{-t_k}$ is unbounded, which is a contradiction. Hence, $\dim_H(\mathcal{G}(F^{\alpha})) \geq t_*$, which is the required result. □

The following theorem is an immediate application of the Theorem 5.34.

Theorem 5.35. *Consider $F : I \to \mathcal{K}_c(\mathbb{R})$ to be a set-valued map. If $|\alpha| < \min\{a_i : i \in \Sigma_N\}$, then $\dim_H(\mathcal{G}(F^{\alpha})) = 1$.*

Proof. Using Proposition 5.30 for every pair $(x, A), (y, B) \in I \times \mathcal{K}_c(\mathbb{R})$, we have

$$D_{\mathcal{G}}(W_i(x, A), W_i(y, B)) \leq a_i D_{\mathcal{G}}((x, A), (y, B)) \text{ for } i \in \Sigma_N.$$

Since $\sum_{i=1}^{N} a_i = 1$, then by Theorem 5.34, $\dim_H(\mathcal{G}(F^\alpha)) \leq 1$. This concludes the proof.

□

Theorem 5.36. *If $F : [0, 1] \to \mathcal{K}(\mathbb{R})$ is a set-valued Lipschitz map having Lipschitz constant l and the graph of F is as defined in (5.4), then $\dim_H(\mathcal{G}(F)) = 1$.*

Proof. To prove this theorem, it will be sufficient to define a bi-Lipschitz map between $[0, 1]$ and $\mathcal{G}(F)$. Define $T : [0, 1] \to \mathcal{G}(F)$ such that $T(x) = (x, F(x))$. Then, we have

$$\begin{aligned} D_{\mathcal{G}}(Tx, Ty) &= D_{\mathcal{G}}((x, F(x)), (y, F(y))) \\ &= |x - y| + H_d(Fx, Fy) \\ &\leq |x - y| + l|x - y| \\ &\leq (1 + l)|x - y|, \end{aligned}$$

$$\text{that is, } D_{\mathcal{G}}(Tx, Ty) \leq (1 + l)|x - y| \tag{5.5}$$

and

$$\begin{aligned} D_{\mathcal{G}}(Tx, Ty) &= D_{\mathcal{G}}((x, Fu), (y, Fy)) \\ &= |x - y| + H_d(Fx, Fy) \end{aligned}$$

$$\text{that is, } D_{\mathcal{G}}(Tx, Ty) \geq \frac{1}{2}|x - y|. \tag{5.6}$$

Equations (5.5) and (5.6) will prove the bi-Lipschitz nature of T.

Hence, $\dim_H(\mathcal{G}(F)) = 1$. □

Theorem 5.37. *Let $F, S \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ be Lipschitz functions such that $S(x_0) - F(x_0) = S(x_N) - F(x_N)$, and let $\alpha \in (-1, 1)$. Then, $\dim_H(\mathcal{G}(F^\alpha)) = 1$ provided that $|\alpha| < a := \min\{a_i : i \in \Sigma_N\}$.*

Proof. The proof follows by Theorem 5.36 and Theorem 5.7. □

Lemma 5.38. *Let $F, T : [0, 1] \to \mathcal{K}(\mathbb{R})$ be set-valued Lipschitz maps with Lipschitz constant l, then $\dim_H(\mathcal{G}(F + T)) \leq \dim_H(\mathcal{G}(T))$, where $(F + T)(x) := F(x) + T(x)$ and $F(x) + T(x)$ denotes the Minkowski sum of $F(x)$ and $T(x)$.*

Proof. To establish the proof of this lemma, it will be sufficient to show the existence of a Lipschitz map from $\mathcal{G}(T)$ to $\mathcal{G}(F + T)$. Define $\Phi : \mathcal{G}(T) \to \mathcal{G}(F + T)$ such that $\Phi(x, T(x)) = (x, F(x) + T(x))$. It is easy to see that Φ is well defined and onto. Now to get its Lipschitz behavior, we have

$$\begin{aligned}
D_{\mathcal{G}}(\Phi(x, T(x)), \Phi(y, T(y))) &= D_{\mathcal{G}}((x, F(x) + T(x)), (y, F(y) + T(y))) \\
&= |x - y| + H_d(F(x) + T(x), F(y) + T(y)) \\
&\leq |x - y| + H_d(F(x), F(y)) + H_d(T(x), T(y)) \\
&\leq |x - y| + l|x - y| + H_d(T(x), T(y)) \\
&\leq (1 + l) \{|x - y| + H_d(T(x), T(y))\}.
\end{aligned}$$

That is, $D_{\mathcal{G}}(\Phi(x, T(x)), \Phi(y, T(y))) \leq (1+l) D_{\mathcal{G}}((x, T(x)), (y, T(y)))$. Hence, Φ being a Lipschitz map implies that

$$\dim_H(\mathcal{G}(F + T)) \leq \dim_H(\mathcal{G}(T)) \text{ and } \dim_B(\mathcal{G}(F + T)) \leq \dim_B(\mathcal{G}(T)).$$

This completes the proof.

□

In view of the Lipschitz invariance property of dimension, one may conclude that the upcoming theorem holds for all aforementioned dimensions.

Theorem 5.39. *Consider* $1 \leq \beta$. *Then, set* $\mathcal{S}_\beta := \{F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R})) : \dim_H(\mathcal{G}(F)) = \beta\}$ *is dense in* $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$.

Proof. Let $F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ and $\epsilon > 0$. Using the density of $\mathcal{L}ip(I, \mathcal{K}(\mathbb{R}))$ in $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$, there exists G in $\mathcal{L}ip(I, \mathcal{K}(\mathbb{R}))$ such that

$$\|F - G\|_\infty < \frac{\epsilon}{2}.$$

Further, we consider a nonvanishing function $H \in \mathcal{S}_\beta$. Let $H_* = G + \frac{\epsilon}{2\|H\|_\infty} H$, which immediately gives

$$\|G - H_*\|_\infty \leq \frac{\epsilon}{2}.$$

This together with Lemma 5.38 implies that $\dim(Gr(H_*)) = \dim(Gr(H)) = \beta$. Hence, we have $H_* \in \mathcal{S}_\beta$ and

$$\|F - H_*\|_\infty \leq \|F - G\|_\infty + \|G - H_*\|_\infty < \epsilon.$$

This completes the proof. □

Before proving our next result, let us note the following lemma as a prelude.

Lemma 5.40. *Consider* $A, B,$ *and* C *to be compact subsets of* $\mathbb{R}$. *Then,*

$$H_d(AB, CB) \leq \sup_{b \in B} |b| H_d(A, C),$$

where $YZ = \{yz : y \in Y \in \mathcal{K}(\mathbb{R}),\ z \in Z \in \mathcal{K}(\mathbb{R})\}$.

Proof. We have

$$\begin{aligned}
H_d(AB, CB) &= \max\Big\{ \sup_{ab\in AB} \inf_{cb'\in CB} |ab - cb'|, \sup_{cb'\in CB} \inf_{ab\in AB} |cb' - ab| \Big\} \\
&\leq \max\Big\{ \sup_{ab\in AB} \inf_{cb\in Cb} |ab - cb|, \sup_{cb'\in CB} \inf_{ab'\in Ab'} |cb' - ab'| \Big\} \\
&\leq \max\Big\{ \sup_{ab\in AB} \inf_{cb\in Cb} |b||a - c|, \sup_{cb'\in CB} \inf_{ab'\in Ab'} |b'||c - a| \Big\} \\
&\leq \max\Big\{ \sup_{a\in A, b\in B} (|b| \inf_{cb'\in CB} |a - c|), \sup_{c\in C, b'\in B} (|b'| \inf_{ab'\in Ab'} |c - a|) \Big\} \\
&\leq \sup_{b\in B} |b| \max\Big\{ \sup_{a\in A} \inf_{c\in C} |a - c|, \sup_{c\in C} \inf_{a\in A} |c - a| \Big\} \\
&= \sup_{b\in B} |b| H_d(A, C),
\end{aligned}$$

proving the assertion. □

Next, we define the multiplication of set-valued maps $F, L : W \subseteq \mathbb{R} \rightrightarrows \mathbb{R}$ by $(FT)(y) = F(y)T(y)$.

Lemma 5.41. *Consider $F, T : [0, 1] \to \mathcal{K}(\mathbb{R})$ to be set-valued Lipschitz maps with Lipschitz constant l. Then,*

$$\dim_H(\mathcal{G}(FT)) \leq \dim_H(\mathcal{G}(T)).$$

Proof. Define $\Phi : \mathcal{G}(T) \to \mathcal{G}(FT)$ such that

$$\Phi\big((x, T(x))\big) = \big(x, F(x)T(x)\big).$$

Choose $M = \max\{1 + l \sup_{z\in \bigcup_{x\in[0,1]} Tx} |z|, \sup_{v\in \bigcup_{y\in[0,1]} Fy} |v|\}$.

Notice that Φ is well-defined and surjective. To prove our lemma, it is enough to prove Φ is a Lipschitz map. For this, we have

$$\begin{aligned} D_{\mathcal{G}}(\Phi(x,Tx),\Phi(y,Ty)) &= D_{\mathcal{G}}((x,FxTx),(y,FyTy)) \\ &= |x-y| + H_d(FxTx,FyTy) \\ &\leq |x-y| + H_d(FxTx,FyTx) + H_d(FyTx,FyTy) \\ &\leq |x-y| + \sup_{z\in Tx}|z|H_d(Fx,Fy) + \sup_{v\in Fy}|v|H_d(Tx,Ty) \\ &\leq |x-y| + \sup_{z\in Tx}|z|l|x-y| + \sup_{v\in Fy}|v|H_d(Tx,Ty) \\ &\leq M\left\{|x-y| + H_d(Tx,Ty)\right\}. \end{aligned}$$

Hence, $D_{\mathcal{G}}(\Phi(x,Tx),\Phi(y,Ty)) \leq MD_{\mathcal{G}}((x,Tx),(y,Ty))$.

This completes the proof. □

Remark 5.42. In the Lemma 5.41, equality may not generally hold. For instance, consider T to be a Weierstrass function whose Hausdorff dimension is strictly greater than 1 (refer [86]) and F to be the zero function. Then, we obtain $1 = \dim_H(\mathcal{G}(FT)) < \dim_H(\mathcal{G}(T))$.

Definition 5.43. Consider W to be a bounded and closed interval of $\mathbb{R}$ and $F : W \rightrightarrows \mathbb{R}$ is a set-valued map. The maximum range of F over the rectangle W is defined as

$$R_F[W] = \sup_{x,y\in W} \sup_{w,z\in F(x)\cup F(y)} |w-z|.$$

As indicated in the introductory section, next, we now provide a set-valued analog of [36, Proposition 11.1].

Proposition 5.44. *Assume $F : [a,b] \rightrightarrows \mathbb{R}$ be a set-valued continuous map, $0 < \eta < x-y$, and $\frac{x-y}{\eta} \leq m \leq 1 + \frac{x-y}{\eta}$ for some $m \in \mathbb{N}$. If $N_\eta(G_F)$ is the number of η-boxes* *that intersect the graph of F, then*

$$\frac{1}{\eta}\sum_{j=1}^{m} R_F[A_j] \leq N_\eta(G_F) \leq 2m + \frac{1}{\eta}\sum_{j=1}^{m} R_F[A_j],$$

where $A_j = [j\eta, (j+1)\eta]$.

Proof. The count of squares having side length η in the part above A_j intersecting the graph of F is at least $\frac{R_F[A_j]}{\eta}$ and at most $2 + \frac{R_F[A_j]}{\eta}$, using the continuity of F. Taking the sum over all such parts yields the required bounds. □

Example 5.45. Consider $F : [0,1] \rightrightarrows \mathbb{R}$ to be a set-valued map defined as $F(x) = [-1,1]$. By Proposition 5.44, we have

$$\overline{\dim}_B(G_F) = \overline{\lim_{\eta\to 0}} \frac{\log N_\eta(G_F)}{-\log(\eta)} \leq \overline{\lim_{\eta\to 0}} \frac{\log\left(2m + \frac{1}{\eta}\sum_{j=1}^{m} R_F[A_i]\right)}{-\log(\eta)} \leq \overline{\lim_{\eta\to 0}} \frac{\log\left(2m + \frac{1}{\eta}\sum_{j=1}^{m} 2\right)}{-\log(\eta)} = 2,$$

because $R_F[A_j] = 2$ for each $j \in \Sigma_m$ and $A_j = [j\eta, (j+1)\eta]$. Similarly,

$$\underline{\dim}_B(G_F) = \underline{\lim_{\eta\to 0}} \frac{\log N_\eta(G_F)}{-\log(\eta)} \geq \underline{\lim_{\eta\to 0}} \frac{\log\left(\frac{1}{\eta}\sum_{j=1}^{m} R_F[A_i]\right)}{-\log(\eta)} = \underline{\lim_{\eta\to 0}} \frac{\log\left(\frac{1}{\eta}\sum_{j=1}^{m} 2\right)}{-\log(\eta)} = 2.$$

Therefore, $\dim_B(G_F) = 2$. This shows that Proposition 5.44 is useful in estimating or finding box dimensions of set-valued functions.

5.5 Conclusion

In this chapter, we have introduced the term α-fractal function, corresponding to set-valued maps (Theorem 5.2). We have observed that a set-valued α-fractal function is generally not interpolatory, unlike a single-valued α-fractal function. However, under certain conditions, it can exhibit interpolatory behavior (Remark 5.4, Note

5.5). We have also examined some properties of this fractal function (Theorem 5.7, Theorem 5.12). Additionally, we have established the existence of a fractal polynomial that approximates a convex set-valued map (Theorem 5.19). We have introduced the concept of constrained approximation for set-valued maps (Theorem 5.24). Furthermore, we have defined the graph of a set-valued map (Definition 5.28) and calculated the fractal dimension of this graph for a certain class of set-valued maps (Theorem 5.36, Lemma 5.38, and Lemma 5.41).

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