Chapter 5

Approximating fixed point results for enriched $\acute{C}$irić-Reich-Rus contraction and enriched Kannan contraction in quasi-Banach space

This chapter presents a few convergence results for the Krasnoselskii-Mann iteration used for enriched $\acute{C}$irić-Reich-Rus contraction and enriched Kannan contraction in the quasi-Banach space. Also, it presents Maia-type fixed point results for enriched $\acute{C}$irić-Reich-Rus contraction and enriched Kannan contraction mappings in the quasi-Banach space.

5.1 Introduction

The purpose of this chapter is to present some approximating fixed point results using Krasnoselskii-Mann iteration for enriched $\acute{C}$irić-Reich-Rus contraction and enriched Kannan contraction in the framework of quasi-Banach space, which are the actual generalization of the previous results in the Banach space in literatures [20, 21]. Also, we have established the Maia-type fixed point results for enriched $\acute{C}$irić-Reich-Rus contraction and enriched Kannan contraction mappings in the quasi-Banach space, which extend the results of Berinde [13] in the Banach space. Several examples and numerical calculations are provided to demonstrate the results.

Let $(X, \|.\|)$ be a normed linear space and $T : X \to X$ be a mapping satisfying the following condition, independently considered to establish fixed point results in 1971 by $\acute{C}$irić [2], Reich [78], and Rus [80];

$$\|Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) \quad \text{for all} \quad x, y \in X. \tag{5.1}$$

where $a, b \geq 0$ and $a + 2b < 1$.

If $a = 0$, then (5.1) converts to Kannan contraction condition:

$$\|Tx - Ty\| \leq b(\|x - Tx\| + \|y - Ty\|) \quad \text{for all} \quad x, y \in X. \tag{5.2}$$

For the completeness of a quasi-normed space, the following significant result highlights the distinction between Cauchy sequences in the quasi-Banach space and Cauchy sequences in the Banach space.

Lemma 5.1. [65] *Let $(X, \|.\|)$ be a quasi-norm space with a quasi-triangle constant $C \geq 1$ and X is said to be complete (quasi-Banach space) if and only if for every series such that $\Sigma_{n=1}^{\infty}\|x_n\| < \infty$ we have $\sum_{n=1}^{\infty}\|x_n\| \in X$ and*

$$\left\|\sum_{n=1}^{\infty} x_n\right\| \leq \sum_{n=1}^{\infty} C^{n+1}\|x_n\|.$$

Several well-known examples of quasi-Banach spaces are the space l_p and $L_p(0,1)$ when $0 < p < 1$, which are p-normable. Also, for Ulam's stability of the nonlinear impulsive differential equations of n-th order, which can be assisted by some examples of quasi-Banach spaces, we refer to Zada and Mashal [100]. Our main results will be discussed later by concluding this section with the following useful lemma.

Lemma 5.2. [69] *Let $\{x_n\}$ be a sequence in quasi-normed space $(X, \|.\|)$ and there exists $\gamma \in [0,1)$ such that*

$$\|x_{n+1} - x_n\| \leq \gamma\|x_n - x_{n-1}\|,$$

then $\{x_n\}$ is a Cauchy sequence.

5.1.1 Delineation

The current chapter is structured as follows: in Section 5.2 we have defined enriched Ćirić-Reich-Rus contractions in the quasi-Banach space and obtained the approximating fixed point results and unifying error estimations and in Section 5.3 we have presented the existence and approximation of fixed point results for enriched Kannan contraction in the quasi-Banach space. In Section 5.4, we have presented Maia-type fixed point theorems for enriched Ćirić-Reich-Rus contraction and enriched Kannan contraction mappings in the quasi-Banach space.

5.2 Enriched Ćirić-Reich-Rus contractions in the quasi-Banach space

Definition 5.3. A map $T : X \to X$ on the quasi-Banach space $(X, \|.\|)$ is said to be a (K, a, b)-enriched Ćirić-Reich-Rus contraction if there exists $k \in [0, \infty)$ and $a, b \geq 0$, satisfying $a + 2b < 1$, such that

$$\|k(x-y) + Tx - Ty\| \leq a\|x-y\| + b(\|x - Tx\| + \|y - Ty\|) \quad \text{for all} \;\; x, y \in X. \tag{5.3}$$

Obviously any enriched Ćirić-Reich-Rus contraction is a Ćirić-Reich-Rus contraction with $k = 0$. Also, if $b = 0$, then from (5.3) we obtain an enriched contraction.

An averaged mapping T_λ, where $\lambda \in (0, 1]$ is obtained by considering a self-mapping T on a convex subset E of a linear space X, and it is defined by

$$T_\lambda x = (1-\lambda)x + \lambda Tx, \quad \text{for all} \;\; x \in X,$$

and it satisfies $Fix(T_\lambda) = Fix(T)$, where $Fix(T)$ is the fixed points set of T.

Theorem 5.4. *Suppose $(X, \|.\|)$ be a quasi-Banach space and $T : X \to X$ be a (k, a, b)- enriched Ćirić-Reich-Rus contraction satisfying quasi-triangle constant C with $1 \leq C < \min\{\frac{1}{b}, \frac{1-a}{2b}\}$. Then,*

(i) $Fix(T) = \{x\}$*, for some* $x \in X$*;*

(ii) the iterative scheme

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, n \geq 0,$$

converges to x for any $x_0 \in X$, and for all $\lambda \in (0, 1]$;

(iii) and the following estimation holds good:

$$\|x_{n+i-1} - x\| \leq \frac{\delta^i}{1-\delta}\|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots; \quad i = 1, 2, 3, \ldots; \quad \text{where,} \quad \delta = \frac{a + bC}{1 - bC}.$$

Proof. If $k = 0$, then $\lambda = 1$. Therefore, the Krasnoselskii-Mann iteration converts to the Picard iteration.

Let us choose $k > 0$, therefore, $\lambda = \frac{1}{k+1}$ lies in $(0, 1)$. Then, from the relation (5.3), we have

$$\left\|(\frac{1}{\lambda} - 1)(x - y) + Tx - Ty\right\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|).$$

Since the averaged operator is given by, $T_\lambda(x) = (1 - \lambda)x + \lambda Tx$, therefore, we have

$$\|T_\lambda x - T_\lambda y\| \leq a\lambda\|x - y\| + b(\lambda\|x - Tx\| + \lambda\|y - Ty\|). \tag{5.4}$$

Again, since $a\lambda < a$ and $\|T_\lambda x - x\| = \|\lambda x - \lambda Tx\|$, therefore, (5.4) can be re-written as

$$\|T_\lambda x - T_\lambda y\| \leq a\|x - y\| + b(\|T_\lambda x - x\| + \|T_\lambda y - y\|). \tag{5.5}$$

Now, using the quasi-triangle inequality in (5.5) we obtain

$$\|T_\lambda x - T_\lambda y\| \leq a\|x - y\| + b\,C\Big(\|T_\lambda x - y\| + \|x - y\| + \|T_\lambda y - T_\lambda x\| + \|T_\lambda x - y\|\Big).$$

Thus, we have

$$\|T_\lambda x - T_\lambda y\| \leq \frac{a + bC}{1 - bC}\|x - y\| + \frac{2bC}{1 - bC}\|T_\lambda x - y\|. \tag{5.6}$$

Let us choose $x = x_{n-1}$ and $y = x_n$. Then, $x_{n+1} = T_\lambda y$ and $x_n = T_\lambda x$ and (5.6) becomes,

$$\|x_{n+1} - x_n\| \leq \frac{a + bC}{1 - bC}\|x_n - x_{n-1}\|. \tag{5.7}$$

Since $1 \leq C < \min\{\frac{1}{b}, \frac{1-a}{2b}\}$ therefore, $C < \frac{1}{b}$ and $C < \frac{1-a}{2b}$, which implies $1 - bC > 0$ and $\frac{a+bC}{1-bC} < 1$. Let us choose $\delta = \frac{a+bC}{1-bC}$ then, $\delta < 1$. Now, (5.7) can be re-written as

$$\|x_{n+1} - x_n\| \leq \delta\|x_n - x_{n-1}\|, \quad \text{where } \delta \in [0, 1). \tag{5.8}$$

Therefore, from Lemma 5.2 we can conclude that $\{x_n\}$ is a Cauchy sequence in the quasi-Banach space X.

Let us consider $m > n$.

Now,

$$\begin{aligned}\|x_{m+n} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \Big(\delta^{n+m-1} + \delta^{n+m-2} + \cdots + \delta^n\Big)\|x_1 - x_0\| \\ &= \delta^n\Big(1 + \delta + \delta^2 + \cdots + \delta^{m-1}\Big)\|x_1 - x_0\|.\end{aligned}$$

Thus, we have

$$\|x_{m+n} - x_n\| \leq \frac{\delta^n}{1-\delta}\Big(1 - \delta^m\Big)\|x_1 - x_0\|. \tag{5.9}$$

Again,

$$\begin{aligned}\|x_{m+n} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \Big(\delta^m + \delta^{m-1} + \cdots + \delta\Big)\|x_n - x_{n-1}\| \\ &= \frac{\delta}{1-\delta}\Big(1 - \delta^m\Big)\|x_n - x_{n-1}\|. \qquad (5.10)\end{aligned}$$

Since the sequence $\{x_n\}$ is a Cauchy sequence in the quasi-Banach space $(X, \|.\|)$ and therefore, it converges. Let us denote

$$\lim_{n\to\infty} x_n = x. \tag{5.11}$$

Now, we prove that x is a fixed point of T_λ. We have

$$\begin{aligned}\|x - T_\lambda x\| &\leq C\Big(\|x - x_{n+1}\| + \|x_{n+1} - T_\lambda x\|\Big) \\ &= C\Big(\|x - x_{n+1}\| + \|T_\lambda x_n - T_\lambda x\|\Big) \\ &\leq C\Big(\|x - x_{n+1}\| + \frac{a + bC}{1 - bC}\|x_n - x\| + \frac{2bC}{1 - bC}\|x_{n+1} - x\|\Big). \quad \text{(by using (5.6))}\end{aligned}$$

Now taking the limit as $n \to \infty$ we get,

$$\|x - T_\lambda x\| \leq 0,$$

which is absurd. Therefore, $x = T_\lambda x$, i.e., x is a fixed point of T_λ. In order to prove that T_λ has a unique fixed point, let us consider that T_λ has two fixed points, say x and y. Then $T_\lambda x = x$ and $T_\lambda y = y$.

Now,

$$\begin{aligned}\|y - x\| &= \|T_\lambda y - T_\lambda x\| \\ &\leq \delta \|y - x\|. \quad \text{(by using (5.8))}\end{aligned}$$

If $\|y - x\| \neq 0$, then $\delta \geq 1$, which is a contradiction. Therefore, $x = y$. Since, $Fix(T_\lambda) = Fix(T)$, therefore, $Fix(T) = \{x\}$, (i) is proved and the conclusion (ii) follows by (5.11).

To prove (iii) we let $m \to \infty$ in (5.9) and (5.10) to get

$$\|x_n - x\| \leq \frac{\delta^n}{1-\delta} \|x_1 - x_0\| \tag{5.12}$$

and

$$\|x_n - x\| \leq \frac{\delta}{1-\delta} \|x_n - x_{n-1}\| \tag{5.13}$$

respectively, and then we merge (5.12) and (5.13) to obtain the unifying error estimation as

$$\|x_{n+i-1} - x\| \leq \frac{\delta^i}{1-\delta} \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots; \quad i = 1, 2, 3, \ldots$$

Hence the desired result. □

Example 5.5. Let $X = \mathbb{R}^2$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$, consider the functional $\|.\|_{a,p}$

$$\|x\|_{s,p} = \begin{cases} \|x\|_p = (|x_1|^p + \|x_2\|^p)^{\frac{1}{p}}, & \text{if } x_2 \neq 0 \\ s|x_1| & \text{if } x_2 = 0, \end{cases}$$

where $p \geq 1$ and $s \neq 1$. Here $\|.\|_{s,p}$ is a quasi-norm on X, with $C = \max\{s, \frac{1}{s}\}$; see Maligranda [65]. Let $E = [0,1] \times [0,1]$ and T be an operator on E defined by

$$T(x_1, x_2) = (-x_1 - x_2, 0) \quad \text{for all} \quad (x_1, x_2) \in E.$$

Suppose $x = (x_1, x_2) = (1, 0)$ and $y = (y_1, y_2) = (0, 0)$. Therefore, $Tx = (-1, 0)$ and $Ty = (0, 0)$. Now, $\|Tx - Ty\| = \|(-1, 0) - (0, 0)\| = \|(-1, 0)\| = s$
Further,

$$\begin{aligned} a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) &= a\|(1, 0)\| + b\Big(\|(2, 0)\| + \|(0, 0)\|\Big) \\ &= as + 2bs \end{aligned}$$

If T is a Ćirić-Reich-Rus contraction then, $s \leq as + 2bs$, implying $a + 2b \geq 1$, (assume $s \neq 0$) which is a contradiction. Therefore, T is not Ćirić-Reich-Rus contraction
Now,

$$\begin{aligned} \|k(x - y) + Tx - Ty\| &= \|k(x_1 - y_1, x_2 - y_2) + (-x_1 - x_2, 0) - (-y_1 - y_2, 0)\| \\ &= \|(x_1 - y_1)(k - 1) - (x_2 - y_2), k(x_2 - y_2)\| \\ &= |(x_1 - y_1)(k - 1) - (x_2 - y_2)| + |k(x_2 - y_2)| \\ &\leq |(x_1 - y_1)(k - 1)| + |x_2 - y_2| + |k(x_2 - y_2)| \\ &= |k - 1||x_1 - y_1| + k|x_2 - y_2| + |x_2 - y_2|. \end{aligned}$$

Again,

$$\begin{aligned} a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) &= a\|(x_1, x_2) - (y_1, y_2)\| + b\Big(\|(x_1, x_2) - (-x_1 - x_2, 0)\| + \\ &\qquad \|(y_1, y_2) - (-y_1 - y_2, 0)\|\Big) \\ &= a\|(x_1 - y_1, x_2 - y_2)\| + b\Big(\|(2x_1 + x_2, x_2)\| + \|(2y_1 + y_2, y_2)\|\Big) \\ &= a\Big(|x_1 - y_1| + |x_2 - y_2|\Big) + b\Big(|2x_1 + x_2| + |x_2| + |2y_1 + y_2| + |y_2|\Big) \\ &= a(|x_1 - y_1| + |x_2 - y_2|) + 2b(x_1 + x_2 + y_1 + y_2). \end{aligned}$$

If we choose $a = \max\{k, |k - 1|\}$ and choose b such that $|x_2 - y_2| \leq 2b(x_1 + x_2 + y_1 + y_2)$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in E$ and $a + 2b < 1$ then,

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) \text{ for all } x, y \in E.$$

Therefore, T is an enriched Ćirić-Reich-Rus contraction on E. Now, we choose quasi-triangle constant C with $1 \leq C < \min\{\frac{1}{b}, \frac{1-a}{2b}\}$. Thereby, from the Theorem 5.4 we can conclude that an iterative scheme $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\ n \geq 0$, where $\lambda \in (0,1]$ converges to a fixed point of T.

5.3 Enriched Kannan contraction in the quasi-Banach space

Now we define enriched Kannan contraction by putting $a = 0$ in (5.3) of enriched Ćirić-Reich-Rus contraction in the quasi-Banach space as follows:

Definition 5.6. A map $T : X \to X$ on the quasi-Banach space $(X, \|.\|)$ is said to be a (k, b)-enriched Kannan contraction if there exists $k \in [0, \infty)$ and $b \geq 0$, satisfying $b < \frac{1}{2}$, such that

$$\|k(x-y) + Tx - Ty\| \leq b(\|x - Tx\| + \|y - Ty\|) \ \text{ for all } \ x, y \in X. \tag{5.14}$$

Obviously, any enriched Kannan contraction is a Kannan contraction with $k = 0$.

Theorem 5.7. *Let $(X, \|.\|)$ be a quasi-Banach space and $T : X \to X$ be a (k, b)- enriched Kannan contraction satisfying quasi-triangle constant C with $1 \leq C < \frac{1}{2b}$. Then,*

(i) $Fix(T) = \{x\}$, for some $x \in X$;

(ii) the iterative scheme

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, n \geq 0,$$

converges to x for any $x_0 \in X$, for all $\lambda \in (0, 1]$;

(iii) and the following estimation holds good:

$$\|x_{n+i-1} - x\| \leq \frac{\delta^i}{1-\delta}\|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots; \quad i = 1, 2, 3, \ldots; \quad \textit{where}, \quad \delta = \frac{bC}{1 - bC}.$$

Proof. The proof is similar to the Theorem 5.4 by putting $a = 0$, so we omit it. □

5.4 Maia-type fixed point theorems

The following fixed point result can be used in the situation where T is not a contraction, but there exists $N \in \mathbb{N}$ such that T^N is a certain contraction.

Theorem 5.8. *Suppose $T : X \to X$ be an operator on the quasi-Banach space $(X, \|.\|)$ and there exists $N \in \mathbb{N}$ such that T^N is (k, a, b)-enriched Ćirić-Reich-Rus contraction satisfying quasi-triangle constant C with $1 \leq C < \min\{\frac{1}{b}, \frac{1-a}{2b}\}$. Then,*

(i) $Fix(T) = \{x\}$, for some $x \in X$;

(ii) the iterative scheme

$$x_{n+1} = (1 - \lambda)x_n + \lambda T^N x_n, n \geq 0,$$

converges to x for any $x_0 \in X$, for all $\lambda \in (0, 1]$.

Proof. Since T^N is (k, a, b)-enriched Ćirić-Reich-Rus contraction therefore, from Theorem 5.4 we obtain $Fix(T^N) = \{x\}$ for some $x \in X$.
Now,

$$T^N(T(x)) = T^{N+1}(x) = T(T^N(x)) = T(x),$$

which implies $T(x) \in Fix(T^N)$. Due to the uniqueness of the fixed point of T^N, we obtain $T(x) = x$ and so $x \in Fix(T)$. The rest of the part of the proof follows by the Theorem 5.4. □

In addition to the contraction mapping principle, the Maia-type fixed point theorem [64] is one of the most interesting generalizations, which was obtained by splitting the assumptions among two metrics on the same set.

Theorem 5.9. [81] *Let X be a non-empty set and d and ρ be two metrics on X and T be a self-map on X satisfying the following conditions:*

(i) X is complete space with respect to the metric d;

(ii) T is continuous in regard to d;

(iii) T is a contraction with respect to ρ.

Then, T is a Picard operator.

Now, our aim is to generalize the Theorem 5.4, and hence, we deduce the Maia-type fixed point results for (k, a, b)-enriched Ćirić-Reich-Rus contraction in the quasi-Banach space.

Theorem 5.10. *Assume that X is a quasi-Banach space with norm* $\|.\|$ *and metric d satisfy the following relation:*

$$d(x, y) \leq \|x - y\| \quad \text{for all} \quad x, y \in X. \tag{5.15}$$

Suppose

(i) T is continuous in regard to d;

(ii) X is complete space with respect to the metric d;

(iii) T is a (k, a, b)*- enriched Ćirić-Reich-Rus contraction with respect to* $\|.\|$ *satisfying quasi-triangle constant* $1 \leq C < \min\{\frac{1}{b}, \frac{1-a}{2b}\}$.

Then the following conclusions hold:

(i) $Fix(T) = \{x\}$, *for some* $x \in X$;

(ii) the iterative scheme

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \; n \geq 0,$$

converges to x for any $x_0 \in X$, *and for all* $\lambda \in (0, 1]$;

(iii) and the following estimation holds good:

$$d(x_{n+i-1}, x) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots; \quad i = 1, 2, 3, \ldots; \quad \text{where,} \quad \delta = \frac{a + bC}{1 - bC}.$$

Proof. Let us choose $k > 0$, therefore, $\lambda = \frac{1}{k+1}$ lies in $(0, 1)$ (the case $k = 0$ is immediate). Therefore, from the inequality (5.3), we have

$$\left\| (\frac{1}{\lambda} - 1)(x - y) + Tx - Ty \right\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|).$$

Since the averaged operator is given by, $T_\lambda(x) = (1-\lambda)x + \lambda Tx$, therefore, we have

$$\|T_\lambda x - T_\lambda y\| \leq a\lambda\|x-y\| + b(\lambda\|x-Tx\| + \lambda\|y-Ty\|). \tag{5.16}$$

Again, since $a\lambda < a$ and $\|T_\lambda x - x\| = \|\lambda x - \lambda Tx\|$, therefore, (5.16) can be re-written as,

$$\|T_\lambda x - T_\lambda y\| \leq a\|x-y\| + b(\|T_\lambda x - x\| + \|T_\lambda y - y\|). \tag{5.17}$$

Now, using the quasi-triangle inequality in (5.17) we obtain

$$\|T_\lambda x - T_\lambda y\| \leq a\|x-y\| + b\,C\Big(\|T_\lambda x - y\| + \|x-y\| + \|T_\lambda y - T_\lambda x\| + \|T_\lambda x - y\|\Big).$$

Therefore, we have

$$\|T_\lambda x - T_\lambda y\| \leq \frac{a+bC}{1-bC}\|x-y\| + \frac{2bC}{1-bC}\|T_\lambda x - y\|. \tag{5.18}$$

Let us choose $x = x_{n-1}$ and $y = x_n$. Then, $x_{n+1} = T_\lambda y$ and $x_n = T_\lambda x$ and (5.18) becomes,

$$\|x_{n+1} - x_n\| \leq \frac{a+bC}{1-bC}\|x_n - x_{n-1}\|, \tag{5.19}$$

and the Krasnoselskii-Mann iteration process is exactly the Picard iteration associated with the averaged operator T_λ, i.e.,

$$x_{n+1} = T_\lambda x_n,\ n \geq 1. \tag{5.20}$$

Since $1 \leq C < \min\{\frac{1}{b}, \frac{1-a}{2b}\}$ therefore, $C < \frac{1}{b}$ and $C < \frac{1-a}{2b}$, which implies $1 - bC > 0$ and $\frac{a+bC}{1-bC} < 1$. Let us choose $\delta = \frac{a+bC}{1-bC}$ then, $\delta < 1$.
Now, (5.19) can be re-written as

$$\|x_{n+1} - x_n\| \leq \delta\|x_n - x_{n-1}\|, \quad \text{where } \delta \in [0,1). \tag{5.21}$$

By (5.21), one can routinely obtain the following two estimations:

$$\|x_{m+n} - x_n\| \leq \frac{\delta^n}{1-\delta}\Big(1-\delta^m\Big)\|x_1 - x_0\| \tag{5.22}$$

and,

$$\|x_{m+n} - x_n\| \leq \frac{\delta}{1-\delta}\Big(1-\delta^m\Big)\|x_n - x_{n-1}\|. \tag{5.23}$$

Now, by (5.15) and (5.22) we have

$$d(x_{m+n}, x_n) \leq \|x_{m+n} - x_n\| \leq \frac{\delta^n}{1-\delta}\Big(1-\delta^m\Big)\|x_1 - x_0\|, \tag{5.24}$$

which shows that $\{x_n\}$ is a Cauchy sequence in (X, d) and hence it is convergent due to completeness of X. Let us denote

$$x = \lim_{n\to\infty} x_n. \tag{5.25}$$

Since T is continuous with respect to d therefore, T_λ is also continuous with respect to d. Thereby, taking the limit on both sides of (5.20), we obtain $x \in Fix(T_\lambda)$.

In order to prove that T_λ has a unique fixed point, let us consider that T_λ has two fixed points, say x and y. Then $T_\lambda x = x$ and $T_\lambda y = y$.

Now,

$$\begin{aligned}\|y - x\| &= \|T_\lambda y - T_\lambda x\| \\ &\leq \delta\|y - x\| \quad \text{(by using (5.21))}.\end{aligned}$$

If $\|y - x\| \neq 0$, then $\delta \geq 1$, which is a contradiction. Therefore, $x = y$. Since, $Fix(T_\lambda) = Fix(T)$, therefore, $Fix(T) = \{x\}$, (i) is proved and the conclusion (ii) follows by (5.25).

To prove (iii) we first observe that by combining (5.22), (5.23), and (5.15), one can obtain

$$d(x_{m+n}, x_n) \leq \frac{\delta^n}{1-\delta}\Big(1-\delta^m\Big)d(x_1, x_0) \tag{5.26}$$

and,

$$d(x_{m+n}, x_n) \leq \frac{\delta}{1-\delta}\Big(1-\delta^m\Big)d(x_n, x_{n-1}). \tag{5.27}$$

Now we let $m \to \infty$ in (5.26) and (5.27) respectively, and then we merge them to obtain the unifying error estimation

$$d(x_{n+i-1}, x) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots; \quad i = 1, 2, 3, \dots$$

Hence the desired result. □

Now, our aim is to extend Theorem 5.7, and hence, we obtain the Maia fixed point theorem for enriched Kannan contraction in the quasi-Banach space.

Theorem 5.11. *Let X be a quasi-Banach space endowed with a metric d and a norm satisfying the following condition*

$$d(x, y) \leq \|x - y\| \quad \textit{for all} \quad x, y \in X. \tag{5.28}$$

Suppose

(i) X is a complete space with respect to the metric d;

(ii) T is continuous with regards to d;

(iii) T is a (k, a, b)- enriched Kannan contraction with respect to $\|.\|$ satisfying quasi-triangle constant C with $1 \leq C < \frac{1}{2b}$.

Then the following conclusions hold:

(i) $Fix(T) = \{x\}$, for some $x \in X$;

(ii) the iterative scheme

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \; n \geq 0,$$

converges to x for any $x_0 \in X$ and for all $\lambda \in (0, 1]$;

(iii) and the following estimation holds good:

$$d(x_{n+i-1}, x) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots; \quad i = 1, 2, 3, \dots; \quad \textit{where,} \quad \delta = \frac{bC}{1-bC}.$$

Proof. It is a particular case of Theorem 5.10 by putting $a = 0$, so we omit it. □

5.5 Concluding remarks

In this chapter, we have presented the concept of enriched Ćirić-Reich-Rus contraction and Kannan contraction in the quasi-Banach space and obtained the approximating fixed point results using the Krasnoselskii-Mann iteration, which is the generalization of the main result in Berinde and Păcurar [20, 21] concerning the approximating fixed points of enriched Ćirić-Reich-Rus contraction and Kannan contraction in the Banach space.

Furthermore, we have obtained the Maia-type fixed point results for enriched Ćirić-Reich-Rus contraction and enriched Kannan contraction mappings in the quasi-Banach space, which extend Maia-type fixed point results in the Banach space [13].

It will be interesting to see if one can define enriched Ćirić contraction and enriched Hardy-Rogers contraction in the quasi-Banach space and generalize the approximating fixed point results for these two contractions in the quasi-Banach space from the Banach space.

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