

Chapter 3

Convolutions of generalized distributions Involving the Weinstein Transform

3.1 Introduction

Convolution is a mathematical operation that combines two functions to produce a third function that represents the integral of the pointwise multiplication of the original functions. In the context of generalized distributions, the convolution operation plays a pivotal role in many areas of mathematics, physics, and engineering. The theory of generalized distributions and convolution provides a powerful mathematical framework for analyzing and solving problems in diverse fields. Its flexibility and generality make it indispensable in theoretical studies as well as practical applications, contributing significantly to the advancement of mathematical analysis and its applications in science and engineering.

Convolution finds extensive applications in various domains. In signal processing, Simmart et al. [62] show that convolution is used for filtering and modulation operations. In probability theory [39], it plays a crucial role in defining the probability distributions of sums of random variables. Moreover, the convolution of distributions inherits certain properties from the distributions being convolved.

Ultradistributions have been introduced by Beurling [5], Björck [8], Roumieu [53] and Hörmander [22] as generalizations of Schwartz distributions. A unification of Beurling-Björck theory and Roumieu theory has been given by Komatsu [29]. Hörmander [22] introduced the spaces of types $D_\omega, D'_\omega, \mathcal{E}_\omega, \mathcal{E}'_\omega$ and examined their various properties by utilising the theory of Fourier transform. Björck [8] studied many properties of convolutions of a test function and a distribution and gave many important results. The Hankel transform of ultradistributions in Roumieu setting has been investigated by Pathak and Pandey [42]. Later on, Pathak and Pandey [46] found Gevrey-type ultradistributions which are suitable for the study of Hankel transform and Hankel convolution transform. S types of spaces have also been introduced by Gel'fand and Shilov [12], and by exploiting the theory of the Fourier transform, many properties have been studied.

The Weinstein transform contains efficient calculus regarding convolution. Due to its nice mathematical background, it has the scope of applications in many areas of the mathematical sciences and engineering. Recently, many results regarding the Weinstein convolution have been done by many authors. In this connections, Mehraj [30] proved Paley-Wiener theorem for the Weinstein transform; Mejjali et al. [31, 32] examined continuous property of the Weinstein wavelet transform and introduced Weinstein Gabor transform. Salhi [33] found the uncertainty principle, Nahia and Salem [58, 59] introduced a mean value property and discussed spherical harmonics. Salem and Nasr [56] discussed Heisenberg type inequalities by exploiting the Weinstein transform and Weinstein convolution.

In the present chapter, motivated by the above results, the author provides various properties of convolutions of generalized distributions by utilising the theory of the Weinstein transform.

3.2 Convolutions of generalized distributions

In this section, by utilizing the theory of the Weinstein transform, we examine various properties of the convolutions of generalized distributions.

Definition 3.2.1. Let $\mathcal{E}_\omega(\Omega)$ be the set of all complex valued functions ϕ in Ω such that

$$\phi\psi \in D_\omega(\Omega) \quad (\forall \psi \in D_\omega(\Omega)).$$

Definition 3.2.2. Let $D'_\omega(\Omega)$ be the space of all linear functionals u on $D_\omega(\Omega)$, where Ω is open subset of \mathbb{R}_+^{n+1} and $\omega \in M$.

Theorem 3.2.3. *Let $u \in D'_\omega(\Omega)$ then*

$$\int_{\mathbb{R}_+^{n+1}} u(\xi)\phi(\xi) d\mu_\beta(\xi) = (u\#_\beta\check{\phi})(0) \quad \forall \phi \in D_\omega(\Omega), \quad (3.2.1)$$

where $\check{\phi}(x) = \phi(-x)$.

Proof: Let $u \in D'_\omega(\Omega)$ and $\phi \in D_\omega(\Omega)$ then from (1.4.7), we have

$$(u\#_\beta\phi)(x) = \int_{\mathbb{R}_+^{n+1}} \phi(y)u(x, y) d\mu_\beta(y).$$

From (1.4.7), we have

$$(u\#_{\beta}\phi)(x) = \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} u(z) D(x, y, z) d\mu_{\beta}(z) \right) d\mu_{\beta}(y).$$

Taking (1.4.8), we get

$$\begin{aligned} (u\#_{\beta}\phi)(x) &= \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} u(z) \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1}\xi_{n+1}) \right. \right. \\ &\quad \times e^{i\langle y', \xi' \rangle} J_{\beta}(y_{n+1}\xi_{n+1}) e^{-i\langle z', \xi' \rangle} J_{\beta}(z_{n+1}\xi_{n+1}) d\mu_{\beta}(\xi) \left. \left. \right) \right. \\ &\quad \times d\mu_{\beta}(z) \left. \right) d\mu_{\beta}(y). \end{aligned}$$

Using Fubini's theorem, we get

$$\begin{aligned} (u\#_{\beta}\phi)(x) &= \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} \phi(y) e^{i\langle y', \xi' \rangle} J_{\beta}(y_{n+1}\xi_{n+1}) d\mu_{\beta}(y) \right) \left(\int_{\mathbb{R}_+^{n+1}} u(z) \right. \\ &\quad \times e^{-i\langle z', \xi' \rangle} J_{\beta}(z_{n+1}\xi_{n+1}) d\mu_{\beta}(z) \left. \right) e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1}\xi_{n+1}) d\mu_{\beta}(\xi). \end{aligned}$$

Using (1.4.6), we get

$$(u\#_{\beta}\phi)(x) = \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(-\xi) (\mathcal{F}_w u)(\xi) e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1}\xi_{n+1}) d\mu_{\beta}(\xi).$$

Therefore,

$$(u\#_{\beta}\phi)(x) = \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) (\mathcal{F}_w\phi)(-\xi) e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1}\xi_{n+1}) d\mu_{\beta}(\xi).$$

Putting $x = 0$, we get

$$(u\#_{\beta}\phi)(0) = \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) (\mathcal{F}_w\phi)(-\xi) d\mu_{\beta}(\xi).$$

Thus

$$(u\#_{\beta}\check{\phi})(0) = \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_{\omega}u)(\xi)(\mathcal{F}_{\omega}\phi)(\xi) d\mu_{\beta}(\xi).$$

By using Parseval formula of the Weinstein transform (1.4.3), we get

$$(u\#_{\beta}\check{\phi})(0) = \int_{\mathbb{R}_+^{n+1}} u(\xi)\phi(\xi) d\mu_{\beta}(\xi).$$

Hence

$$(u\#_{\beta}\check{\phi})(0) = \int_{\mathbb{R}_+^{n+1}} u(\xi)\phi(\xi) d\mu_{\beta}(\xi). \quad (3.2.2)$$

Definition 3.2.4. Let $u \in L^1_{loc}(\Omega)$ then for u with the element in $D'_{\omega}(\Omega)$ which is given by

$$\langle u, \phi \rangle = \int_{\mathbb{R}_+^{n+1}} u(\xi)\phi(\xi) d\mu_{\beta}(\xi) \quad \forall \phi \in D_{\omega}(\Omega). \quad (3.2.3)$$

Definition 3.2.5. Let $\omega \in M$. If $u \in D'_{\omega}$, $\phi \in D_{\omega}$ then the convolution $u\#_{\beta}\phi$ is defined by

$$(u\#_{\beta}\phi)(x) = \langle u(y), \phi(x, y) \rangle = \langle u, \tau_x\check{\phi} \rangle. \quad (3.2.4)$$

Theorem 3.2.6. Let $\omega \in M$. Let $\phi, \psi \in D_{\omega}$ and $u \in D'_{\omega}$ then

$$((u\#_{\beta}\phi)\#_{\beta}\psi) = (u\#_{\beta}(\phi\#_{\beta}\psi)).$$

Proof: Let $\omega \in M$, and $\phi, \psi \in D_{\omega}$ for given $\epsilon > 0$ we form the Riemann sum

$$f_{\epsilon}(x) = \epsilon^{n+1} \sum_t (\tau_{\epsilon t}\phi)(x)\psi(\epsilon t) = \epsilon^{n+1} \sum_t \phi(\epsilon t, x)\psi(\epsilon t). \quad (3.2.5)$$

where t is integer co-ordinates. We claim that $f_\epsilon(x)$ converges to $(\phi \#_\beta \psi)(x)$ in D_ω as $\epsilon \rightarrow 0$.

$$\begin{aligned} (\mathcal{F}_w f_\epsilon)(\xi) &= \mathcal{F}_w \left(\epsilon^{n+1} \sum_t (\tau_{\epsilon t} \phi)(x) \psi(\epsilon t) \right) (\xi) \\ &= \epsilon^{n+1} \sum_t \mathcal{F}_w \left((\tau_{\epsilon t} \phi)(x) \psi(\epsilon t) \right) (\xi) \\ &= \epsilon^{n+1} \sum_t \psi(\epsilon t) \mathcal{F}_w (\tau_{\epsilon t} \phi)(\xi). \end{aligned}$$

Utilising (2.2.20), we get

$$(\mathcal{F}_w f_\epsilon)(\xi) = \epsilon^{n+1} \sum_t \psi(\epsilon t) e^{-i\langle \xi', \epsilon t' \rangle} J_\beta(\xi_{n+1} \epsilon t_{n+1}) (\mathcal{F}_w \phi)(\xi) = R_\epsilon(\xi) (\mathcal{F}_w \phi)(\xi), \quad (3.2.6)$$

where

$$R_\epsilon(\xi) = \epsilon^{n+1} \sum_t \psi(\epsilon t) e^{-i\langle \xi', \epsilon t' \rangle} J_\beta(\xi_{n+1} \epsilon t_{n+1}),$$

is the Riemann sum for the integral defining $\mathcal{F}_w \psi$ which tends to 0 as $\epsilon \rightarrow 0$.

Next,

$$\|f_\epsilon - (\phi \#_\beta \psi)\|_{\lambda, \omega} = \int_{\mathbb{R}_+^{n+1}} e^{\lambda \omega(\xi)} |\mathcal{F}_w (f_\epsilon - (\phi \#_\beta \psi))(\xi)| d\mu_\beta(\xi).$$

For linear property of the Weinstein transform, we get

$$\|f_\epsilon - (\phi \#_\beta \psi)\|_{\lambda, \omega} = \int_{\mathbb{R}_+^{n+1}} e^{\lambda \omega(\xi)} |(\mathcal{F}_w f_\epsilon)(\xi) - \mathcal{F}_w (\phi \#_\beta \psi)(\xi)| d\mu_\beta(\xi).$$

Then from (1.4.13), we get

$$\begin{aligned} \|f_\epsilon - (\phi \#_\beta \psi)\|_{\lambda, \omega} &= \int_{\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |R_\epsilon(\xi)(\mathcal{F}_w\phi)(\xi) - (\mathcal{F}_w\phi)(\xi)(\mathcal{F}_w\psi)(\xi)| d\mu_\beta(\xi) \\ &\leq \int_{\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |(\mathcal{F}_w\phi)(\xi)| |R_\epsilon(\xi) - (\mathcal{F}_w\psi)(\xi)| d\mu_\beta(\xi) \\ &= \int_{\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |(\mathcal{F}_w\phi)(\xi)| |(\mathcal{F}_w\psi)(\xi) - R_\epsilon(\xi)| d\mu_\beta(\xi), \end{aligned}$$

then, above tends to 0 as $(\mathcal{F}_w\phi)(\xi) \rightarrow 0$ when $|\xi| \rightarrow \infty$.

Therefore

$$f_\epsilon \rightarrow \phi \#_\beta \psi \text{ as } \epsilon \rightarrow 0. \quad (3.2.7)$$

Now from (3.2.4), we have

$$(u \#_\beta (\phi \#_\beta \psi))(x) = \langle u(y), (\phi \#_\beta \psi)(x, y) \rangle.$$

Taking (3.2.7), we get

$$(u \#_\beta (\phi \#_\beta \psi))(x) = \langle u(y), f_\epsilon(x, y) \rangle.$$

From (3.2.5), the above implies

$$\begin{aligned} (u \#_\beta (\phi \#_\beta \psi))(x) &= \left(\langle u(y), \lim_{\epsilon \rightarrow 0} \epsilon^{n+1} \sum_t (\tau_{\epsilon t} \phi)(x, y) \psi(\epsilon t) \rangle \right) \\ &= \left(\langle u(y), \lim_{\epsilon \rightarrow 0} \epsilon^{n+1} \sum_t \phi(\epsilon t, x, y) \psi(\epsilon t) \rangle \right) \\ &= \lim_{\epsilon \rightarrow 0} \epsilon^{n+1} \sum_t \langle u(y), \phi(\epsilon t, x, y) \rangle \psi(\epsilon t). \end{aligned}$$

From (3.2.4), the above yields

$$\begin{aligned} (u\#_{\beta}(\phi\#_{\beta}\psi))(x) &= \lim_{\epsilon \rightarrow 0} \epsilon^{n+1} \sum_t (u\#_{\beta}\phi)(\epsilon t, x) \psi(\epsilon t) \\ &= \lim_{\epsilon \rightarrow 0} \epsilon^{n+1} \sum_t \tau_{\epsilon t}(u\#_{\beta}\phi)(x) \psi(\epsilon t). \end{aligned}$$

Therefore, from (3.2.5), we have

$$(u\#_{\beta}(\phi\#_{\beta}\psi))(x) = ((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(x).$$

Lemma 3.2.7. *Let $\omega \in M$. If $u \in D'_{\omega}$ and $\phi \in D_{\omega}$ then*

$$(\mathcal{F}_{\omega}v)(\eta) = ((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(0), \quad (3.2.8)$$

where $v(\eta) = \psi(\eta)(u\#_{\beta}\phi)(\eta)$ and $\check{\psi}_{\xi}(\eta) = e^{-i\langle \eta', \xi' \rangle} J_{\beta}(\eta_{n+1}\xi_{n+1})\psi(\eta)$.

Proof: By (1.4.12), we have

$$((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(x) = \int_{\mathbb{R}_+^{n+1}} (u\#_{\beta}\phi)(x, y) \check{\psi}_{\xi}(y) d\mu_{\beta}(y).$$

Using (1.4.7), the above yields

$$\begin{aligned} ((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(x) &= \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} (u\#_{\beta}\phi)(z) D_{\beta}(x, y, z) d\mu_{\beta}(z) \right) \\ &\quad \times \check{\psi}_{\xi}(y) d\mu_{\beta}(y). \end{aligned}$$

From (1.4.8), we have

$$\begin{aligned} ((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(x) &= \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} (u\#_{\beta}\phi)(z) \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \eta' \rangle} J_{\beta}(x_{n+1}\eta_{n+1}) \right. \right. \\ &\quad \times e^{i\langle y', \eta' \rangle} J_{\beta}(y_{n+1}\eta_{n+1}) e^{-i\langle z', \eta' \rangle} J_{\beta}(z_{n+1}\eta_{n+1}) d\mu_{\beta}(\eta) \Big) \\ &\quad \left. \times d\mu_{\beta}(z) \right) \check{\psi}_{\xi}(y) d\mu_{\beta}(y). \end{aligned}$$

By Fubini's theorem, we find that

$$\begin{aligned} ((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(x) &= \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} (u\#_{\beta}\phi)(z) e^{-i\langle z', \eta' \rangle} J_{\beta}(z_{n+1}\eta_{n+1}) d\mu_{\beta}(z) \right) \\ &\quad \times \left(\int_{\mathbb{R}_+^{n+1}} e^{i\langle y', \eta' \rangle} J_{\beta}(y_{n+1}\eta_{n+1}) \check{\psi}_{\xi}(y) d\mu_{\beta}(y) \right) e^{-i\langle x', \eta' \rangle} \\ &\quad \times J_{\beta}(x_{n+1}\eta_{n+1}) d\mu_{\beta}(\eta). \end{aligned}$$

In view of (1.4.1), we obtain

$$\begin{aligned} ((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(x) &= \int_{\mathbb{R}_+^{n+1}} \mathcal{F}_w(u\#_{\beta}\phi)(\eta) (\mathcal{F}_w\psi_{\xi})(\eta) e^{-i\langle x', \eta' \rangle} \\ &\quad \times J_{\beta}(x_{n+1}\eta_{n+1}) d\mu_{\beta}(\eta). \end{aligned}$$

Putting $x = 0$, we have

$$((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(0) = \int_{\mathbb{R}_+^{n+1}} \mathcal{F}_w(u\#_{\beta}\phi)(\eta) (\mathcal{F}_w\psi_{\xi})(\eta) d\mu_{\beta}(\eta).$$

Exploiting Parseval formula of the Weinstein transform (1.4.3), we get

$$((u\#_{\beta}\phi)\#_{\beta}\check{\psi}_{\xi})(0) = \int_{\mathbb{R}_+^{n+1}} (u\#_{\beta}\phi)(\eta) \psi_{\xi}(\eta) d\mu_{\beta}(\eta).$$

Substituting the value of $\psi_\xi(\eta)$, the above expression yeilds

$$\begin{aligned}
 ((u\#_\beta\phi)\#_\beta\check{\psi}_\xi)(0) &= \int_{\mathbb{R}_+^{n+1}} (u\#_\beta\phi)(\eta)e^{-i\langle\eta',\xi'\rangle}J_\beta(\eta_{m+1}\xi_{n+1})\psi(\eta)d\mu_\beta(\eta) \\
 &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle\eta',\xi'\rangle}J_\beta(\eta_{m+1}\xi_{n+1})\psi(\eta)(u\#_\beta\phi)(\eta)d\mu_\beta(\eta) \\
 &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle\eta',\xi'\rangle}J_\beta(\eta_{m+1}\xi_{n+1})v(\eta)d\mu_\beta(\eta) \\
 &= (\mathcal{F}_wv)(\eta).
 \end{aligned}$$

Therefore,

$$(\mathcal{F}_wv)(\eta) = (u\#_\beta\phi\#_\beta\check{\psi}_\xi)(0).$$

Lemma 3.2.8. *Let $\phi \in D_\omega$ and $u \in D'_\omega$. If supp ϕ is contained in fixed compact set then there exists a constant C such that*

$$|\langle u, \phi \rangle| \leq C\|\phi\|_{\lambda,\omega}. \quad (3.2.9)$$

Proof: Let $u \in D'_\omega$, $\phi \in D_\omega$ then from (3.2.3), we write

$$\langle u, \phi \rangle = \int_{\mathbb{R}_+^{n+1}} u(\xi)\phi(\xi)d\mu_\beta(\xi).$$

Taking Parseval formula (1.4.3) of the Weinstein transform, we have

$$\begin{aligned}
 \langle u, \phi \rangle &= \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_wu)(\xi)(\mathcal{F}_w\phi)(\xi)d\mu_\beta(\xi) \\
 &= \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\xi)(\mathcal{F}_wu)(\xi)d\mu_\beta(\xi).
 \end{aligned}$$

Then,

$$\begin{aligned}
 |\langle u, \phi \rangle| &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) e^{\lambda\omega(\xi)} e^{-\lambda\omega(\xi)} (\mathcal{F}_w \phi)(\xi) d\mu_\beta(\xi) \right| \\
 &\leq \sup_{\xi \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w \phi)(\xi) e^{\lambda\omega(\xi)}| \int_{\mathbb{R}_+^{n+1}} |e^{-\lambda\omega(\xi)} (\mathcal{F}_w u)(\xi)| d\mu_\beta(\xi) \\
 &\leq \sup_{\xi \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w \phi)(\xi)| e^{\lambda\omega(\xi)} \int_{\mathbb{R}_+^{n+1}} |e^{-\lambda\omega(\xi)} (\mathcal{F}_w u)(\xi)| d\mu_\beta(\xi) \\
 &= C \|\phi\|_{\lambda, \omega},
 \end{aligned}$$

where $C = \int_{\mathbb{R}_+^{n+1}} |e^{-\lambda\omega(\xi)} (\mathcal{F}_w u)(\xi)| d\mu_\beta(\xi)$.

Therefore

$$|\langle u, \phi \rangle| \leq C \|\phi\|_{\lambda, \omega}.$$

Theorem 3.2.9. *Let $\omega \in M$. If $u \in D'_\omega$ and $\phi \in D_\omega$ then,*

$$u \#_\beta \phi \in \mathcal{E}_\omega. \quad (3.2.10)$$

Proof: Let $v(x) = \psi(x)(u \#_\beta \phi)(x)$ and

$$\psi_\xi(x) = e^{-i\langle x', \xi' \rangle} J_\beta(x_{n+1} \xi_{n+1}) \psi(x) \quad (3.2.11)$$

From (3.2.8), we have

$$(\mathcal{F}_w v)(\eta) = (u \#_\beta \phi) \#_\beta \check{\psi}_\xi(0).$$

Using Theorem 3.2.6 we get

$$(\mathcal{F}_w v)(\eta) = u \#_\beta (\phi \#_\beta \check{\psi}_\xi)(0).$$

Therefore, from (3.2.1) and (3.2.3), above yields

$$(\mathcal{F}_w v)(\eta) = \langle u, (\phi \#_{\beta} \check{\psi}_{\xi}) \rangle$$

So that,

$$|(\mathcal{F}_w v)(\eta)| = |\langle u, (\phi \#_{\beta} \check{\psi}_{\xi}) \rangle| = |\langle u, (\check{\phi} \#_{\beta} \psi_{\xi}) \rangle|.$$

From (3.2.9), then we obtain

$$|(\mathcal{F}_w v)(\eta)| \leq C \| \check{\phi} \#_{\beta} \psi_{\xi} \|_{\lambda, \omega}, \quad (3.2.12)$$

for some $\lambda > 0$ and C is independent of η .

In view of (2.2.14), we have

$$|(\mathcal{F}_w v)(\eta)| \leq C \sup_{\eta \in \mathbb{R}_+^{n+1}} |\mathcal{F}_w(\check{\phi} \#_{\beta} \psi_{\xi})(\eta)| e^{\lambda \omega(\eta)}.$$

Using (1.4.13), we obtain

$$|(\mathcal{F}_w v)(\eta)| \leq C \sup_{\eta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w(\check{\phi})(\eta)(\mathcal{F}_w \psi_{\xi})(\eta))| e^{\lambda \omega(\eta)}. \quad (3.2.13)$$

Next, we have

$$(\mathcal{F}_w \psi_{\xi})(\eta) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \eta' \rangle} J_{\beta}(x_{n+1} \eta_{n+1}) \psi_{\xi}(x) d\mu_{\beta}(x).$$

From (3.2.11), we find

$$(\mathcal{F}_w \psi_{\xi})(\eta) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \eta' \rangle} J_{\beta}(x_{n+1} \eta_{n+1}) e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1} \xi_{n+1}) \psi(x) d\mu_{\beta}(x). \quad (3.2.14)$$

By using (1.4.1) and (1.4.8), we get

$$\begin{aligned}
 (\mathcal{F}_w \psi_\xi)(\eta) &= \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \zeta' \rangle} J_\beta(x_{n+1} \zeta_{n+1}) D_\beta(\eta, \zeta, \xi) d\mu_\beta(\zeta) \right) \\
 &\quad \times \psi(x) d\mu_\beta(x).
 \end{aligned}$$

From Fubini's theorem we find

$$\begin{aligned}
 (\mathcal{F}_w \psi_\xi)(\eta) &= \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \zeta' \rangle} J_\beta(x_{n+1} \zeta_{n+1}) \psi(x) d\mu_\beta(x) \right) \\
 &\quad \times D_\beta(\eta, \zeta, \xi) d\mu_\beta(\zeta) \\
 &= \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w \psi)(\zeta) D_\beta(\eta, \zeta, \xi) d\mu_\beta(\xi).
 \end{aligned}$$

Then,

$$\begin{aligned}
 |(\mathcal{F}_w \psi_\xi)(\eta)| &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w \psi)(\zeta) D_\beta(\eta, \zeta, \xi) d\mu_\beta(\xi) \right| \\
 &\leq \sup_{\zeta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w \psi)(\zeta)| \cdot \left| \int_{\mathbb{R}_+^{n+1}} D_\beta(\eta, \zeta, \xi) d\mu_\beta(\xi) \right| \\
 &\leq \sup_{\zeta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w \psi)(\zeta)|, \text{ as } \int_{\mathbb{R}_+^{n+1}} D_\beta(\eta, \zeta, \xi) d\mu_\beta(\xi) = 1.
 \end{aligned}$$

Therefore,

$$|(\mathcal{F}_w \psi_\xi)(\eta)| \leq \sup_{\eta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w \psi)(\eta)|. \tag{3.2.15}$$

Using (3.2.13) and (3.2.15), we can find

$$\begin{aligned}
 & |(\mathcal{F}_w v)(\eta)| \\
 & \leq C \sup_{\eta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w(\check{\phi})(\eta)(\mathcal{F}_w \psi)(\eta)|e^{\lambda\omega(\eta)} \\
 & \leq C \left(\sup_{\eta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w(\check{\phi})(\eta)|e^{\lambda\omega(\eta)} \right) \left(\sup_{\eta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w \psi)(\eta)|e^{\lambda\omega(\eta)} \right) e^{-\lambda\omega(\eta)} \\
 & = C \|\check{\phi}\|_{\lambda, \omega} \|\psi\|_{\lambda, \omega} e^{-\lambda\omega(\eta)}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 |(\mathcal{F}_w v)(\eta)| & \leq C \|\check{\phi}\|_{\lambda, \omega} \|\psi\|_{\lambda, \omega} e^{-\lambda\omega(\eta)} \\
 & < \infty.
 \end{aligned}$$

Therefore,

$$v(x) = \psi(x)(u \#_{\beta} \phi)(x) \in D_{\omega}.$$

Hence,

$$u \#_{\beta} \phi \in \mathcal{E}_{\omega}.$$

Lemma 3.2.10. *Let ϕ and ψ in D_{ω} . Then*

$$\phi \#_{\beta} \psi = \psi \#_{\beta} \phi. \tag{3.2.16}$$

Proof: Let ϕ and $\psi \in D_{\omega}$. Then

$$(\phi \#_{\beta} \psi)(\xi) = \int_{\mathbb{R}_+^{n+1}} \phi(\eta) \psi(\eta, \xi) d\mu_{\beta}(\eta).$$

From (1.4.7), we have

$$\begin{aligned} (\phi \#_{\beta} \psi)(\xi) &= \int_{\mathbb{R}_+^{n+1}} \phi(\eta) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) D_{\beta}(\eta, \xi, z) d\mu_{\beta}(z) \right) d\mu_{\beta}(\eta) \\ &= \int_{\mathbb{R}_+^{n+1}} \phi(\eta) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) D_{\beta}(z, \xi, \eta) d\mu_{\beta}(z) \right) d\mu_{\beta}(\eta). \end{aligned}$$

By Fubini's theorem, we can obtain

$$\begin{aligned} (\phi \#_{\beta} \psi)(\xi) &= \int_{\mathbb{R}_+^{n+1}} \psi(z) \left(\int_{\mathbb{R}_+^{n+1}} \phi(\eta) D_{\beta}(z, \xi, \eta) d\mu_{\beta}(\eta) \right) d\mu_{\beta}(z) \\ &= \int_{\mathbb{R}_+^{n+1}} \psi(z) \phi(z, \xi) d\mu_{\beta}(z) \\ &= (\psi \#_{\beta} \phi)(\xi). \end{aligned}$$

Therefore,

$$\phi \#_{\beta} \psi = \psi \#_{\beta} \phi.$$

Theorem 3.2.11. *Let $\phi, \psi \in D_{\omega}$, then for any multi-index α we have*

$$(\Delta_{W, \beta}^n)^{\alpha} \phi \#_{\beta} \psi = \phi \#_{\beta} (\Delta_{W, \beta}^n)^{\alpha} \psi = (\Delta_{W, \beta}^n)^{\alpha} (\phi \#_{\beta} \psi). \quad (3.2.17)$$

Proof: Let $\phi, \psi \in D_{\omega}$. Then from convolution of the Weinstein transform (1.4.12), we have

$$((\Delta_{W, \beta}^n)^{\alpha} \phi \#_{\beta} \psi)(\eta) = \int_{\mathbb{R}_+^{n+1}} \psi(\xi) (\Delta_{W, \beta}^n)^{\alpha} \phi(\xi, \eta) d\mu_{\beta}(\xi).$$

Using (1.4.7), we get

$$\begin{aligned}
 & ((\Delta_{W,\beta}^n)_\eta^\alpha \phi \#_\beta \psi)(\eta) \\
 &= \int_{\mathbb{R}_+^{n+1}} \psi(\xi) (\Delta_{W,\beta}^n)_\eta^\alpha \left(\int_{\mathbb{R}_+^{n+1}} \phi(z) D_\beta(\xi, \eta, z) d\mu_\beta(z) \right) d\mu_\beta(\xi) \\
 &= (\Delta_{W,\beta}^n)_\eta^\alpha \left(\int_{\mathbb{R}_+^{n+1}} \psi(\xi) \left(\int_{\mathbb{R}_+^{n+1}} \phi(z) D_\beta(\xi, \eta, z) d\mu_\beta(z) \right) d\mu_\beta(\xi) \right) \\
 &= (\Delta_{W,\beta}^n)_\eta^\alpha \int_{\mathbb{R}_+^{n+1}} \psi(\xi) \phi(\xi, \eta) d\mu_\beta(\xi) \\
 &= (\Delta_{W,\beta}^n)_\eta^\alpha (\phi \#_\beta \psi)(\eta).
 \end{aligned}$$

Therefore,

$$(\Delta_{W,\beta}^n)_\eta^\alpha \phi \#_\beta \psi = (\Delta_{W,\beta}^n)_\eta^\alpha (\phi \#_\beta \psi).$$

Now,

$$(\phi \#_\beta (\Delta_{W,\beta}^n)_\eta^\alpha \psi)(\eta) = \int_{\mathbb{R}_+^{n+1}} \phi(\xi) (\Delta_{W,\beta}^n)_\eta^\alpha (\psi)(\xi, \eta) d\mu_\beta(\xi).$$

In veiw of (1.4.7), we find

$$\begin{aligned}
 & (\phi \#_\beta (\Delta_{W,\beta}^n)_\eta^\alpha \psi)(\eta) \\
 &= \int_{\mathbb{R}_+^{n+1}} \phi(\xi) (\Delta_{W,\beta}^n)_\eta^\alpha \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) D_\beta(\xi, \eta, z) d\mu_\beta(z) \right) d\mu_\beta(\xi) \\
 &= (\Delta_{W,\beta}^n)_\eta^\alpha \left(\int_{\mathbb{R}_+^{n+1}} \phi(\xi) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) D_\beta(\xi, \eta, z) d\mu_\beta(z) \right) d\mu_\beta(\xi) \right) \\
 &= (\Delta_{W,\beta}^n)_\eta^\alpha \int_{\mathbb{R}_+^{n+1}} \phi(\xi) \psi(\xi, \eta) d\mu_\beta(\xi) \\
 &= (\Delta_{W,\beta}^n)_\eta^\alpha (\phi \#_\beta \psi)(\eta).
 \end{aligned}$$

This implies that

$$\phi \#_\beta (\Delta_{W,\beta}^n)_\eta^\alpha \psi = (\Delta_{W,\beta}^n)_\eta^\alpha \phi \#_\beta \psi = (\Delta_{W,\beta}^n)_\eta^\alpha (\phi \#_\beta \psi).$$

Definition 3.2.12. Let ω be the real-valued continuous function defined on \mathbb{R}_+^{n+1} , then the class of all ω is denoted by M , which satisfies the following conditions:

$$0 = \omega(0) = \lim_{x \rightarrow 0} \omega(x) \leq \omega(x + y) \leq \omega(x) + \omega(y), \forall x, y \in \mathbb{R}_+^{n+1}. \quad (3.2.18)$$

$$J_n(\omega) = \int_{|y| \geq 1} \frac{\omega(y)}{|y|^{n+1}} d\mu_\beta(y) < \infty. \quad (3.2.19)$$

$$a + b \log(1 + |y|) \leq \omega(y), \forall y \in \mathbb{R}_+^{n+1}, a \in \mathbb{R}, \text{ and } b > 0. \quad (3.2.20)$$

Definition 3.2.13. Let $\omega \in M$ and $\lambda \geq 0$. Then for each multi-index α the space $S_\omega(\mathbb{R}_+^{n+1})$ is defined to be the set of all complex-valued infinitely differentiable function ϕ on \mathbb{R}_+^{n+1} such that

$$p_{\alpha, \lambda}(\phi) = \sup_{x \in \mathbb{R}_+^{n+1}} e^{\lambda \omega(x)} |(\Delta_{W, \beta}^n)^\alpha \phi(x)| < \infty \quad (3.2.21)$$

and

$$\pi_{\alpha, \lambda}(\phi) = \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda \omega(\xi)} |(\Delta_{W, \beta}^n)^\alpha (\mathcal{F}_w \phi)(\xi)| < \infty. \quad (3.2.22)$$

$S_\omega(\mathbb{R}_+^{n+1})$ is called the Schwartz space and it forms a vector space over the field of complex numbers.

Definition 3.2.14. $S'_\omega(\mathbb{R}_+^{n+1})$ is defined as the set of all linear continuous maps f from $S_\omega(\mathbb{R}_+^{n+1})$ to \mathbb{C} . We say that $S'_\omega(\mathbb{R}_+^{n+1})$ is the dual of $S_\omega(\mathbb{R}_+^{n+1})$.

Lemma 3.2.15. Let $\phi, \psi \in L^1_\beta(\mathbb{R}_+^{n+1})$. Then

$$\mathcal{F}_w(\phi\psi) = \mathcal{F}_w\phi \#_\beta \mathcal{F}_w\psi. \quad (3.2.23)$$

Proof: Let $\xi \in \mathbb{R}_+^{n+1}$. Then from (1.4.6), we have

$$\mathcal{F}_w^{-1}(\phi \#_{\beta} \psi)(\xi) = \mathcal{F}_w(\phi \#_{\beta} \psi)(-\xi).$$

By using (1.4.13), we get

$$\begin{aligned} \mathcal{F}_w^{-1}(\phi \#_{\beta} \psi)(\xi) &= \mathcal{F}_w \phi(-\xi) \mathcal{F}_w \psi(-\xi) \\ &= (\mathcal{F}_w^{-1} \phi)(\xi) (\mathcal{F}_w^{-1} \psi)(\xi). \end{aligned}$$

Therefore,

$$\mathcal{F}_w^{-1}(\phi \#_{\beta} \psi)(\xi) = (\mathcal{F}_w^{-1} \phi)(\xi) (\mathcal{F}_w^{-1} \psi)(\xi).$$

Now, we replace $\phi \rightarrow \mathcal{F}_w \phi$ and $\psi \rightarrow \mathcal{F}_w \psi$ we get

$$\begin{aligned} \mathcal{F}_w^{-1}(\mathcal{F}_w \phi \#_{\beta} \mathcal{F}_w \psi)(\xi) &= \mathcal{F}_w^{-1}(\mathcal{F}_w \phi)(\xi) \mathcal{F}_w^{-1}(\mathcal{F}_w \psi)(\xi) \\ &= \phi \psi. \end{aligned}$$

Therefore,

$$\mathcal{F}_w^{-1}(\mathcal{F}_w(\phi) \#_{\beta} \mathcal{F}_w(\psi))(\xi) = \phi(\xi) \psi(\xi) = (\phi \psi)(\xi).$$

Hence, by the properties of the Weinstein transform (1.4.2) and (1.4.6), we have

$$\mathcal{F}_w \phi \#_{\beta} \mathcal{F}_w \psi = \mathcal{F}_w(\phi \psi).$$

Theorem 3.2.16. *Let $\omega \in M$, then S_{ω} is topological algebra under point-wise multiplication .*

Proof: Let $\phi, \psi \in S_\omega$ then we have

$$p_{\alpha,\lambda}(\phi\psi) = \sup_{x \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(x)} |(\Delta_{W,\beta}^n)_x^\alpha(\phi(x)\psi(x))|. \quad (3.2.24)$$

Using (1.4.15), we have

$$\begin{aligned} (\Delta_{W,\beta}^n)_x^\alpha[\phi(x)\psi(x)] &= \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \\ &\quad \times \frac{1}{\rho'!} E'_{\beta,r} x_{n+1}^{r-\alpha} (D_x^{\rho'+q} \phi(x)) \cdot (D_x^{\rho'+2\delta'+r-q} \psi(x)). \end{aligned}$$

Then from (6.2.10), we find

$$\begin{aligned} p_{\alpha,\lambda}(\phi\psi) &= \sup_{x \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(x)} \left| \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \right. \\ &\quad \left. \times \frac{1}{\rho'!} E'_{\beta,r} x_{n+1}^{r-\alpha} (D_x^{\rho'+q} \phi(x)) \cdot (D_x^{\rho'+2\delta'+r-q} \psi(x)) \right| \\ &\leq \sup_{x \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(x)} \left| \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \right. \\ &\quad \left. \times \frac{1}{\rho'!} E'_{\beta,r} (D_x^{\rho'+q} \phi(x)) e^{k\omega(x)} (D_x^{\rho'+2\delta'+r-q} \psi(x)) \right|, \text{ for some } k \in \mathbb{N} \\ &\leq \sup_{x \in \mathbb{R}_+^{n+1}} \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \\ &\quad \times \frac{1}{\rho'!} E'_{\beta,r} e^{\frac{\lambda\omega(x)}{2}} |(D_x^{\rho'+q} \phi(x))| e^{\frac{\lambda+2k}{2}\omega(x)} |(D_x^{\rho'+2\delta'+r-q} \psi(x))| \\ &\leq \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \frac{1}{\rho'!} E'_{\beta,r} \\ &\quad \times \sup_{x \in \mathbb{R}_+^{n+1}} e^{\frac{\lambda\omega(x)}{2}} |(D_x^{\rho'+q} \phi(x))| \sup_{x \in \mathbb{R}_+^{n+1}} e^{\frac{\lambda+2k}{2}\omega(x)} |(D_x^{\rho'+2\delta'+r-q} \psi(x))|. \end{aligned}$$

In view of (3.2.21) and (1.4.16), we get

$$\begin{aligned}
 p_{\alpha,\lambda}(\phi\psi) &\leq \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'|\leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \\
 &\quad \times p_{\rho'+q, \frac{\lambda}{2}}(\phi) p_{\rho'+2\delta'+r-q, \frac{\lambda+2k}{2}}(\psi) \\
 &< \infty.
 \end{aligned}$$

This implies that

$$p_{\alpha,\lambda}(\phi\psi) < \infty.$$

Next, we have to prove that $\pi_{\alpha,\lambda}(\phi\psi) < \infty$. For this we take

$$\pi_{\alpha,\lambda}(\phi\psi) = \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |(\Delta_{W,\beta}^n)_\xi^\alpha [\mathcal{F}_w(\phi\psi)(\xi)]|. \quad (3.2.25)$$

From (3.2.23), we find

$$\pi_{\alpha,\lambda}(\phi\psi) = \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |(\Delta_{W,\beta}^n)_\xi^\alpha [(\mathcal{F}_w\phi) \#_\beta (\mathcal{F}_w\psi)(\xi)]|. \quad (3.2.26)$$

In view of (3.2.17), above yields

$$\pi_{\alpha,\lambda}(\phi\psi) = \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |(\mathcal{F}_w\phi) \#_\beta (\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\psi)(\xi)|. \quad (3.2.27)$$

The right hand side of (3.2.27), can be evaluated as

$$\begin{aligned}
 &|(\mathcal{F}_w\phi) \#_\beta ((\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\psi)(\xi))| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\eta) ((\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\psi)(\eta, \xi)) d\mu_\beta(\eta) \right|
 \end{aligned}$$

By using (1.4.7), we get

$$\begin{aligned}
 & |(\mathcal{F}_w\phi)\#_\beta((\Delta_{W,\beta}^n)_\xi^\alpha(\mathcal{F}_w\psi))(\xi)| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\eta) \left((\Delta_{W,\beta}^n)_\xi^\alpha \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\psi)(z) D_\beta(\eta, \xi, z) d\mu_\beta(z) \right) d\mu_\beta(\eta) \right| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\eta) \left(\int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\psi)(z) (\Delta_{W,\beta}^n)_\xi^\alpha D_\beta(\eta, \xi, z) d\mu_\beta(z) \right) d\mu_\beta(\eta) \right|.
 \end{aligned}$$

Using (1.4.8), we obtain

$$\begin{aligned}
 & |(\mathcal{F}_w\phi)\#_\beta((\Delta_{W,\beta}^n)_\xi^\alpha(\mathcal{F}_w\psi))(\xi)| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\eta) \left(\int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\psi)(z) (\Delta_{W,\beta}^n)_\xi^\alpha \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle \eta', y' \rangle} \right. \right. \right. \\
 &\quad \times J_\beta(\eta_{n+1}y_{n+1}) e^{i\langle \xi', y' \rangle} J_\beta(\xi_{n+1}y_{n+1}) e^{-i\langle z', y' \rangle} J_\beta(z_{n+1}y_{n+1}) \\
 &\quad \left. \left. \left. \times d\mu_\beta(y) \right) d\mu_\beta(z) \right) d\mu_\beta(\eta) \right| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\eta) \left(\int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\psi)(z) \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle \eta', y' \rangle} J_\beta(\eta_{n+1}y_{n+1}) \right. \right. \right. \\
 &\quad \times (-\|y\|^2)^\alpha e^{i\langle \xi', y' \rangle} J_\beta(\xi_{n+1}y_{n+1}) e^{-i\langle z', y' \rangle} J_\beta(z_{n+1}y_{n+1}) \\
 &\quad \left. \left. \left. \times d\mu_\beta(y) \right) d\mu_\beta(z) \right) d\mu_\beta(\eta) \right| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\eta) \left(\int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\psi)(z) (\Delta_{W,\beta}^n)_z^\alpha D_\beta(\eta, \xi, z) \right. \right. \\
 &\quad \left. \left. \times d\mu_\beta(z) \right) d\mu_\beta(\eta) \right|.
 \end{aligned}$$

Integrating by parts we get

$$\begin{aligned}
 & |(\mathcal{F}_w\phi)\#_\beta((\Delta_{W,\beta}^n)_\xi^\alpha(\mathcal{F}_w\psi))(\xi)| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w\phi)(\eta) \left(\int_{\mathbb{R}_+^{n+1}} (-1)^\alpha ((\Delta_{W,\beta}^n)_z^\alpha \mathcal{F}_w\psi)(z) D_\beta(\eta, \xi, z) \right. \right. \\
 &\quad \left. \left. \times d\mu_\beta(z) \right) d\mu_\beta(\eta) \right|
 \end{aligned}$$

Using (1.4.5), we get

$$\begin{aligned}
 & |(\mathcal{F}_w \phi) \#_{\beta} ((\Delta_{W,\beta}^n)_{\xi}^{\alpha} (\mathcal{F}_w \psi))(\xi)| \\
 & \quad \times d\mu_{\beta}(z) \Big) d\mu_{\beta}(\eta)| \\
 & \leq \int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w \phi)(\eta)| \left(\left| \int_{\mathbb{R}_+^{n+1}} ((\Delta_{W,\beta}^n)_z^{\alpha} \mathcal{F}_w \psi)(z) D_{\beta}(\eta, \xi, z) \right. \right. \\
 & \quad \left. \left. \times d\mu_{\beta}(z) \right) d\mu_{\beta}(\eta) \right| \\
 & \leq \sup_{z \in \mathbb{R}_+^{n+1}} (|(\Delta_{W,\beta}^n)_z^{\alpha} \mathcal{F}_w \psi)(z)| \int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w \phi)(\eta)| \left(\int_{\mathbb{R}_+^{n+1}} D_{\beta}(\eta, \xi, z) \right. \\
 & \quad \left. \times d\mu_{\beta}(z) \right) d\mu_{\beta}(\eta) \\
 & \leq \sup_{z \in \mathbb{R}_+^{n+1}} (|(\Delta_{W,\beta}^n)_z^{\alpha} \mathcal{F}_w \psi)(z)| \int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w \phi)(\eta)| d\mu_{\beta}(\eta).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & \left| (\mathcal{F}_w \phi) \#_{\beta} ((\Delta_{W,\beta}^n)_{\xi}^{\alpha} (\mathcal{F}_w \psi))(\xi) \right| \\
 & \leq \sup_{z \in \mathbb{R}_+^{n+1}} (|(\Delta_{W,\beta}^n)_z^{\alpha} \mathcal{F}_w \psi)(z)| \int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w \phi)(\eta)| d\mu_{\beta}(\eta). \quad (3.2.28)
 \end{aligned}$$

Using (3.2.27) and (3.2.28), we get

$$\begin{aligned}
 \pi_{\alpha,\lambda}(\phi\psi) & \leq \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |((\Delta_{W,\beta}^n)_{\xi}^{\alpha} \mathcal{F}_w \psi)(\xi)| \\
 & \quad \times \int_{\mathbb{R}_+^{n+1}} e^{l\omega(\eta)} e^{-l\omega(\eta)} |(\mathcal{F}_w \phi)(\eta)| d\mu_{\beta}(\eta).
 \end{aligned}$$

In view of (3.2.22), we get

$$\pi_{\alpha,\lambda}(\phi\psi) \leq \pi_{\alpha,\lambda}(\psi) \sup_{\eta \in \mathbb{R}_+^{n+1}} |(\mathcal{F}_w \phi)(\eta)| e^{l\omega(\eta)} \int_{\mathbb{R}_+^{n+1}} e^{-l\omega(\eta)} d\mu_{\beta}(\eta).$$

Again from (3.2.22), we can find

$$\begin{aligned} \pi_{\alpha,\lambda}(\phi\psi) &\leq \pi_{\alpha,\lambda}(\psi)\pi_{0,l}(\phi) \int_{\mathbb{R}_+^{n+1}} e^{-l\omega(\eta)} d\mu_\beta(\eta) \\ &< \infty. \end{aligned}$$

Hence,

$$\phi\psi \in S_\omega.$$

Theorem 3.2.17. *Let $\omega \in M$, then S_ω is topological algebra under convolution.*

Proof: Let ϕ and $\psi \in S_\omega$, then

$$p_{\alpha,\lambda}(\phi\#_\beta\psi) = \sup_{x \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(x)} |(\Delta_{W,\beta}^n)_x^\alpha(\phi\#_\beta\psi)(x)|. \quad (3.2.29)$$

Using (3.2.17), we get

$$p_{\alpha,\lambda}(\phi\#_\beta\psi) = \sup_{x \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(x)} |(\phi\#_\beta(\Delta_{W,\beta}^n)_x^\alpha\psi)(x)|. \quad (3.2.30)$$

Now,

$$|(\phi\#_\beta(\Delta_{W,\beta}^n)_x^\alpha\psi)(x)| = \left| \int_{\mathbb{R}_+^{n+1}} \phi(y)(\Delta_{W,\beta}^n)_x^\alpha\psi(x,y) d\mu_\beta(y) \right|.$$

In veiw of (1.4.12), we obtain

$$\begin{aligned} |(\phi\#_\beta(\Delta_{W,\beta}^n)_x^\alpha\psi)(x)| &= \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z)(\Delta_{W,\beta}^n)_x^\alpha D_\beta(x,y,z) \right. \right. \\ &\quad \left. \left. d\mu_\beta(z) \right) d\mu_\beta(y) \right|. \end{aligned}$$

Taking (1.4.8), we get

$$\begin{aligned}
 & |(\phi \#_{\beta} (\Delta_{W,\beta}^n)_x \psi)(x)| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) (\Delta_{W,\beta}^n)_x \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1} \xi_{n+1}) \right. \right. \right. \\
 &\quad \left. \left. \left. \times e^{i\langle y', \xi' \rangle} J_{\beta}(y_{n+1} \xi_{n+1}) e^{-i\langle z', \xi' \rangle} J_{\beta}(z_{n+1} \xi_{n+1}) d\mu_{\beta}(\xi) \right) d\mu_{\beta}(z) \right) d\mu_{\beta}(y) \right| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) \left(\int_{\mathbb{R}_+^{n+1}} (-\|\xi\|^2)^{\alpha} e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1} \xi_{n+1}) \right. \right. \right. \\
 &\quad \left. \left. \left. \times e^{i\langle y', \xi' \rangle} J_{\beta}(y_{n+1} \xi_{n+1}) e^{-i\langle z', \xi' \rangle} J_{\beta}(z_{n+1} \xi_{n+1}) d\mu_{\beta}(\xi) \right) d\mu_{\beta}(z) \right) d\mu_{\beta}(y) \right| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) \left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \xi' \rangle} J_{\beta}(x_{n+1} \xi_{n+1}) e^{i\langle y', \xi' \rangle} J_{\beta} \right. \right. \right. \\
 &\quad \left. \left. \left. \times (y_{n+1} \xi_{n+1}) (-\|\xi\|^2)^{\alpha} e^{-i\langle z', \xi' \rangle} J_{\beta}(z_{n+1} \xi_{n+1}) d\mu_{\beta}(\xi) \right) d\mu_{\beta}(z) \right) d\mu_{\beta}(y) \right| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} \psi(z) (\Delta_{W,\beta}^n)_z D_{\beta}(x, y, z) d\mu_{\beta}(z) \right) d\mu_{\beta}(y) \right| \\
 &= \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} (-1)^{\alpha} (\Delta_{W,\beta}^n)_z \psi(z) D_{\beta}(x, y, z) \right) d\mu_{\beta}(y) \right| \\
 &\leq \sup_{z \in \mathbb{R}_+^{n+1}} |(\Delta_{W,\beta}^n)_z \psi(z)| \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) \left(\int_{\mathbb{R}_+^{n+1}} D_{\beta}(x, y, z) d\mu_{\beta}(z) \right) d\mu_{\beta}(y) \right|.
 \end{aligned}$$

Therefore,

$$|(\phi \#_{\beta} (\Delta_{W,\beta}^n)_x \psi)(x)| \leq \sup_{z \in \mathbb{R}_+^{n+1}} |(\Delta_{W,\beta}^n)_z \psi(z)| \cdot \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) d\mu_{\beta}(y) \right|.$$

Using above expression in (3.2.30), we get

$$\begin{aligned}
 p_{\alpha,\lambda}(\phi\#\beta\psi) &\leq \sup_{z\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(z)} |(\Delta_{W,\beta}^n)_z^\alpha \psi(z)| \left| \int_{\mathbb{R}_+^{n+1}} \phi(y) d\mu_\beta(y) \right| \\
 &= p_{\lambda,\alpha}(\psi) \left| \int_{\mathbb{R}_+^{n+1}} e^{l\omega(y)} \phi(y) e^{-l\omega(y)} d\mu_\beta(y) \right| \\
 &\leq p_{\lambda,\alpha}(\psi) \sup_{y\in\mathbb{R}_+^{n+1}} |e^{l\omega(y)} \phi(y)| \left| \int_{\mathbb{R}_+^{n+1}} e^{-l\omega(y)} d\mu_\beta(y) \right| \\
 &\leq p_{\lambda,\alpha}(\psi) \sup_{y\in\mathbb{R}_+^{n+1}} e^{l\omega(y)} |\phi(y)| \left| \int_{\mathbb{R}_+^{n+1}} e^{-l\omega(y)} d\mu_\beta(y) \right| \\
 &= p_{\lambda,\alpha}(\psi) p_{l,0}(\phi) \left| \int_{\mathbb{R}_+^{n+1}} e^{-l\omega(y)} d\mu_\beta(y) \right| \\
 &< \infty.
 \end{aligned}$$

Next, we have to prove that $\pi_{\alpha,\lambda}(\phi\#\beta\psi) < \infty$. For this we take

$$\pi_{\alpha,\lambda}(\phi\#\beta\psi) = \sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |(\Delta_{W,\beta}^n)_\xi^\alpha \mathcal{F}_w(\phi\#\beta\psi)(\xi)|.$$

From (1.4.13), we have

$$\pi_{\alpha,\lambda}(\phi\#\beta\psi) = \sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |(\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\phi(\xi)\mathcal{F}_w\psi(\xi))|.$$

By using (1.4.15), we get

$$\begin{aligned}
 & \pi_{\alpha,\lambda}(\phi \#_{\beta} \psi) \\
 &= \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} \left| \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \right. \\
 & \times \frac{1}{\rho'!} E'_{\beta,r} \xi^{r-\alpha} (D_{\xi}^{\rho'+q}(\mathcal{F}_w \phi)(\xi)) (D_{\xi}^{\rho'+2\delta'+r-q}(\mathcal{F}_w \psi)(\xi)) \Big| \\
 & \leq \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} \left| \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \right. \\
 & \times \frac{1}{\rho'!} E'_{\beta,r} e^{k\omega(\xi)} (D_{\xi}^{\rho'+q}(\mathcal{F}_w) \phi(\xi)) (D_{\xi}^{\rho'+2\delta'+r-q}(\mathcal{F}_w) \psi(\xi)) \Big|
 \end{aligned}$$

for some $k \in \mathbb{N}$. Therefore we have

$$\begin{aligned}
 & \pi_{\alpha,\lambda}(\phi \#_{\beta} \psi) \\
 & \leq \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \\
 & \times \frac{1}{\rho'!} E'_{\beta,r} \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\frac{\lambda\omega(\xi)}{2}} |(D_{\xi}^{\rho'+q}(\mathcal{F}_w \phi)(\xi))| \\
 & \times \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\frac{(\lambda+2k)\omega(\xi)}{2}} |(D_{\xi}^{\rho'+2\delta'+r-q}(\mathcal{F}_w \psi)(\xi))|.
 \end{aligned}$$

In veiw of (3.2.22) and (1.4.16), we get

$$\begin{aligned}
 \pi_{\alpha,\lambda}(\phi \#_{\beta} \psi) & \leq \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{q=0}^r \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1, \delta_2, \dots, \delta_n \geq 0} \binom{\alpha}{j} \binom{r}{q} \binom{\alpha-j}{\delta_1, \delta_2, \dots, \delta_n} \\
 & \times \pi_{\rho'+q, \frac{\lambda}{2}}(\phi) \pi_{\rho'+2\delta'+r-q, \frac{\lambda+2k}{2}}(\psi) \\
 & < \infty.
 \end{aligned}$$

Hence,

$$\phi \#_{\beta} \psi \in S_{\omega}.$$
