CHAPTER 5

TEMPERATURE-RATE DEPENDENT THEORY OF POROTHERMOELASTICITY

5.1 Theory of Temperature-Rate Dependent Porothermoelasticity for Anisotropic Medium[1]

5.1.1 Introduction

The present chapter is focused on some aspects of porothermoelasticity based on temperature-rate dependent theory. The work is divided into different sections. This section is mainly devoted to the development of the temperature-rate dependent theory of porothermoelasticity for anisotropic medium. It must be mentioned here that the concept of porothermoelasticity is concerned with the study of the thermal and mechanical behaviour of elastic materials with porosity. This approach combines the theory of heat conduction with poroelastic constitutive equations by incorporating the coupling of the temperature field with the stresses and pore pressure. The mathematical modeling of the porothermoelastic material to various practical problems has attracted researchers in recent years due to its wide applications to geomechanics, mining and related areas where the coupled thermal and poro-mechanical processes play an essential role. Hence, because of the useful applications of the subject, this sub-chapter is motivated to derive

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the set of governing equations for the theory of temperature-rate dependent fluid saturated anisotropic poroelastic medium from the fundamental laws of thermodynamics.

The porothermoelasticity theory was first developed by Biot (1956a), where he considered the general class of solid material consisting of innumerable interconnected fluid saturated cavities and used Lagrange's equations to derive the stress-strain relation and the coupled equations of motion in the context of generalized poroelasticity. Biot's theory of porothermoelasticity has gained serious attention of researchers since last four decades. The work reported by Biot (1962a); Biot and Temple (1972); Rice and Cleary (1976); Kurashige (1989); Bear et al. (1992); Fourie and Du Plessis (2003) are worth to be mentioned here. However, the first porothermoelasticity theory for the isotropic medium, including one relaxation parameter and predicting the finite speed of thermal disturbance, has been established by Youssef (2007) and later on by Sherief and Hussein (2012). Theoretical development on porothermoelasticity and some interesting investigations in this respect have been already discussed in chapter 1 of the present thesis. Further, it is to be mentioned here that some studies on the temperature-rate dependent theory of porothermoelasticity for isotropic medium are recently reported by researchers to study the dynamical problems. Saeed et al. (2020) have solved one dimensional problem of isotropic medium using FEM. Alzahrani and Abbas (2020a), Guo Xiong (2021) have analyzed different thermomechanical problems in this context. The authors have directly stated the governing equations for the homogeneous and isotropic poroelastic medium in these studies. However, the theoretical foundation for the temperature-rate dependent theory of porothermoelasticity for the anisotropic medium is not available in the literature. Therefore, it is worth establishing a mathematical foundation for the temperature-rate dependent porothermoelasticity theory for the general anisotropic medium.

The objective of the present subchapter is to establish the basic governing equations and constitutive relations for the generalized temperature-rate dependent theory of porothermoelasticity (TRDPTE) for the general anisotropic medium by using the fundamental laws of thermodynamics. In order to formulate the theory, suitable porothermoelastic assumptions are made based on the temperature-rate dependent theory, and generalized forms of the fundamental laws of thermodynamics are used in the context of porothermoelasticity. Firstly, a general nonlinear theory of porothermoelasticity for the homogeneous and anisotropic medium is constructed. Further, some linearity assumptions are made to obtain the linear theory of TRDPTE for homogeneous and anisotropic porothermoelastic medium.

5.1.2 Formulation of the Theory

In this section, a porothermoelasticity theory involving the effects of temperature as well as temperature-rate is developed. Following Biot (1956a; 1962b; 1964) and Youssef (2007), the state of stress (σ_{ij}) at a point for the porothermoelastic materials is assumed as

$$\sigma_{ij} = (1-\gamma)\tau^*_{ij} + \gamma p \delta_{ij}. \tag{5.1.1}$$

The porothermoelastic materials involve two different phases of the material, solid and liquid. Here, σ_{ij} is the effective stress, τ^*_{ij} and p denoting the stress on the solid phase and fluid pressure, respectively, and γ is the porosity of the material. The subscripted indices i, j vary from 1 to 3. From Eq. (5.1.1), the effective state of stress can be rewritten as the sum of stress due to solid phase (τ_{ij}) and fluid phase (τ), i.e.

$$\sigma_{ij} = \tau_{ij} + \tau\delta_{ij}, \tag{5.1.2}$$

where $\tau_{ij} = (1-\gamma)\tau^*_{ij}$ and $\tau = \gamma p$.

Now, by assuming that u_i is the displacement due to solid phase and U_i is the displacement for the fluid phase, the state of strain due to solid phase (e_{ij}) and fluid phase (ε) are defined in the usual way as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \text{ and } \varepsilon = U_{i,i}. \tag{5.1.3}$$

Next, by denoting ρ^{s*} and ρ^{f*} as the densities due to solid and fluid phases, and following (Biot (1956a; 1962b; 1964) and Youssef (2007)), we may write that

$$\rho = (1-\gamma)\rho^{s*} + \gamma\rho^{f*}, \tag{5.1.4}$$

where ρ is the effective density of the material.

Alternatively, Eq. (5.1.4) can be written as

$$\rho = \rho^{s} + \rho^{f}, \tag{5.1.5}$$

where $\rho^s = (1-\gamma)\rho^{s*}$ and $\rho^f = \gamma\rho^{f*}$. Throughout this subchapter the superscripted s is used to denote the solid phase and f to denote the fluid phase of the physical quantities.

The mass coefficients corresponding to two different phases are given by

$$\rho_{11} = \rho^{s} - \rho_{12} \text{ and } \rho_{22} = \rho^{f} - \rho_{12}, \tag{5.1.6}$$

where ρ_{12} is the dynamic coupling constant.

Following Youssef (2007), Sherief and Hussein (2012) and Ezzat and Ezzat (2016), the equations of motion in the present context can be given as

$$\tau_{ji,j} + \rho^{s} b_i^{s} = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i, \tag{5.1.7}$$

$$\tau_{,i} + \rho^{f} b_i^{f} = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i, \tag{5.1.8}$$

where b_i^s and b_i^f are the body forces related to solid and fluid phases, receptively.

Following Sherief and Hussein (2012) and Ezzat and Ezzat (2016), the kinetic energy of the porothrmoelastic system in the present context is stated as

$$K.E. = \frac{1}{2}(\rho_{11}\dot{u}_i\dot{u}_i + 2\rho_{12}\dot{u}_i\dot{U}_i + \rho_{22}\dot{U}_i\dot{U}_i). \tag{5.1.9}$$

Therefore, in view of Eq. (5.1.9), the first law of thermodynamics can be written in the present context as

$$\frac{d}{dt}\left[\int_V \frac{1}{2}(\rho_{11}\dot{u}_i\dot{u}_i + 2\rho_{12}\dot{u}_i\dot{U}_i + \rho_{22}\dot{U}_i\dot{U}_i)dV + \int_V \rho I dV\right]$$
$$= \int_V (\rho^s b^s_j \dot{u}_j + \rho^f b^f_i \dot{U}_i)dV - \int_V (q^s_{i,i} + q^f_{i,i})dV$$
$$+ \int_V \rho(h^s + h^f)dV + \int_A (\tau_{ij} n_j \dot{u}_i + \tau n_j \dot{U}_i)dA, \tag{5.1.10}$$

where A is the bounding surface of volume V occupying the bulk material. I denotes the internal energy per unit mass, n_i denotes the unit normal vector to the surface A along x_i direction, q^s_i and q^f_i are the components of heat flux and h^s and h^f are the external heat sources, respectively.

Now, from the principle of virtual work done, the rate of change of kinetic energy is always equal to the rate of change of all forces including internal and external forces, i.e.,

$$\frac{d}{dt}(K.E.) = \int_V (\rho^s b^s_j \dot{u}_j + \rho^f b^f_i \dot{U}_i)dV + \int_A (\tau_{ij} n_j \dot{u}_i + \tau n_i \dot{U}_i)dA - \int_V (\tau_{ij}\dot{e}_{ij} + \tau\dot{\varepsilon})dV. \tag{5.1.11}$$

Using Eq. (5.1.10), in Eq. (5.1.10), it yields

$$\frac{d}{dt}\int_V \rho I dV = \int_V (\tau_{ij}\dot{e}_{ij} + \tau\dot{\varepsilon})dV - \int_V (q^s_{i,i} + q^f_{i,i})dV + \int_V \rho(h^s + h^f)dV. \tag{5.1.12}$$

Eq. (5.1.12) holds for any arbitrary volume, therefore this equation can be simplified as

$$\rho\dot{I} = \tau_{ij}\dot{e}_{ij} + \tau\dot{\varepsilon} - q^s_{i,i} - q^f_{i,i} + \rho(h^s + h^f). \tag{5.1.13}$$

Now, by following Green and Laws (1972), the generalized second law of thermodynamics in the context of porothermoelasticity for the solid phase can be stated as

$$\frac{d}{dt}\int_V \rho\eta^s \geq \int_V \frac{\rho h^s}{T^s}dV - \int_A \frac{q^s_i n_i}{T^s}dA, \tag{5.1.14}$$

where T^s is the generalized temperature which requires the constitutive relations with restriction $T^s > 0$ and can be assumed to be the functions of temperature, temperature rate, or temperature gradient of solid phase. On the equilibrium, T^s will be equal to the reference temperature (T_0).

Further, by using the Gauss divergence theorem, the Eq. (5.1.14) yields

$$(\rho T^s \dot{\eta}^s - \rho h^s) + q^s_{i,i} - \frac{q^s_i}{T^s} T^s_{,i} \geq 0. \tag{5.1.15}$$

Similarly, for the fluid phase, the entropy production inequality can be obtained as

$$(\rho T^f \dot{\eta}^f - \rho h^f) + q^f_{i,i} - \frac{q^f_i}{T^f} T^f_{,i} \geq 0. \tag{5.1.16}$$

Here, T^f is the generalized temperature corresponding to fluid phase, and $T^f > 0$ can be assumed to be the functions of temperature, temperature-rate, or temperature gradient of fluid phase. On equilibrium, $T^f = T_0$.

Adding Eqs. (5.1.15) and (5.1.16) it results

$$(\rho T^f \dot{\eta}^f + \rho T^s \dot{\eta}^s - \rho h^f - \rho h^s) + q^s_{i,i} + q^f_{i,i} - \frac{q^s_i}{T^s} T^s_{,i} - \frac{q^f_i}{T^f} T^f_{,i} \geq 0. \tag{5.1.17}$$

Now, the Helmholtz free energy function (F) per unit mass is introduced in the present context of porothermoelasticity as

$$\rho F = \rho I - \rho T^s \eta^s - \rho T^f \eta^f$$

i.e.,

$$\rho \dot{I} = \rho \dot{F} + \rho \dot{T}^s \eta^s + \rho \dot{T}^f \eta^f + \rho T^s \dot{\eta}^s + \rho T^f \dot{\eta}^f. \tag{5.1.18}$$

Comparing the Eq. (5.1.13) with Eq. (5.1.18) it gives

$$\rho(T^s \dot{\eta}^s + T^f \dot{\eta}^f) - \rho(h^s + h^f) = -(\rho \dot{F} + \rho \dot{T}^s \eta^s + \rho \dot{T}^f \eta^f) + \tau_{ij} \dot{e}_{ij} + \tau \dot{\varepsilon} - q^s_{i,i} - q^f_{i,i}, \tag{5.1.19}$$

and, using the Eq. (5.1.19) into Eq. (5.1.17) provides

$$-(\rho \dot{F} + \rho \dot{T}^s \eta^s + \rho \dot{T}^f \eta^f) + \tau_{ij} \dot{e}_{ij} + \tau \dot{\varepsilon} - \frac{q^s_i}{T^s} T^s_{,i} - \frac{q^f_i}{T^f} T^f_{,i} \geq 0. \tag{5.1.20}$$

5.1.3 Constitutive Assumptions

We consider that the porothermoelastic system is characterized by the internal energy (I), entropy (η^s, η^f), and the heat flux vectors (q^s_i, q^f_i). To derive the theory, we postulate that these response functions are dependent on the field variables $e_{ij}, \varepsilon, \theta^s, \theta^f, \dot{\theta}^s, \dot{\theta}^f, \theta^s_{,i}, \theta^f_{,i}$ i.e.

$$\begin{aligned} I &= \tilde{I}(e_{ij}, \varepsilon, \theta^s, \theta^f, \dot{\theta}^s, \dot{\theta}^f, \theta^s_{,i}, \theta^f_{,i}), \\ \eta^s &= \tilde{\eta}^s(e_{ij}, \varepsilon, \theta^s, \dot{\theta}^s, \theta^s_{,i}), \\ \eta^f &= \tilde{\eta}^f(e_{ij}, \varepsilon, \theta^f, \dot{\theta}^f, \theta^f_{,i}), \\ q^s_i &= \tilde{q}^s_i(e_{ij}, \varepsilon, \theta^s, \dot{\theta}^s, \theta^s_{,i}), \\ q^f_i &= \tilde{q}^f_i(e_{ij}, \varepsilon, \theta^f, \dot{\theta}^f, \theta^f_{,i}) \ . \end{aligned} \tag{5.1.21}$$

Therefore, the Helmholtz free energy function can also be given as

$$F = \tilde{F}(e_{ij}, \varepsilon, \theta^s, \theta^f, \dot{\theta}^s, \dot{\theta}^f, \theta^s_{,i}, \theta^f_{,i}). \tag{5.1.22}$$

In order to derive the governing equations for temperature-rate dependent theory of porothermoelasticity, the generalized temperatures for the solid and fluid phases are assumed as the functions of $\theta^s, \dot{\theta}^s$ and $\theta^f, \dot{\theta}^f$. Therefore, in view of the Eqs. (5.1.21-5.1.22), the Eq. (5.1.20) can be rewritten as

$$\begin{aligned} &\left(-\rho\frac{\partial \tilde{F}}{\partial e_{ij}} + \tau_{ij}\right)\dot{e}_{ij} + \left(-\rho\frac{\partial \tilde{F}}{\partial \varepsilon} + \tau\right)\dot{\varepsilon} - \left(\rho\frac{\partial \tilde{F}}{\partial \theta^s} + \rho\eta^s\frac{\partial T^s}{\partial \theta^s}\right)\dot{\theta}^s - \left(\rho\frac{\partial \tilde{F}}{\partial \dot{\theta}^s} + \rho\eta^s\frac{\partial T^s}{\partial \dot{\theta}^s}\right)\ddot{\theta}^s \\ &\quad - \left(\rho\frac{\partial \tilde{F}}{\partial \theta^f} + \rho\eta^f\frac{\partial T^f}{\partial \theta^f}\right)\dot{\theta}^f - \left(\rho\frac{\partial \tilde{F}}{\partial \dot{\theta}^f} + \rho\eta^s\frac{\partial T^f}{\partial \dot{\theta}^f}\right)\ddot{\theta}^f - \left(\rho\frac{\partial \tilde{F}}{\partial \theta^s_{,i}} + \frac{q^s_i}{T^s}\frac{\partial T^s}{\partial \dot{\theta}^s}\right)\dot{\theta}^s_{,i} \\ &\quad - \left(\rho\frac{\partial \tilde{F}}{\partial \theta^f_{,i}} + \frac{q^f_i}{T^f}\frac{\partial T^f}{\partial \dot{\theta}^f}\right)\dot{\theta}^f_{,i} - \frac{q^s_i}{T^s}\frac{\partial T^s}{\partial \theta^s}\theta^s_{,i} - \frac{q^f_i}{T^f}\frac{\partial T^f}{\partial \theta^f}\theta^f_{,i} \geq 0. \end{aligned} \tag{5.1.23}$$

Based on the assumptions, the present system are independent from $\dot{e}_{ij}$, $\dot{\varepsilon}$, $\ddot{\theta}^s$, $\ddot{\theta}^f$,$\dot{\theta}^s_{,i}$ and $\dot{\theta}^f_{,i}$, which implies that the coefficients of these terms in Eq. (5.1.23) must vanish. Therefore, the following relations can be derived from the inequality (5.1.23)

$$\tau_{ij} = \rho\frac{\partial \tilde{F}}{\partial e_{ij}}, \tag{5.1.24}$$

$$\tau = \rho\frac{\partial \tilde{F}}{\partial \varepsilon}, \tag{5.1.25}$$

$$\frac{\partial \tilde{F}}{\partial \dot{\theta}^s} + \eta^s\frac{\partial T^s}{\partial \dot{\theta}^s} = 0, \tag{5.1.26}$$

$$\frac{\partial \tilde{F}}{\partial \dot{\theta}^f} + \eta^f \frac{\partial T^f}{\partial \dot{\theta}^f} = 0, \tag{5.1.27}$$

$$\rho \frac{\partial \tilde{F}}{\partial \theta^s_{,i}} + \frac{q^s_i}{T^s} \frac{\partial T^s}{\partial \dot{\theta}^s} = 0, \tag{5.1.28}$$

$$\rho \frac{\partial \tilde{F}}{\partial \theta^f_{,i}} \dot{\theta}^f_{,i} + \frac{q^f_i}{T^f} \frac{\partial T^f}{\partial \dot{\theta}^f} = 0, \tag{5.1.29}$$

$$\left(\frac{\partial \tilde{F}}{\partial \theta^s} + \eta \frac{\partial T^s}{\partial \theta^s} \right) \dot{\theta}^s + \left(\frac{\partial \tilde{F}}{\partial \theta^f} + \eta^f \frac{\partial T^f}{\partial \theta^f} \right) \dot{\theta}^f + \frac{q^s_i}{T^s} \frac{\partial T^s}{\partial \theta^s} \theta^s_{,i} \frac{q^f_i}{T^f} \frac{\partial T^f}{\partial \theta^f} \theta^f_{,i} \leq 0. \tag{5.1.30}$$

Also, in view of Eqs. (5.1.13) and (5.1.19), the energy equation can be obtained as

$$q^s_{i,i} + q^f_{i,i} + \rho T^s \dot{\eta}^s + \rho T^f \dot{\eta}^f - \rho(h^s + h^f) = 0. \tag{5.1.31}$$

Hence, the Eqs. (5.1.24-5.1.31) represent constitutive equations for the nonlinear temperature-rate dependent porothermoelasticity.

5.1.4 Linear theory of Temperature-Rate Dependent Porothermoelasticity

Now, to present the linearized equations corresponding to the constitutive relations (5.1.24-5.1.29), the Maclaurin's series expansion of the free energy function $\tilde{F}(e_{ij}, \varepsilon, \theta^s, \theta^f, \dot{\theta}^s, \dot{\theta}^f, \theta^s_{,i}, \theta^f_{,i})$ is taken about it's natural state and retaining the terms up to second order, yields

$$
\begin{aligned}
\rho\tilde{F} = {} & F|_{\mathbf{0}} + (\theta^s - T_0)\ \frac{\partial F}{\partial \theta^s}\bigg|_{\mathbf{0}} + (\theta^f - T_0)\ \frac{\partial F}{\partial \theta^f}\bigg|_{\mathbf{0}} + \dot{\theta}^f\ \frac{\partial F}{\partial \dot{\theta}^f}\bigg|_{\mathbf{0}} + \dot{\theta}^f\ \frac{\partial F}{\partial \dot{\theta}^f}\bigg|_{\mathbf{0}} \\
& + e_{ij}\ \frac{\partial F}{\partial e_{ij}}\bigg|_{\mathbf{0}} + \varepsilon\ \frac{\partial F}{\partial \varepsilon}\bigg|_{\mathbf{0}} + \theta^s_{,i}\ \frac{\partial F}{\partial \theta^s_{,i}}\bigg|_{\mathbf{0}} + \theta^f_{,i}\ \frac{\partial F}{\partial \theta^f_{,i}}\bigg|_{\mathbf{0}} + \frac{1}{2}\Bigg[(\theta^s - T_0)^2\ \frac{\partial^2 F}{\partial (\theta^s)^2}\bigg|_{\mathbf{0}} \\
& + (\theta^f - T_0)^2\ \frac{\partial^2 F}{\partial (\theta^f)^2}\bigg|_{\mathbf{0}} + (\dot{\theta}^s)^2\ \frac{\partial^2 F}{\partial (\dot{\theta}^s)^2}\bigg|_{\mathbf{0}} + (\dot{\theta}^f)^2\ \frac{\partial^2 F}{\partial (\dot{\theta}^f)^2}\bigg|_{\mathbf{0}} + e_{ij}e_{kl}\ \frac{\partial^2 F}{\partial e_{ij}\partial e_{kl}}\bigg|_{\mathbf{0}} + \varepsilon^2\ \frac{\partial^2 F}{\partial \varepsilon^2}\bigg|_{\mathbf{0}} \\
& + \theta^s_{,i}\theta^s_{,j}\ \frac{\partial^2 F}{\partial \theta^s_{,i}\partial \theta^s_{,j}}\bigg|_{\mathbf{0}} + \theta^f_{,i}\theta^f_{,j}\ \frac{\partial^2 F}{\partial \theta^f_{,i}\partial \theta^f_{,j}}\bigg|_{\mathbf{0}} + 2(\theta^s - T_0)(\theta^f - T_0)\ \frac{\partial^2 F}{\partial \theta^s \partial \theta^f}\bigg|_{\mathbf{0}} \\
& + 2(\theta^s - T_0)\dot{\theta}^s\ \frac{\partial^2 F}{\partial \theta^s \partial \dot{\theta}^s}\bigg|_{\mathbf{0}} + 2(\theta^s - T_0)\dot{\theta}^f\ \frac{\partial^2 F}{\partial \theta^s \partial \dot{\theta}^f}\bigg|_{\mathbf{0}} + 2\dot{\theta}^s(\theta^f - T_0)\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial \theta^f}\bigg|_{\mathbf{0}} \\
& + 2\dot{\theta}^s\theta^s_{,i}\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial \theta^s_{,i}}\bigg|_{\mathbf{0}} + 2(\theta^s - T_0)e_{ij}\ \frac{\partial^2 F}{\partial \theta^s \partial e_{ij}}\bigg|_{\mathbf{0}} + 2(\theta^s - T_0)\varepsilon\ \frac{\partial^2 F}{\partial \theta^s \partial \varepsilon}\bigg|_{\mathbf{0}} + 2\dot{\theta}^s e_{ij}\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial e_{ij}}\bigg|_{\mathbf{0}} \\
& + 2\dot{\theta}^s\varepsilon\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial \varepsilon}\bigg|_{\mathbf{0}} + 2\dot{\theta}^s\theta^f_{,i}\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial \theta^f_{,j}}\bigg|_{\mathbf{0}} + 2\dot{\theta}^f e_{ij}\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial e_{ij}}\bigg|_{\mathbf{0}} + 2\dot{\theta}^f\varepsilon\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial \varepsilon}\bigg|_{\mathbf{0}} \\
& + 2\dot{\theta}^f\theta^f_{,i}\ \frac{\partial^2 F}{\partial \dot{\theta}^s \partial \theta^f_{,i}}\bigg|_{\mathbf{0}} + 2\dot{\theta}^f\theta^s_{,i}\ \frac{\partial^2 F}{\partial \dot{\theta}^f \partial \theta^s_{,i}}\bigg|_{\mathbf{0}} + 2e_{ij}\varepsilon\ \frac{\partial^2 F}{\partial e_{ij}\partial \varepsilon}\bigg|_{\mathbf{0}} + 2e_{ij}\theta^s_{,i}\ \frac{\partial^2 F}{\partial e_{ij}\partial \theta^s_{,i}}\bigg|_{\mathbf{0}} \\
& + 2e_{ij}\theta^f_{,i}\ \frac{\partial^2 F}{\partial e_{ij}\partial \theta^f_{,i}}\bigg|_{\mathbf{0}} + 2\varepsilon\theta^s_{,i}\ \frac{\partial^2 F}{\partial \varepsilon \partial \theta^s_{,i}}\bigg|_{\mathbf{0}} + 2\varepsilon\theta^f_{,i}\ \frac{\partial^2 F}{\partial \varepsilon \partial \theta^f_{,i}}\bigg|_{\mathbf{0}} + 2\theta^s_{,i}\theta^f_{,j}\ \frac{\partial^2 F}{\partial \theta^s_{,i}\partial \theta^f_{,j}}\bigg|_{\mathbf{0}}\Bigg],
\end{aligned}
\tag{5.1.32}
$$

where the subscripted $\mathbf{0}$ notation denotes the natural state $(0, 0, T_0, T_0, 0, 0, 0, 0)$, T_0 being the reference temperature. Now, by following Sherief and Hussein (2012) and Ezzat and Ezzat (2016), we define the coefficients of Eq. (5.1.32) as $\frac{\partial^2 F}{\partial e_{ij}\partial e_{kl}}\Big|_{\mathbf{0}} = C_{ijkl}$, the elasticity tensor, $\frac{\partial^2 F}{\partial \varepsilon^2}\Big|_{\mathbf{0}} = C$, the porosity constant, $\frac{\partial^2 F}{\partial e_{ij}\partial \varepsilon}\Big|_{\mathbf{0}} = D_{ij}$, the poroelasticity coupling tensor, $\frac{\partial^2 F}{\partial \theta^s \partial e_{ij}}\Big|_{0} = \beta^s_{ij}$, the thermoelastic coupling tensor of solid phase, $\frac{\partial^2 F}{\partial \theta^f \partial e_{ij}}\Big|_{0} = \beta^f_{ij}$, the thermoelastic coupling tensor of fluid phase, $\frac{\partial^2 F}{\partial \theta^s \partial \varepsilon}\Big|_{\mathbf{0}} = \beta^s$, $\frac{\partial^2 F}{\partial \dot{\theta}^s \partial \theta^s_{,i}}\Big|_{\mathbf{0}} = C^s_i$, the center of symmetry vector of solid phase, $\frac{\partial^2 F}{\partial \dot{\theta}^f \partial \theta^f_{,i}}\Big|_{\mathbf{0}} = C^f_i$, the center of symmetry vector of fluid phase, the thermoelastic coupling constant of solid phase on fluid phase, $\frac{\partial^2 F}{\partial \theta^f \partial \varepsilon}\Big|_{\mathbf{0}} = \beta^f$ which is the thermoelastic coupling constant of fluid phase on solid phase, $\frac{\partial^2 F}{\partial (\theta^f)^2}\Big|_{\mathbf{0}} = -\rho c^f_E / T_0$, $\frac{\partial^2 F}{\partial (\theta^s)^2}\Big|_{\mathbf{0}} = -\rho c^s_E / T_0$, $\frac{\partial^2 F}{\partial \theta^s \partial \dot{\theta}^s}\Big|_{\mathbf{0}} = \frac{t^s_1 \rho c^s_E}{T_0}$, $\frac{\partial^2 F}{\partial \theta^f \partial \dot{\theta}^f}\Big|_{\mathbf{0}} = \frac{t^f_1 \rho c^f_E}{T_0}$, $\frac{\partial^2 F}{\partial (\dot{\theta}^s)^2}\Big|_{\mathbf{0}} = -\frac{t^s_1 t^s_2 \rho c^s_E}{T_0}$, $\frac{\partial^2 F}{\partial (\dot{\theta}^s)^2}\Big|_{\mathbf{0}} = -\frac{t^f_1 t^f_2 \rho c^f_E}{T_0}$, where t^s_1, t^s_2 and t^f_1, t^f_2 are denoted as the relaxation parameters for the solid phase and fluid phases, respectively.

Further, we denote $\frac{\partial^2 F}{\partial\theta^s_{,i}\partial\theta^s_{,j}}\Big|_{\mathbf{0}} = \frac{t^s_1}{T_0}K^s_{ij}$ as the conductivity tensor of solid phase, $\frac{\partial^2 F}{\partial\theta^f_{,i}\partial\theta^f_{,j}}\Big|_{\mathbf{0}} = \frac{t^f_1}{T_0}K^f_{ij}$ as the conductivity tensor of fluid phase.

For the linearity of the formulation, we may take $1/T^s = 1/T^f \simeq 1/T_0$. Now, in view of Eq. (5.1.30) and by substituting these values in Eq. (5.1.32), the simplified form of Eq. (5.1.32) is obtained as

$$\begin{aligned}\rho F = {} & \frac{1}{2}C_{ijkl}e_{ij}e_{kl} + \frac{1}{2}D_{ij}e_{ij}\varepsilon - \beta^s_{ij}(\theta^s - T_0 + t^s_1\dot{\theta}^s)e_{ij} - \beta^f_{ij}(\theta^f - T_0 + t^f_1\dot{\theta}^f)e_{ij} \\ & - \beta^s(\theta^s - T_0 + t^s_1\dot{\theta}^s)\varepsilon - \frac{\rho c^s_E((\theta^s - T_0)^2 + t^s_1 t^s_2(\dot{\theta}^s)^2 + 2t^s_1(\theta^s - T_0)\dot{\theta}^s)}{2} \\ & - \beta^f(\theta^f - T_0 + t^s_1\dot{\theta}^f)\varepsilon - \frac{\rho c^f_E((\theta^f - T_0)^2 + t^f_1 t^f_2(\dot{\theta}^f)^2 + 2t^f_1(\theta^f - T_0)\dot{\theta}^f)}{2} \\ & + \frac{t^s_1}{T_0}C^s_i\theta^s_{,i}\dot{\theta}^s + \frac{t^f_1}{T_0}C^f_i\theta^f_i\dot{\theta}^f + \frac{t^s_1}{T_0}\frac{1}{2}K^s_{ij}\theta^s_{,i}\theta^s_{,j} + \frac{t^f_1}{T_0}\frac{1}{2}K^f_{ij}\theta^f_{,i}\theta^f_{,j}. \end{aligned} \tag{5.1.33}$$

Similarly, the Maclaurin's expansion of the generalized temperatures (T^s and T^f) can be be written as

$$T^s(\theta^s, \dot{\theta}^s) = T^s(\theta^s, 0) + \dot{\theta}^s\frac{\partial T^s(T_0, 0)}{\partial\dot{\theta}^s} + (\theta^s - T_0)\dot{\theta}^s\frac{\partial^2 T(T_0, 0)}{\partial\theta^s\partial\dot{\theta}^s} + (\dot{\theta}^s)^2\frac{\partial^2 T^s(T_0, 0)}{\partial\dot{\theta}^2}, \tag{5.1.34}$$

$$T^f(\theta^f, \dot{\theta}^f) = T^f(\theta^f, 0) + \dot{\theta}^f\frac{\partial T^f(T_0, 0)}{\partial\dot{\theta}^f} + (\theta^f - T_0)\dot{\theta}^f\frac{\partial^2 T^f(T_0, 0)}{\partial\theta^f\partial\dot{\theta}^f} + (\dot{\theta}^f)^2\frac{\partial^2 T^f(T_0, 0)}{\partial(\dot{\theta}^f)^2}. \tag{5.1.35}$$

Initially, both θ^s and θ^f are taken as the same and equal to the reference temperature T_0. Further, by following the assumption $T^s(\theta^s, 0) = \theta^s$, $\frac{\partial T^s(T_0,0)}{\partial\theta^s} = t^s_1$, $T^f(\theta^f, 0) = \theta^f$ and $\frac{\partial T^f(T_0,0)}{\partial\theta} = t^f_1$, Eqs. (5.1.34-5.1.35) can be written as (Green and Lindsay (1972))

$$T^s = \vartheta^s + t^s_1[\dot{\vartheta}^s + \chi^s_1(\vartheta^s)\dot{\vartheta}^s + \chi^s_2(\dot{\vartheta}^s)^2], \tag{5.1.36}$$

$$T^f = \vartheta^f + t^f_1[\dot{\vartheta}^f + \chi^f_1(\vartheta^f)\dot{\vartheta}^f + \chi^f_2(\dot{\vartheta}^f)^2], \tag{5.1.37}$$

where $\vartheta^s = (\theta^s - T_0)$ and $\vartheta^f = (\theta^f - T_0)$ are the temperatures above reference temperatures T_0.

Hence, by substituting these values of F, T^s and T^f from Eqs. (5.1.33),(5.1.36-5.1.37) in Eqs. (5.1.24-5.1.30) implies that

$$\tau_{ij} = C_{ijkl}e_{kl} + D_{ij}\varepsilon - \beta^s_{ij}(\vartheta^s + t^s_1\dot{\vartheta}^s) - \beta^f_{ij}(\vartheta^f + t^f_1\dot{\vartheta}^f), \tag{5.1.38}$$

$$\tau = C\varepsilon + D_{ij}e_{ij} - \beta^s(\vartheta^s + t^s_1\dot{\vartheta}^s) - \beta^f(\vartheta^f + t^f_1\dot{\vartheta}^f), \tag{5.1.39}$$

$$\rho T_0\eta^s = T_0(\beta^s_{ij}e_{ij} + \beta^s\varepsilon) + \rho c^s_E(\vartheta^s + t^s_2\dot{\vartheta}^s) - C^s_i\vartheta^s_{,i}, \tag{5.1.40}$$

$$\rho T_0\eta^f = T_0(\beta^f_{ij}e_{ij} + \beta^f\varepsilon) + \rho c^f_E(\vartheta^f + t^f_2\dot{\vartheta}^f) - C^f_i\vartheta^f_{,i}, \tag{5.1.41}$$

$$q^s_i = -(C^s_i\dot{\vartheta}^s + K^s_{ij}\vartheta^s_{,j}), \tag{5.1.42}$$

$$q^f_i = -(C^f_i\dot{\vartheta}^f + K^f_{ij}\vartheta^f_{,j}), \tag{5.1.43}$$

$$\rho c^s_E(t^s_1 - t^s_2)(\dot{\vartheta}^s)^2 + 2C^s_i\dot{\vartheta}^s\vartheta^s_{,i} + \rho c^f_E(t^f_1 - t^f_2)(\dot{\vartheta}^f)^2 + 2C^f_i\dot{\vartheta}^f\vartheta^f_{,i} + K^s_{ij}\vartheta^s_{,i}\vartheta^s_{,j} + K^f_{ij}\vartheta^f_{,i}\vartheta^f_{,j} \geq 0, \tag{5.1.44}$$

Furthermore, in view of Eq. (5.1.31), the energy equation for the solid and fluid phases may be written explicitly as

$$q^s_{i,i} = -\rho T_0\dot{\eta}^s + \rho h^s + K_1(\vartheta^s - \vartheta^f), \tag{5.1.45}$$

$$q^f_{i,i} = -\rho T_0\dot{\eta}^f + \rho h^f - K_1(\vartheta^s - \vartheta^f), \tag{5.1.46}$$

where, K_1 is the dynamic coupling constant (Youssef (2007)).

Next, by combining Eqs. (5.1.40,5.1.42,5.1.45) and (5.1.41,5.1.43,5.1.46), we obtain the heat conduction equations for solid and fluid phases as

$$K^s_{ij}\vartheta^s_{,ji} = \rho c^s_E(\dot{\vartheta}^s + t^s_2\ddot{\vartheta}^s) + T_0(\beta^s_{ij}\dot{e}_{ij} + \beta^s\dot{\varepsilon}) - C^s_i\dot{\vartheta}^s_{,i} + \rho h^s + K_1(\vartheta^s - \vartheta^f), \tag{5.1.47}$$

$$K^f_{ij}\vartheta^f_{,ji} = \rho c^f_E(\dot{\vartheta}^f + t^f_2\ddot{\vartheta}^f) + T_0(\beta^f_{ij}\dot{e}_{ij} + \beta^f\dot{\varepsilon}) - C^f_i\dot{\vartheta}^f_{,i} + \rho h^s - K_1(\vartheta^s - \vartheta^f). \tag{5.1.48}$$

Hence, the Eqs. (5.1.7-5.1.8), (5.1.38-5.1.43) along with the relations (5.1.45-5.1.48) constitute the basic governing equations for the temperature-rate dependent porothermoelasticity theory for an anisotropic medium.

5.1.5 Conclusion

The subchapter successfully established the temperature-rate dependent porothermoelasticity (TRDPTE) theory for anisotropic medium. Here, The basic governing equations of the theory are derived by following fundamental laws of thermodynamics. The major achievement of this subchapter is that here a most general nonlinear set of governing equations corresponding to TRDPTE theory is derived, and further from this nonlinear set of equations, a linear theory of porothermoelasticity is formulated. Due to the generality of formulation, the present theory can be applied to any problem involving the thermomechanical interactions inside a poroelastic medium.

5.2 Some Theorems on the Temperature-Rate Dependent theory of Porothermoelasticity [2]

5.2.1 Introduction

The main objective of this subchapter is to establish some theoretical results for the temperature-rate dependent porothermoelasticity (TRDTE) theory. In the previous subchapter, the temperature-rate dependent theory of porothermoelasticity is formulated from the fundamental laws of generalized thermodynamics. During the last few years, the porothermoelasticity theory has gained serious attention from researchers. Some authors have also investigated the various porothermoelastic problems under the TRDTE theory. The development of the theoretical results for mathematical models provides a strong foundation for didderent numerical methods, which further helps to analyze the corresponding mathematical model. The mathematical modeling of the porothermoelasticity under temperature-rate dependent theory is derived in the previous subchapters, and there is a scope to investigate this theory and understand the applicability of this theory from the mathematical point of view.

The uniqueness theorem has been studied by many researchers in the context of thermoelasticity theories. Weiner (1957) and Sherief and Dhaliwal (1980) have proved the uniqueness theorem for an isotopic medium. Ignaczak (1963) has discussed the uniqueness results for the different thermoelasticity theory. Later on, the uniqueness theorem in the context of porothermoelasticity theory has been established by Sherif and Hussein (2012) and Ezzat and Ezzat (2016) for different porothermoelastic problems under the LS model. The variational principle plays an important role in the theoretical foundation of the numerical techniques for solving the various thermomechanical problems. It has applications for both the unified development of the theory and the

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approximate solution of fully coupled initial-boundary value problems in coupled thermoelasticity and porothermoelasticity theory. It also provides an alternative way to determine the state of the dynamics of a problem under consideration. Therefore, these principles lend themselves readily to numerical solutions for different finite element approaches like extended Ritz method as developed by Nickell and Sackman (1968) in the context of classical thermoelasticity theory, the variational method for both space and time as discussed by Darrall and Dargush (2018) for dynamical thermoelastic problems etc. The finite element method (FEM) is widely used to solve most engineering problems. Therefore, several authors have presented the variational results in the context of different thermomechanical theories. Sherief and Dhaliwal (1980) and Dhaliwal and Sherief (1980) have derived the variational principle of generalized thermoelasticity theory for isotropic medium and anisotropic medium. He (2005) has derived the variational principle for the micromorpic theory of thermoelasticity. El-Karamany and Ezzat (2011a) have developed the convolutional type variational principle for the fractional order thermoelasticity theory with two-temperature. Youssef and Al-Lehaibi (2010) have presented a variational principle for the fractional order generalized thermoelasticity. A detailed discussion of the variational principles for thermoelasticity has been given in the books by Lebon (1980) and Hetnarski et al. (2009).

Reciprocity theorems have become increasingly important lately because of their uses in the numerical solution of boundary value problems by the boundary element method. It provides an important tool for finding the methods to integrate elasticity equations with Green's function. The BEM method is frequently used by the researchers as an alternative to finite element method in many engineering applications. The reciprocity theorem also acts as an essential tool in the developing the boundary element method (BEM) since the BEM needs some theoretical preparation, namely, reciprocity theorem and fundamental solutions, as a starting point. Anwar and Sherief (1988; 1994), Chen and Dargush (1995), Dargush and Banerjee (1991), Cheng and Predeleanu (1987), Cheng and Detournay (1998)), Nowacki (1975) have reported interesting applications of the reciprocity theorem to the solution of engineering problems. Hence, the theoretical results (variational principle and reciprocity theorem) have direct applications in the development of some numerical methods that can be used to investigate a system numerically.

The present subchapter attempts to deal with the temperature-rate dependent porothermoelasticity theory for anisotropic medium and establishes some important results. We consider the governing equations for temperature-rate dependent porothermoelasticity theory (TRDPTE) for the anisotropic medium, formulated in the previous subchapter. Firstly, the uniqueness of the solution is proved for a mixed boundary initial value problem in the context of the TRDPTE theory for homogeneous and anisotropic medium. Further, a variational principle is established on the basis of the present theory for the fluid saturated the porothermoelastic medium. Here, the principle of virtual work done is applied to derive the expressions for the generalized strain energy, Biot's free energy, and the total work done by the system during the thermomechanical process. Further, in view of this, the variational principle is established in the context of the temperature-rate dependent porothermoelasticity. Finally, a reciprocity theorem is proved for a general problem of the porothermoelasticity theory of anisotropic medium.

5.2.2 Basic Governing Equations

From the previous subchapter, the basic governing equations of porothermoelasticity in context of temperature-rate dependent theory for anisotropic medium can be considered as follows:

The equations of motion:

$$\tau_{ji,j} + \rho^s b_i^s = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i, \tag{5.2.1}$$

$$\tau_{,j} + \rho^f b_i^f = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \tag{5.2.2}$$

Stress-strain-temperature relations:

$$\tau_{ij} = C_{ijkl}e_{kl} + D_{ij}\varepsilon - \beta^s_{ij}\left(1 + t^s_1\frac{\partial}{\partial t}\right)\theta^s - \beta^f_{ij}\left(1 + t^f_1\frac{\partial}{\partial t}\right)\theta^f, \tag{5.2.3}$$

$$\tau = D_{ij}e_{ij} + C\varepsilon - \beta^s\left(1 + t^s_1\frac{\partial}{\partial t}\right)\theta^s - \beta^f\left(1 + t^f_1\frac{\partial}{\partial t}\right)\theta^f. \tag{5.2.4}$$

The heat conduction laws:

$$q^s_i = -K^s_{ij}\theta^s_{,j} \tag{5.2.5}$$

$$q^f_i = -K^f_{ij}\theta^f_{,j}. \tag{5.2.6}$$

The energy equations in the absence of external heat source:

$$q^s_{i,i} = -\rho\vartheta_0\dot{\eta}^s, \tag{5.2.7}$$

$$q^f_{i,i} = -\rho T_0\dot{\eta}^f. \tag{5.2.8}$$

The entropy equations:

$$\rho T_0\eta^s = -\rho c^s_E(\theta^s + t^s_2\dot{\theta}^s) - T_0(\beta^s_{ij}e_{ij} + \beta^s\varepsilon), \tag{5.2.9}$$

$$\rho T_0\eta^f = -\rho c^f_E(\theta^f + t^f_2\dot{\theta}^f) - T_0(\beta^f_{ij}e_{ij} + \beta^f\varepsilon). \tag{5.2.10}$$

where τ_{ij} and τ denote the components of stress tensor and fluid pressure corresponding to the solid and fluid phases, respectively; b^ω_i is the external body force per unit volume; C_{ijkl} is the elasticity tensor, D_{ij} is poroelastic coupling tensor; C is the bulk modulus of fluid phase; β^ω_{ij} is the thermoelasticity tensor, β^ω is the thermoelasticity constant due to the presence of porosity; q^ω_i is heat flux vector; K^ω_{ij} is the conductivity tensor; θ^ω is temperature above reference temperature T_0, such that $\frac{|\theta^\omega|}{T_0} \ll 1$; c^ω_E is the specific heat at constant strain ρ^ω is the density of material; η^ω is the entropy, where, super scripted $\omega(=s$ or $f)$ will be used with s denoting the case of solid phase and f denoting the fluid phase, for material parameters and field variables.

Now, the heat conduction equation corresponding to the solid phase can be obtained by combining Eqs. (5.2.5,5.2.7,5.2.9) as

$$K^s_{ij}\theta^s_{ij} = \rho c^s_E \left(1 + t^s_2 \frac{\partial}{\partial t}\right)\dot{\theta}^s + T_0\left(\beta^s_{ij}\dot{e}_{ij} + \beta^s\dot{\varepsilon}\right). \tag{5.2.11}$$

Similarly, from Eqs. (5.2.6,5.2.8,5.2.10), the heat conduction equation corresponding to the fluid phase can be found as

$$K^f_{ij}\theta^f_{ij} = \rho c^f_E \left(1 + t^f_2 \frac{\partial}{\partial t}\right)\dot{\theta}^f + T_0\left(\beta^f_{ij}\dot{e}_{ij} + \beta^f\dot{\varepsilon}\right). \tag{5.2.12}$$

5.2.3 Uniqueness Theorem

In this section, a uniqueness theorem on the present porothermoelasticity for homogeneous anisotropic poroelastic medium is established. Consider a homogeneous porothermoelastic medium of volume V bounded by the surface area A. Here, the restrictions on material parameters are assumed as follows:

$$\begin{gathered}\rho > 0, c^s_E > 0, c^f_E > 0, K^s_{ij}\vartheta^s_i\vartheta^s_j \geq 0, K^f_{ij}\vartheta^f_{,i}\vartheta^f_{,j} \geq 0,\\ C_{ijkl}C\delta_{ij}\delta_{kl} \geq D_{ij}D_{ij},\ \rho_{12} < (\rho_{11}\rho_{22})^{1/2}, \rho_{11} > 0, \rho_{22} > 0\\ C_{ijkl}e_{ij}e_{kl} \geq 0, t^s_1 > t^s_2 > 0, t^f_1 > t^f_2 > 0, C > 0, D_{ij}e_{ij} > 0.\end{gathered} \tag{5.2.13}$$

Initial and boundary conditions:

For the set of Eqs. (5.2.1-5.2.12) the general non-homogeneous initial conditions are taken as

$$\begin{gathered}u_i(x,0) = u_{i0},\ U_i(x,0) = U_{i0},\ \dot{u}_i(x,0) = \dot{u}_{i0},\ \dot{U}_i(x,0) = \dot{U}_{i0},\ \text{ for all}\, x \in V,\\ \vartheta^s(x,0) = \vartheta^s_{x0},\ \vartheta^f(x,0) = \vartheta^f_{x0},\ \dot{\vartheta}^s(x,0) = \dot{\vartheta}^s_{x0},\ \dot{\vartheta}^f(x,0) = \dot{\vartheta}^f_{x0},\ \text{ for all}\, x \in V.\end{gathered} \tag{5.2.14}$$

Further, we assume the following boundary conditions:

$$\begin{gathered}u_i(x,t) = \hat{u}_i,\ U_i(x,t) = \hat{U}_i,\ \text{ for all } x \in A\,\text{and}\, t \geq 0,\\ \vartheta^s(x,t) = \hat{\vartheta}^s,\ \vartheta^f(x,t) = \hat{\vartheta}^f,\ \text{ for all } x \in A\,\text{and}\, t \geq 0.\end{gathered} \tag{5.2.15}$$

Then the uniqueness theorem is established in the next theorem.

Theorem: Consider a regular and homogeneous anisotropic porothermoelastic material occupying the space V with the surface boundary area A. Then there exists at most one set of single valued functions $\tau_{ij}, \tau, e_{ij}, \varepsilon$ of class C^1 (C^i denotes the set of all functions having continuous i^{th} order derivatives) and$\{u_i, U_i, \vartheta^s, \vartheta^f, q_i^s, q_i^f\}$ of class C^2 such that Eqs. (5.2.1-5.2.12) with the initial conditions (5.2.14) and boundary conditions (5.2.15) are satisfied.

Proof: To prove the theorem, we assume a counter argument that there exist two different sets of solutions i.e. $\{\tau_{ij}^{(1)}, \tau^{(1)}, e_{ij}^{(1)}, \varepsilon^{(1)},\ u_i^{(1)}, U_i^{(1)}, \vartheta^{s(1)}, \vartheta^{f(1)}, q_i^{s(1)}, q_i^{f(1)}\}$ and $\{\tau_{ij}^{(2)}, \tau^{(2)}, e_{ij}^{(2)}, \varepsilon^{(2)}, u_i^{(2)}, U_i^{(2)}, \vartheta^{s(2)}, \vartheta^{f(2)},\ q_i^{s(2)}, q_i^{f(2)}\}$ to the Eqs. (5.2.1-5.2.12) with the initial conditions (5.2.14) and boundary conditions (5.2.15).

As the present partial differential equations are linear in nature, therefore in the absence of external heat sources (h^s, h^f) and body forces (b_i^s, b_i^f), the set $\{\bar{\tau}_{ij}, \bar{\tau}, \bar{e}_{ij}, \bar{\varepsilon}, \bar{u}_i, \bar{U}_i, \bar{\vartheta}^s, \bar{\vartheta}^f, \bar{q}_i^s, \bar{q}_i^f\}$= $\{\tau_{ij}^{(1)} - \tau_{ij}^{(2)}, \tau^{(1)} - \tau^{(2)}, e_{ij}^{(1)} - e_{ij}^{(2)}, \varepsilon^{(1)} - \varepsilon^{(2)}, u_i^{(1)} - u_i^{(2)}, U_i^{(1)} - U_i^{(2)}, \vartheta^{s(1)} - \vartheta^{s(2)}, \vartheta^{f(1)} - \vartheta^{f(2)}, q_i^{s(1)} - q_i^{s(2)}, q_i^{f(1)} - q_i^{f(2)}\}$ will also satisfy the Eqs. (5.2.1-5.2.12) along with the homogeneous counter parts of initial and boundary conditions (5.2.14-5.2.15).

Now, we take the integral

$$\int_V \bar{\tau}_{ij}\dot{\bar{e}}_{ij}dV = \int_V \bar{\tau}_{ij}\dot{\bar{u}}_{i,j}dV = \int_V (\bar{\tau}_{ij}\dot{\bar{u}}_i)_{,j}dV - \int_V \bar{\tau}_{ij,j}\dot{\bar{u}}_i dV. \tag{5.2.16}$$

Therefore, by applying the divergence theorem in Eq. (5.2.16) and using the homogeneous boundary conditions, we obtain

$$\int_V (\bar{\tau}_{ij}\dot{\bar{e}}_{ij} + \bar{\tau}_{ij,j}\dot{\bar{u}}_i)dV = 0. \tag{5.2.17}$$

Substituting the values of $\bar{\tau}_{ij}$ and $\bar{\tau}_{ij,j}$ from Eqs. (5.2.3) and (5.2.1), respectively, we find that

$$\begin{aligned}\int_V [C_{ijkl}\overline{e}_{kl}\dot{\bar{e}}_{ij} + D_{ij}\overline{\varepsilon}\dot{\bar{e}}_{ij} - \beta_{ij}^s(\overline{\vartheta}^s + t_1^s\dot{\overline{\vartheta}}^s)\dot{\bar{e}}_{ij} \\ -\beta_{ij}^f(\overline{\vartheta}^f + t_1^f\dot{\overline{\vartheta}}^f)\dot{\bar{e}}_{ij} + \rho_{11}\ddot{\bar{u}}_i\dot{\bar{u}}_i + \rho_{12}\ddot{\overline{U}}_i\dot{\bar{u}}_i] = 0.\end{aligned} \tag{5.2.18}$$

Similarly, if we consider the integral

$$\int_V \bar{\tau}\dot{\bar{\varepsilon}}dV = \int_V (\bar{\tau}\dot{\bar{U}}_i)_{,i}dV - \int_V \bar{\tau}_{,i}\dot{\bar{U}}_i dV, \tag{5.2.19}$$

then by applying the divergence theorem in Eq. (5.2.19) along with homogeneous boundary conditions, we get

$$\int_V (\bar{\tau}\dot{\bar{\varepsilon}} + \bar{\tau}_{,i}\dot{\bar{U}}_i)dV = 0. \tag{5.2.20}$$

Using the values of $\bar{\tau}$ and $\bar{\tau}_{,i}$ from Eqs. (5.2.4) and (5.2.2), the Eq. (5.2.20) yields

$$\begin{aligned}\int_V [C\bar{\varepsilon}\dot{\bar{\varepsilon}} + D_{ij}\overline{e}_{ij}\dot{\bar{\varepsilon}} - \beta^s(\overline{\vartheta}^s + t_1^s\dot{\overline{\vartheta}}^s)\dot{\bar{\varepsilon}} - \\ \beta^f(\overline{\vartheta}^f + t_1^s\dot{\overline{\vartheta}}^f)\dot{\bar{\varepsilon}} + \rho_{12}\ddot{\bar{u}}_i\dot{\bar{U}}_i + \rho_{22}\ddot{\overline{U}}_i\dot{\bar{U}}_i]dV = 0.\end{aligned} \tag{5.2.21}$$

Combining Eqs. (5.2.19) and (5.2.21), we get

$$\begin{aligned}\frac{d}{dt}\int_V \left[C_{ijkl}\overline{e}_{kl}\overline{e}_{ij} + D_{ij}\bar{e}_{ij}\bar{\varepsilon} + C\bar{\varepsilon}^2 + \rho_{11}\dot{\bar{u}}_i\dot{\bar{u}}_i + \rho_{12}\dot{\bar{u}}_i\dot{\bar{U}}_i + \rho_{22}\dot{\bar{U}}_i\dot{\bar{U}}_i\right] dV \\ - \int_V [(\beta_{ij}^s\dot{\bar{e}}_{ij} + \beta^s\dot{\bar{\varepsilon}})(\overline{\vartheta}^s + t_1^s\dot{\overline{\vartheta}}^s) + (\beta_{ij}^f\dot{\bar{e}}_{ij} + \beta^f\dot{\bar{\varepsilon}})(\overline{\vartheta}^f + t_1^f\dot{\overline{\vartheta}}^f)]dV = 0.\end{aligned} \tag{5.2.22}$$

Using the Eqs. (5.2.11-5.2.12) in Eq. (5.2.22), we obtain

$$\begin{aligned}\frac{d}{dt}\int_V \left[C_{ijkl}\overline{e}_{kl}\overline{e}_{ij} + D_{ij}\bar{e}_{ij}\bar{\varepsilon} + C\bar{\varepsilon}^2 + \rho_{11}\dot{\bar{u}}_i\dot{\bar{u}}_i + \rho_{12}\dot{\bar{u}}_i\dot{\bar{U}}_i + \rho_{22}\dot{\bar{U}}_i\dot{\bar{U}}_i\right] dV \\ - \int_V [K_{ij}^s\vartheta_{,ji}^s(\overline{\vartheta}^s + t_1^s\dot{\overline{\vartheta}}^s) - \rho c_E^s(\dot{\overline{\vartheta}}^s + t_2^s\ddot{\vartheta}^s)(\overline{\vartheta}^s + t_1^s\dot{\overline{\vartheta}}^s)]dV \\ - \int_V [K_{ij}^f\vartheta_{,ji}^f(\overline{\vartheta}^f + t_1^f\dot{\overline{\vartheta}}^f) - \rho c_E^f(\dot{\overline{\vartheta}}^f + t_2^f\ddot{\vartheta}^f)(\overline{\vartheta}^f + t_1^f\dot{\overline{\vartheta}}^f)]dV = 0,\end{aligned} \tag{5.2.23}$$

or,

$$\begin{aligned}
&\frac{d}{dt}\int_V \left[C_{ijkl}\overline{e}_{kl}\overline{e}_{ij} + 2D_{ij}\overline{e}_{ij}\overline{\varepsilon} + C\overline{\varepsilon}^2 + \rho_{11}\dot{\bar{u}}_i\dot{\bar{u}}_i + 2\rho_{12}\dot{\bar{u}}_i\dot{\bar{U}}_i + \rho_{22}\dot{\bar{U}}_i\dot{\bar{U}}_i\right] dV \\
&\qquad - \int_V [K^s_{ij}\vartheta^s_{,j}(\overline{\vartheta}^s + t^s_1\dot{\overline{\vartheta}}^s)]_{,i}dV + \int_V K^s_{ij}\overline{\vartheta}^s_{,j}(\overline{\vartheta}^s_{,i} + t^s_1\dot{\overline{\vartheta}}^s_{,i})dV \\
&\qquad + \int_V \rho c^s_E[(\dot{\vartheta}^s + t^s_2\ddot{\vartheta}^s)(\overline{\vartheta}^s + t^s_2\dot{\overline{\vartheta}}^s) + (t^s_1 - t^s_2)(\dot{\overline{\vartheta}}^s + t^s_2\ddot{\vartheta}^s)\dot{\overline{\vartheta}}^s]dV \\
&\qquad - \int_V [K^f_{ij}\vartheta^f_{,j}(\overline{\vartheta}^f + t^f_1\dot{\overline{\vartheta}}^f)]_{,i}dV + \int_V K^f_{ij}\overline{\vartheta}^f_{,j}(\overline{\vartheta}^f_{,i} + t^f_1\dot{\overline{\vartheta}}^f_{,i})dV \\
&\qquad + \int_V \rho c^f_E[(\dot{\vartheta}^f + t^f_2\ddot{\vartheta}^f)(\overline{\vartheta}^f + t^f_2\dot{\overline{\vartheta}}^f) + (t^f_1 - t^f_2)(\dot{\vartheta}^f + t^f_2\ddot{\vartheta}^f)\dot{\overline{\vartheta}}^f]dV = 0. \qquad (5.2.24)
\end{aligned}$$

Therefore, in view of the boundary conditions and rearranging the terms of Eq. (5.2.24), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_V (C_{ijkl}\overline{e}_{kl}\overline{e}_{ij} + 2D_{ij}\overline{e}_{ij}\overline{\varepsilon} + C\overline{\varepsilon}^2) + (\rho_{11}\dot{\bar{u}}_i\dot{\bar{u}}_i + 2\rho_{12}\dot{\bar{u}}_i\dot{\bar{U}}_i + \rho_{22}\dot{\bar{U}}_i\dot{\bar{U}}_i)\, dV\right. \\
&\qquad \int_V (t^s_1K^s_{ij}\overline{\vartheta}^s_{,j}\overline{\vartheta}^s_{,i} + t^f_1K^f_{ij}\overline{\vartheta}^f_{,j}\overline{\vartheta}^f_{,i})dV + \int_V \{\rho c^s_E(\overline{\vartheta}^s + t^s_2\dot{\overline{\vartheta}}^s)^2 \\
&\qquad \left. + \rho c^f_E(\overline{\vartheta}^f + t^f_2\dot{\overline{\vartheta}}^f)^2\}dV + \int_V \{(t^s_1 - t^s_2)t^s_2(\dot{\overline{\vartheta}}^s)^2 + (t^f_1 - t^f_2)t^f_2(\dot{\overline{\vartheta}}^f)^2\}dV\right] \\
&\qquad \int_V [(t^s_1 - t^s_2)t^s_2(\dot{\overline{\vartheta}}^s)^2 + (t^f_1 - t^f_2)(\dot{\overline{\vartheta}}^f)^2 + K^s_{ij}\overline{\vartheta}^s_{,j}\overline{\vartheta}^s_{,i} + K^f_{ij}\overline{\vartheta}^f_{,j}\overline{\vartheta}^f_{,i}]dV = 0. \\
&\qquad\qquad (5.2.25)
\end{aligned}$$

Now, in the first integral of Eq. (5.2.25), the terms given in the first small bracket will be positive by the restriction as considered in Eq. (5.2.13) and the remaining terms in the second small bracket of this integral denote the kinetic energy and hence always remain to be positive. Further, in view of Eq. (5.2.13), the rest of the terms of Eq. (5.2.25) are also non-negative. Hence, each of the terms vanishes separately. Therefore, from Eqs. (5.2.3-5.2.5) and (5.2.6) and in view of homogeneous initial conditions, it is clear that each term of the set $\{\bar{\tau}_{ij}, \bar{\tau}, \bar{e}_{ij}, \bar{\varepsilon}, \bar{u}_i, \bar{U}_i, \bar{\vartheta}^s, \bar{\vartheta}^f, \bar{q}^s_i, \bar{q}^f_i\}$ is zero. Which implies that our assumption of more than one solution is wrong. It proves that more that one solution of the present problem (5.2.1-5.2.12) with the given initial and boundary conditions (5.2.14-5.2.15) is not possible. Hence, the proof of the uniqueness theorem is completed.

5.2.4 Formulation of the Variation Principle

A porothermoelastic process is a coupled dynamical process of an exchange of mechanical energy into thermal energy in a poroelastic medium, which is accompanied by strain and temperature changes within the body. This process occurs due to some thermal or mechanical disturbance in the body and can be explained by defining the initial as well as boundary conditions on the system corresponding to the governing equations of the porothermoelastic system. Hence, by applying the variational principle, a thermomechanical system can be defined if thermal and mechanical loadings (surface traction forces and body forces) are known. As mentioned in the previous section, the variational principle has applications in both the development of the theory and the approximate solution of the fully coupled thermomechanical problem. In this section, the variational principle is formulated to explicitly represent an alternative form of the dynamical process of the porothermoelastic system. This variational formulation will be attempted on the basis of a total functional expression where its extremum is equivalent to the governing equations of the coupled porothermoelasticity, namely equations of motion (5.2.1-5.2.2) and the coupled heat conduction Eqs. (5.2.11-5.2.12) corresponding to solid and fluid phases.

Firstly, we consider a porothermoelastic system of volume V that is bounded by the surface A. Then due to a small variation in the present porothermoelastic system from the equilibrium state, the virtual work done by the traction and body forces can be given as

$$\int_A (\tau_{ij} n_j \delta u_i + \tau n_i \delta U_i) dA + \int_V (\rho^s b_i^s \delta u_i + \rho^f b_i^f \delta U_i) dV. \tag{5.2.26}$$

By applying the divergence theorem in the second integral of Eq. (5.2.26), we have

$$\int_A (\tau_{ij} n_j \delta u_i + \tau n_i \delta U_i) dA = \int_V (\tau_{ij,j} \delta u_i + \tau_{,j} \delta U_i) dV + \int_V (\tau_{ij} \delta e_{ij} + \tau \delta \varepsilon) dV. \tag{5.2.27}$$

Next, by using the equations of motion (5.2.1) and (5.2.2) in Eq. (5.2.27), we get

$$\begin{aligned}\int_V (\rho^s b_i^s \delta u_i + \rho^f b_i^f \delta U_i) dV + \int_A (\tau_{ij} n_j \delta u_i + \tau n_i \delta U_i) dA = \\ \int_V (\rho_{11} \ddot{u}_i \delta u_i + \rho_{12} \ddot{U}_i \delta u_i + \rho_{12} \ddot{u}_i \delta U_i + \rho_{22} \ddot{U}_i \delta U_i) dV \\ + \int_V (\tau_{ij} \delta e_{ij} + \tau \delta \varepsilon) dV. \qquad (5.2.28)\end{aligned}$$

Now, by substituting the values of τ_{ij} and τ from Eqs. (5.2.3-5.2.4), the last term of Eq. (5.2.28) can be written as

$$\begin{aligned}\int_V (\tau_{ij} \delta e_{ij} + \tau \delta \varepsilon) dV = \int_V (C_{ijkl} e_{kl} \delta e_{ij} + D_{ij} \varepsilon \delta e_{ij} + D_{ij} e_{ij} \delta \varepsilon + C \varepsilon \delta \varepsilon) dV \\ - \beta_{ij}^s \int_V L_1^s \theta^s \delta e_{ij} dV - \beta^s \int_V L_1^f \theta^s \delta \varepsilon dV \\ - \beta_{ij}^f \int_V L_1^s \theta^s \delta e_{ij} dV - \beta^f \int_V L_1^f \theta^f \delta \varepsilon dV, \qquad (5.2.29)\end{aligned}$$

where $L_1^s = (1 + t_1^s \frac{\partial}{\partial t})$ and $L_1^f = (1 + t_1^f \frac{\partial}{\partial t})$.

Define the work done due to the deformation of the system (W) as

$$W = \frac{1}{2} \int_V (C_{ijkl} e_{kl} e_{ij} + 2 D_{ij} e_{ij} \varepsilon + C \varepsilon^2) dV.$$

Therefore,

$$\delta W = \int_V (C_{ijkl} e_{kl} \delta e_{ij} + D_{ij} \delta e_{ij} \varepsilon + D_{ij} e_{ij} \delta \varepsilon + C \varepsilon \delta \varepsilon) dV. \qquad (5.2.30)$$

Therefore, from Eqs. (5.2.29-5.2.30) and Eq. (5.2.28), we obtain

$$\begin{aligned}\int_V (\rho^s b_i^s \delta u_i + \rho^f b_i^f \delta U_i) dV + \int_A (\tau_{ij} n_j \delta u_i + \tau n_i \delta U_i) dA \\ - \int_V (\rho_{11} \ddot{u}_i \delta u_i + \rho_{12} \ddot{U}_i \delta u_i + \rho_{12} \ddot{u}_i \delta U_i + \rho_{22} \ddot{U}_i \delta U_i) dV = \\ \delta W - \beta_{ij}^s \int_V L_1^s \theta^s \delta e_{ij} dV - \beta^s \int_V L_1^f \theta^s \delta \varepsilon dV \\ - \beta_{ij}^f \int_V L_1^s \theta^s \delta e_{ij} dV - \beta^f \int_V L_1^f \theta^f \delta \varepsilon dV, \qquad (5.2.31)\end{aligned}$$

where, $L_1^s = (1 + t_1^s \frac{\partial}{\partial t})$ and $L_1^f = (1 + t_1^f \frac{\partial}{\partial t})$. Here, the L.H.S. of Eq. (5.2.31) denotes the work done by the body forces, traction forces, and the inertial forces.

Next, we define the entropy flux (S_i) in the present context for the solid and fluid phases as

$$\dot{S}_i^s = \frac{q_i^s}{T_0} \text{ and } \dot{S}_i^f = \frac{q_i^f}{T_0}. \tag{5.2.32}$$

In view of Eqs. (5.2.7-5.2.10), and in the absence of external heat sources, Eq. (5.2.32) yields

$$T_0 S_{i,i}^s + T_0 \beta_{ij}^s e_{ij} + T_0 \beta^s \varepsilon + \rho c_E^s L_2^s \theta^s = 0, \tag{5.2.33}$$

$$T_0 S_{i,i}^f + T_0 \beta_{ij}^f e_{ij} + T_0 \beta^f \varepsilon + \rho c_E^f L_2^f \theta^f = 0, \tag{5.2.34}$$

where we denote $L_2^s = (1 + t_2^s \frac{\partial}{\partial t})$ and $L_2^f = (1 + t_2^f \frac{\partial}{\partial t})$.

Further, by combining Eq. (5.2.32) with Eqs. (5.2.5) and (5.2.6), we obtain

$$\begin{aligned} K_{ij}^s \theta_{,j}^s &= -T_0 \dot{S}_i^s \\ \text{and } K_{ij}^f \theta_{,j}^f &= -T_0 \dot{S}_i^s, \end{aligned}$$

or,

$$\begin{aligned} \theta_{,i}^s &= -T_0 \lambda_{ij}^s \dot{S}_i^s \\ \text{and, } \theta_{,i}^f &= -T_0 \lambda_{ij}^f \dot{S}_i^f, \end{aligned} \tag{5.2.35}$$

where, λ_{ij}^s and λ_{ij}^f are thermal resistivity tensors (Hetnarski et al. (2009)).

In Eq. (5.2.32 a,b) multiplying by δS_i^s and δS_i^f, respectively, and taking volume integral on the equations, we have

$$\int_V \theta_{,i}^s \delta S_i^s dV = -T_0 \lambda_{ij}^s \int_V \dot{S}_i^s \delta S_i^s dV, \tag{5.2.36}$$

$$\int_V \theta_{,i}^f \delta S_i^f dV = -T_0 \lambda_{ij}^s \int_V \dot{S}_i^f \delta S_i^f dV. \tag{5.2.37}$$

Further, in view of Eq. (5.2.33), the L.H.S. of Eq. (5.2.36) implies that

$$\int_V \theta^s_{,i}\delta S^s_i dV = \int_A \theta^s \delta S^s_i n_i dA + \int_V (\beta^s_{ij}\theta^s \delta e_{ij} + \beta^s \theta^s \delta\varepsilon) dV + \frac{\rho c^s_E}{T_0} L^s_2 \int_V \theta^s \delta\theta^s dV. \quad (5.2.38)$$

Similarly, from L.H.S. of Eq. (5.2.37) and Eq. (5.2.34), we get

$$\int_V \theta^f_{,i}\delta S^f_i dV = \int_A \theta^f \delta S^f_i n_i dA + \int_V (\beta^f_{ij}\theta^f \delta e_{ij} + \beta^f \theta^f \delta\varepsilon) dV + \frac{\rho c^f_E}{T_0} L^f_2 \int_V \theta^f \delta\theta^f dV. \quad (5.2.39)$$

Now, we introduce the heat potential function corresponding to the solid and fluid phases as

$$P^{*s} = \frac{\rho c^s_E}{2T_0} L^s_2 \int_V (\theta^s)^2 dV, \quad (5.2.40)$$

$$P^{*f} = \frac{\rho c^f_E}{2T_0} L^f_2 \int_V (\theta^f)^2 dV. \quad (5.2.41)$$

Therefore, we have

$$\delta P^{*s} = \frac{\rho c^s_E}{T_0} L^s_2 \int_V \theta^s \delta\theta^s dV, \quad (5.2.42)$$

$$\delta P^{*f} = \frac{\rho c^f_E}{T_0} L^f_2 \int_V \theta^f \delta\theta^f dV. \quad (5.2.43)$$

Further, we define the generalized Biot's free energy function in the context of porothermoelasticity for the two different phases as

$$V^{*s} = \frac{T_0}{2} \lambda^s_{ij} \frac{d}{dt} \int_V S^s_i S^s_j dV, \quad (5.2.44)$$

$$V^{*f} = \frac{T_0}{2} \lambda^f_{ij} \frac{d}{dt} \int_V S^f_i S^f_j dV, \quad (5.2.45)$$

which implies that

$$\delta V^{*s} = T_0 \lambda^s_{ij} \frac{d}{dt} \int_V S^s_i \delta S^s_j dV, \quad (5.2.46)$$

$$\delta V^{*f} = T_0 \lambda^f_{ij} \frac{d}{dt} \int_V S^f_i \delta S^f_j dV. \quad (5.2.47)$$

Therefore, by using the values from Eqs. (5.2.38-5.2.47) in Eqs. (5.2.36-5.2.37), we obtain

$$\int_A \theta^s \delta S_i^s n_i dA + \int_V (\beta_{ij}^s \theta^s \delta e_{ij} + \beta^s \theta^s \delta \varepsilon) dV + \delta P^{*s} + \delta V^{*s} = 0, \tag{5.2.48}$$

$$\int_A \theta^f \delta S_i^f n_i dA + \int_V (\beta_{ij}^f \theta^f \delta e_{ij} + \beta^f \theta^f \delta \varepsilon) dV + \delta P^{*f} + \delta V^{*f} = 0. \tag{5.2.49}$$

Now by adding Eqs. (5.2.48-5.2.49), we get

$$\begin{aligned}\int_A (\theta^s \delta S_i^s + \theta^f \delta S_i^f) n_i dA + \int_V (\beta_{ij}^s \theta^s \delta e_{ij} + \beta^s \theta^s \delta \varepsilon) dV\\ + \int_V (\beta_{ij}^f \theta^f \delta e_{ij} + \beta^f \theta^f \delta \varepsilon) dV + \delta(L_1^s P^{*s} + L_1^f P^{*f}) + \delta(L_1^s V^{*s} + L_1^f V^{*f}) = 0.\end{aligned} \tag{5.2.50}$$

Therefore, by combining the Eqs. (5.2.31) and (5.2.50), we find that

$$\begin{aligned}\delta P^* + \delta V^* + \delta W = & \int_V (\rho^s b_i^s \delta u_i + \rho^f b_i^f \delta U_i) dV - \int_V \Big[\rho_{11} \ddot{u}_i \delta u_i + \rho_{12} \ddot{u}_i \delta U_i + \rho_{12} \delta u_i \ddot{U}_i \\ & + \rho_{22} \ddot{U}_i \delta U_i\Big] dV + \int_A \{(\tau_{ij} n_j \delta u_i - L_1^s \theta^s n_i \delta S_i^s) \\ & + (\tau n_i \delta U_i - L_1^f \theta^f n_i \delta S_i^f)\} dA,\end{aligned} \tag{5.2.51}$$

where $P^* = (L_1^s P^{*s} + L_1^f P^{*f})$ denotes the total heat potential energy and $V^* = (L_1^s V^{*s} + L_1^f V^{*f})$ denotes the effective Biot's free energy functional in the present context of temperature-rate dependent porothermoelasticity theory.

The sum of potential energy (P^*) and the work deformation (W) can be termed as the generalized strain energy in the present context.

Therefore, the Eq. (5.2.51) can be written as

$$\delta\Omega = \int_V (\rho^s b_i^s \delta u_i + \rho^f b_i^f \delta U_i) dV + \int_A \{(\tau_{ij} n_j \delta u_i - \theta^s n_i \delta S_i^s) + (\tau n_i \delta U_i - \theta^f n_i \delta S_i^f)\} dA, \tag{5.2.52}$$

where $\Omega = P^* + V^* + W$, i.e., the sum of virtual internal work of deformation (W), the heat potential (P^*) and the generalized Biot's free energy function (V^*) is termed as the generalized strain energy functional of the present porothermoelastic system. Also, the terms on the right-hand side of Eq. (5.2.52) expresses variation in potential energy due to external forces i.e., the virtual external work of the body forces, of traction on the boundary, and of heating of the boundary. Therefore, the generalized potential energy due to external forces can be assumed as

$$P_e = -\int_V (\rho^s b_i^s u_i + \rho^f b_i^f U_i) dV - \int_A \{(\tau_{ij} n_j u_i - \theta^s n_i S_i^s) - (\tau n_i \delta U_i - \theta^f n_i S_i^f)\} dA. \quad (5.2.53)$$

Hence, we derived the total energy functional (Γ_T) of present porothermoelastic system as the sum of generalized strain energy functional (Ω) and the generalized potential energy (P_e), i.e. $\Gamma_T = \Omega + P_e$, defined as

$$\begin{aligned}
\Gamma_T = & \frac{1}{2}\int_V (C_{ijkl} e_{kl} e_{ij} + D_{ij} \varepsilon e_{ij} + D_{kj} e_{kj} \varepsilon + C\varepsilon^2) dV \\
& + L_1^s \frac{\rho c_E^s}{2T_0} L_2^s \int_V (\theta^s)^2 \delta\theta^s dV + L_1^f \frac{\rho c_E^f}{2T_0} L_2^f \int_V (\theta^f)^2 dV \\
& \frac{T_0}{2} \lambda_{ij}^s L_1^s \frac{d}{dt} \int_V S_i^s S_j^s dV + \frac{T_0}{2} L_1^f \lambda_{ij}^f \frac{d}{dt} \int_V S_i^f S_j^f dV + \\
& \int_V (\rho^s b_i^s u_i + \rho^f b_i^f U_i) dV - \frac{1}{2} \frac{d^2}{dt^2} \int_V \left[\rho_{11} u_i u_i + 2\rho_{12} u_i U_i + \rho_{22} U_i \delta U_i\right] dV \qquad (5.2.54) \\
& - \int_A \left[(\tau_{ij} n_j u_i - L_1^s \theta^s n_i S_i^s) - (\tau n_i U_i - L_1^f \theta^f n_i S_i^f)\right] dA.
\end{aligned}$$

The variational principle of the present porothermoelastic system defined on its total energy functional Γ_T, states that it will be extemum if the equation of motion (5.2.1-5.2.2) and heat conduction Eqs. (5.2.11-5.2.12) are satisfied identically. We establish this as below.

In view Eqs. (5.2.52-5.2.53), Eq. (5.2.54) yields

$$\delta\Gamma_T = 0. \quad (5.2.55)$$

Eq. (5.2.55) shows that the total energy functional attains its extremum. Therefore, it is left to show that the equations of motion (5.2.1-5.2.2) and the heat conduction equations (5.2.11-5.2.12) are also satisfied.

Now, applying the first variation on Eq. (5.2.54) in view of Eq. (5.2.55) , we obtain

$$
\begin{aligned}
\delta\Gamma_T = & \int_V (C_{ijkl}e_{kl}\delta e_{ij} + D_{ij}\varepsilon\delta e_{ij} + D_{kj}e_{kj}\delta\varepsilon + C\varepsilon^2)dV \\
& + L_1^s \frac{\rho c_E^s}{T_0} L_2^s \int_V \theta^s\delta\theta^s dV + L_1^f \frac{\rho c_E^f}{T_0} L_2^f \int_V \theta^f\delta\theta^f dV \\
& + T_0\lambda_{ij}^s L_1^s \frac{d}{dt}\int_V S_i^s\delta S_j^s dV + T_0 L_1^f \lambda_{ij}^f \frac{d}{dt}\int_V S_i^f\delta S_j^f dV \\
& + \int_V (\rho^s b_i^s\delta u_i + \rho^f b_i^f\delta U_i)dV \\
& - \int_V \left[\rho_{11}\ddot{u}_i\delta u_i + \rho_{12}\ddot{u}_i\delta U_i + \rho_{12}\delta u_i\ddot{U}_i + \rho_{22}U_i\delta\ddot{U}_i\right] dV \qquad (5.2.56) \\
& - \int_A \left[(\tau_{ij}n_j\delta u_i - L_1^s\theta^s n_i\delta S_i^s) - (\tau n_i\delta U_i - L_1^f\theta^f n_i\delta S_i^f)\right] dA = 0.
\end{aligned}
$$

By substituting the values of $\theta^s$and $\theta^f$from Eqs. (5.2.33) and (5.2.34), we have

$$
\begin{aligned}
\delta\Gamma_T = & \int_V (C_{ijkl}e_{kl}\delta u_{i,j} + D_{ij}\varepsilon\delta u_{i,j} + D_{kj}e_{kj}\delta U_{i,i} + C\varepsilon\delta U_{i,i})dV \\
& - L_1^s\int_V \theta^s\delta(S_{i,i}^s + \beta_{ij}^s e_{ij} + \beta^s\varepsilon)dV - L_1^f\int_V \theta^f\delta(S_{i,i}^f + \beta_{ij}^f e_{ij} + \beta^f\varepsilon)dV \\
& + T_0\lambda_{ij}^s L_1^f \frac{d}{dt}\int_V S_i^s\delta S_j^s dV + T_0 L_1^f\lambda_{ij}^f \frac{d}{dt}\int_V S_i^f\delta S_j^f dV \\
& + \int_V (\rho^s b_i^s\delta u_i + \rho^f b_i^f\delta U_i)dV \\
& - \int_V \left[\rho_{11}\ddot{u}_i\delta u_i + \rho_{12}\ddot{u}_i\delta U_i + \rho_{12}\delta u_i\ddot{U}_i + \rho_{22}U_i\delta\ddot{U}_i\right] dV \\
& + \int_A \{(\tau_{ij}n_j\delta u_i - L_1^s\theta^s n_i\delta S_i^s) + (\tau n_i\delta U_i - L_1^f\theta^f n_i\delta S_i^f)\}dA = 0. \qquad (5.2.57)
\end{aligned}
$$

Now by applying the divergence theorem along with Eqs. (5.2.33-5.2.34) in Eq. (5.2.57) we find that

$$
\begin{aligned}
\delta\Gamma_T = & \int_V \left[C_{ijkl}e_{kl,j} + D_{ij}\varepsilon_{,j} - L_1^s\beta_{ij}^s\theta_{,j}^s - L_1^f\beta_{ij}^f\theta_{,j}^f + \rho^s b_i^s - (\rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i)\right]\delta u_i dV \\
& \int_V \left[D_{kj}e_{kj,i} + C\varepsilon_{,i} - L_1^s\beta^s\theta_{,j}^s - L_1^f\beta^f\theta_{,j}^f + \rho^s b_i^s - (\rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i)\right]\delta U_i dV \\
& + L_1^s\int_V (\theta_{,j}^s + T_0\lambda_{ij}^s\dot{S}_i{}^f)\delta S_j^s dV + L_1^f\int_V (\theta_{,j}^f + T_0\lambda_{ij}^f\dot{S}_i{}^f)\delta S_j^f dV \qquad (5.2.58) \\
& = 0.
\end{aligned}
$$

Since δu_i, δU_i, δS_i^s and δS_i^f are the virtual displacements and the virtual entropy flow, respectively, therefore in view of Eqs. (5.2.33-5.2.34), it follows from the Eq. (5.2.58) that the first variation of the functional Γ_T should vanish throughout the volume if the equations of motion and heat conduction equations are satisfied identically.

Hence, we established the variational formulation of the temperature-rate dependent theory of fully coupled porothermoelasticity for anisotropic medium.

5.2.5 Reciprocity Theorem

In this section, we shall prove a reciprocity theorem for general problem of anisotropic porothermoelasticity with temperature-rate terms and two thermal relaxation parameters. In order to establish a reciprocity relation, we consider here an anisotropic homogeneous poroelastic body occupying volume V and bounded by the surface A. This system is assumed to be influenced by the body forces b_i^s, b_i^f, traction forces $\mathit{\Gamma}_i^s$, Γ_i^f and surface temperatures Θ^s, Θ^f corresponding to the solid and fluid phases.

If we consider two different sets of porothermoelastic loading/causes as given by

$$Z_n = \{b_i^{s(n)}, b_i^{f(n)}, \Gamma_i^{s(n)}, \Gamma_i^{f(n)}, \Theta^{s(n)}, \Theta^{f(n)}\},\ n = 1, 2, \tag{5.2.59}$$

then the porothermoelastic configurations due to these porothermoelastic loadings (5.2.59) can be explained by different sets of field variables, i.e., displacement and temperatures corresponding to both solid and fluid phases. The thermoelastic configurations corresponding to two different causes are symbolically denoted as

$$Y_n = \{u_i^{(n)}, U_i^{(n)}, \theta^{s(n)}, \theta^{f(n)}\},\ n = 1, 2. \tag{5.2.60}$$

The relationship between two different porothermoelastic loadings and porothermoelastic configurations as given above is called the reciprocity relation, which can be stated by the following reciprocity theorem.

Theorem: We assume that the stresses τ_{ij}, τ and strain functions e_{ij}, ε are of class C^1

whereas, displacements u_i,U_i and temperatures θ^s, θ^f are of class of C^2 in the whole region $V+A$ for $t>0$. If these functions satisfy the equations of motion (5.2.1-5.2.2), generalized heat conduction laws (5.2.5-5.2.6), and other constitutive relations (5.2.3-5.2.4,5.2.7-5.2.10) along with the homogeneous initial conditions and the boundary conditions as

$$\tau_{ij}n_j=\Gamma^s_i,\ \tau n_i=\Gamma^f_i,\ \theta^s=\Theta^s,\ \theta^f=\Theta^f,\ \ \forall x\in A, t>0, \tag{5.2.61}$$

then, we have

$$\begin{aligned}
T_0\int_A(\Gamma^{s(1)}_i*D_tu^{(2)}_i-\Gamma^{s(2)}_i*D_tu^{(1)}_i)dA+T_0\int_A(\Gamma^{f(1)}_i*D_tU^{(2)}_i-\Gamma^{f(2)}_i*D_tU^{(1)}_i)dA\\
+\rho^sT_0\int_V(b^{s(1)}_i*D_tu^{(2)}_i-b^{s(2)}_i*D_tu^{(1)}_i)dV+\rho^fT_0\int_V(b^{f(1)}_i*D_tU^{(2)}_i-b^{f(2)}_i*D_tU^{(1)}_i)dV\\
+K^s_{ij}\int_A(\Theta^{s(1)}*L^s_1\theta^{s(2)}_{,i}n_j-\Theta^{s(2)}*L^s_1\theta^{s(1)}_{,i}n_j)dA\\
+K^f_{ij}\int_A(\Theta^{f(1)}*L^f_1\theta^{f(2)}_{,i}n_j-\Theta^{f(2)}*L^f_1\theta^{f(1)}_{,i}n_jdA=0,
\end{aligned} \tag{5.2.62}$$

where $D_t=\frac{\partial}{\partial t}$, and the $'*'$ notation denotes the convolution operation, and the convolution of any two functions f_1 and f_2 can be defined as $f_1*f_2=\int_0^t f_1(u)f_2(t-u)du$.

Proof: First, by applying the Laplace transformation on stress-strain-temperature relation (5.2.3) corresponding to two different configurations, we have

$$\bar{\tau}^{(1)}_{ij}=C_{ijkl}\bar{e}^{(1)}_{kl}+D_{ij}\bar{\varepsilon}^{(1)}-\beta^s_{ij}\bar{L}^s_1\bar{\theta}^{s(1)}-\beta^f_{ij}\bar{L}^f_1\bar{\theta}^{f(1)}, \tag{5.2.63}$$

and

$$\bar{\tau}^{(2)}_{ij}=C_{ijkl}\bar{e}^{(2)}_{kl}+D_{ij}\bar{\varepsilon}^{(2)}-\beta^s_{ij}\bar{L}^s_1\bar{\theta}^{s(2)}-\beta^f_{ij}\bar{L}^f_1\bar{\theta}^{f(2)}. \tag{5.2.64}$$

where the over-headed bar notation $\bar{f}(s)$ denotes the Laplace transform of any function $f(t)$.

Now, multiplying by $\bar{e}^{(2)}_{ij}$ in Eq. (5.2.63) and by $\bar{e}^{(1)}_{ij}$ in Eq. (5.2.64), then taking the volume integral over the difference of these two equations, we find that

$$\begin{aligned}\int_V (\bar{\tau}_{ij}^{(1)}\bar{e}_{ij}^{(2)} - \bar{\tau}_{ij}^{(2)}\bar{e}_{ij}^{(1)})dV = \int_V & D_{ij}(\bar{\varepsilon}^{(1)}\bar{e}_{ij}^{(2)} - \bar{\varepsilon}^{(2)}\bar{e}_{ij}^{(1)})dV \\ &+ \beta_{ij}^s \bar{L}_1^s \int_V (\bar{\theta}^{s(2)}\bar{e}_{ij}^{(1)} - \bar{\theta}^{s(1)}\bar{e}_{ij}^{(2)})dV \\ &+ \beta_{ij}^f \bar{L}_1^f \int_V (\bar{\theta}^{f(2)}\bar{e}_{ij}^{(1)} - \bar{\theta}^{f(1)}\bar{e}_{ij}^{(2)})dV. \end{aligned} \tag{5.2.65}$$

By applying the Gauss divergence theorem, the first term of L.H.S. can be evaluated as

$$\int_V \bar{\tau}_{ij}^{(1)}\bar{e}_{ij}^{(2)}dV = \int_V \bar{\tau}_{ij}^{(1)}\bar{u}_{i,j}^{(2)}dV = \int_A \bar{\tau}_{ij}^{(1)}n_j\bar{u}_i^{(2)}dA - \int \bar{\tau}_{ij,j}^{(1)}\bar{u}_i^{(2)}dV. \tag{5.2.66}$$

Hence, using equation of motion (5.2.3) and the boundary conditions (5.2.61) in Eq. (3.4.9), we get

$$\int_V \bar{\tau}_{ij}^{(1)}\bar{e}_{ij}^{(2)}dV = \int_A \bar{\Gamma}_i^{s(1)}\bar{u}_i^{(2)}dA + \rho^s \int_V \bar{b}_i^{(1)}\bar{u}_i^{(2)}dV - s^2 \int_V (\rho_{11}\bar{u}_i^{(1)} + \rho_{12}\bar{U}_i^{(1)})\bar{u}_i^{(2)}. \tag{5.2.67}$$

Therefore, by substituting the value of $\int_V \bar{\tau}_{ij}^{(1)}\bar{e}_{ij}^{(2)}dV$ from Eq. (5.2.67) and the analogous value of $\int_V \bar{\tau}_{ij}^{(2)}\bar{e}_{ij}^{(1)}dV$ in Eq. (5.2.65), we find that

$$\begin{aligned} &\int_A (\bar{\Gamma}_i^{s(1)}\bar{u}_i^{(2)} - \bar{\Gamma}_i^{s(2)}\bar{u}_i^{(1)})dA + \rho^s \int_V (\bar{b}_i^{s(1)}\bar{u}_i^{(2)} - \bar{b}_i^{s(2)}\bar{u}_i^{(1)})dV \\ &+\rho_{12}s^2 \int_V (\bar{u}_i^{(1)}\bar{U}_i^{(2)} - u_i^{(2)}\bar{U}_i^{(1)})dV + D_{ij}\int_V (\bar{\varepsilon}^{(2)}\bar{e}_{ij}^{(1)} - \bar{\varepsilon}^{(1)}\bar{e}_{ij}^{(2)})dV \\ &+\beta_{ij}^s \bar{L}_1^s \int_V (\bar{\theta}^{s(1)}\bar{e}_{ij}^{(2)} - \bar{\theta}^{s(2)}\bar{e}_{ij}^{(1)})dV + \beta_{ij}^f \bar{L}_1^f \int_V (\bar{\theta}^{f(1)}\bar{e}_{ij}^{(2)} - \bar{\theta}^{f(2)}\bar{e}_{ij}^{(1)})dV = 0. \end{aligned} \tag{5.2.68}$$

Next, by using the same argument as above with Eqs. (5.2.2) and (5.2.4), a similar relation corresponding to the fluid phase can be obtained as

$$\begin{aligned} &\int_A (\bar{\Gamma}_i^{f(1)}\bar{U}_i^{(2)} - \bar{\Gamma}_i^{f(2)}\bar{U}_i^{(1)})dA + \rho^f \int_V (\bar{b}_i^{f(1)}\bar{U}_i^{(2)} - \bar{b}_i^{f(2)}\bar{U}_i^{(1)})dV + \\ &\rho_{12}s^2 \int_V (\bar{U}_i^{(1)}\bar{u}_i^{(2)} - U_i^{(2)}\bar{u}_i^{(1)})dV + D_{ij}\int_V (\bar{\varepsilon}^{(1)}e_{ij}^{(2)} - \bar{\varepsilon}^{(2)}e_{ij}^{(1)})dV \\ &+\beta^s \bar{L}_1^s \int_V (\bar{\theta}^{s(1)}\bar{\varepsilon}^{(2)} - \bar{\theta}^{s(2)}\bar{\varepsilon}^{(1)})dV + \beta^f \bar{L}_1^f \int_V (\bar{\theta}^{f(1)}\bar{\varepsilon}^{(2)} - \bar{\theta}^{f(2)}\bar{\varepsilon}^{(2)})dV = 0 \end{aligned} \tag{5.2.69}$$

Eqs. (5.2.68) and (5.2.69) constitute the first and second part of the reciprocity theorem involving the mechanical porothermoelastic loads, i.e., body forces and surface traction force terms only.

Now, in order to find the third and fourth part of the reciprocity theorem, we apply the Laplace transform on the heat conduction Eq. (5.2.9) for two different configurations as

$$K^s_{ij}\bar{\theta}^{s(1)}_{,ij} = \rho c^s_E s \bar{L}^s_2(\bar{\theta}^{s(1)}) + T_0 s(\beta^s_{ij}\bar{e}^{(1)}_{ij} + \beta^s\bar{\varepsilon}^{(1)}), \tag{5.2.70}$$

$$K^s_{ij}\bar{\theta}^{s(2)}_{,ij} = \rho c^s_E s \bar{L}^s_2(\bar{\theta}^{s(2)}) + T_0 s(\beta^s_{ij}\bar{e}^{(2)}_{ij} + \beta^s\bar{\varepsilon}^{(2)}). \tag{5.2.71}$$

Again, multiplying by $\bar{\theta}^{s(2)}$ in Eq. (5.2.70) and by $\bar{\theta}^{s(1)}$ in Eq. (5.2.71) and then by considering the volume integral over the difference of these two equations, we find that

$$\begin{aligned} K^s_{ij}\int_V (\bar{\theta}^{s(1)}_{,ij}\bar{\theta}^{s(2)} - \bar{\theta}^{s(2)}_{,ij}\bar{\theta}^{s(1)})dV = T_0 s\beta^s_{ij}&\int_V (\bar{e}^{(1)}_{ij}\bar{\theta}^{s(2)} - \bar{e}^{(2)}_{ij}\bar{\theta}^{s(1)})dV \\ + T_0 s\beta^f_{ij}&\int_V (\bar{\varepsilon}^{(1)}\bar{\theta}^{s(2)} - \bar{\varepsilon}^{(2)}\bar{\theta}^{s(1)})dV. \end{aligned} \tag{5.2.72}$$

In the Eq. (5.2.72), the first integral of the L.H.S. can be simplified by using the divergence theorem as

$$K^s_{ij}\int_V \bar{\theta}^{s(1)}_{,ij}\bar{\theta}^{s(2)}dV = K^s_{ij}\int_A \bar{\theta}^{s(1)}_{,i} n_j\bar{\theta}^{s(2)}dA - K^s_{ij}\int_V \bar{\theta}^{s(1)}_{,i}\bar{\theta}^{s(2)}_{,j}dV. \tag{5.2.73}$$

Therefore, by using the value of $K^s_{ij}\int_V \bar{\theta}^{s(1)}_{,ij}\bar{\theta}^{s(2)}dV$ from Eq. (5.2.73) and analogous value of $K^s_{ij}\int_V \bar{\theta}^{s(2)}_{,ij}\bar{\theta}^{s(1)}dV$ with the boundary condition (5.2.61), the Eq. (5.2.72) yields

$$\begin{aligned} K^s_{ij}\int_A (\Theta^{s(2)}\theta^{s(1)}_{,i}n_j - \Theta^{s(1)}\theta^{s(2)}_{,i}n_j)dA + \beta^s_{ij}T_0 s&\int_V (\bar{e}^{(2)}_{ij}\bar{\theta}^{s(1)} - \bar{e}^{(1)}_{ij}\bar{\theta}^{s(2)})dV \\ +\beta^s T_0 s&\int_V (\bar{\varepsilon}^{(2)}\bar{\theta}^{s(1)} - \bar{\varepsilon}^{(1)}\bar{\theta}^{s(2)})dV = 0. \end{aligned} \tag{5.2.74}$$

Similarly, we can obtain another relation involving surface boundary temperature term with the help of Eq. (5.2.61) as

$$\begin{aligned}K^f_{ij}\int_A(\Theta^{f(2)}\theta^{f(1)}_{,i}n_j-\Theta^{f(1)}\theta^{f(2)}_{,i}n_jdA+\beta^f_{ij}T_0s\int_V(\bar{e}^{(2)}_{ij}\bar{\theta}^{f(1)}-\bar{e}^{(1)}_{ij}\bar{\theta}^{f(2)})dV\\+\beta^fT_0s\int_V(\bar{\varepsilon}^{(2)}\bar{\theta}^{f(1)}-\bar{\varepsilon}^{(1)}\bar{\theta}^{f(2)})dV=0.\end{aligned} \quad (5.2.75)$$

Now, we eliminate $\beta^s_{ij}T_0\int_V(\bar{e}^{(2)}_{ij}\bar{\theta}^{s(1)}-\bar{e}^{(1)}_{ij}\bar{\theta}^{s(2)})dV$ from Eqs. (5.2.68) and (5.2.74) to obtain

$$\begin{aligned}T_0s\int_A(\bar{\Gamma}^{s(1)}_i\bar{u}^{(2)}_i-\bar{\Gamma}^{s(2)}_i\bar{u}^{(1)}_i)dA+\rho^sT_0s\int_V(\bar{b}^{s(1)}_i\bar{u}^{(2)}_i-\bar{b}^{s(2)}_i\bar{u}^{(1)}_i)dV\\+\rho_{12}T_0s^3\int_V(\bar{u}^{(1)}_i\bar{U}^{(2)}_i-u^{(2)}_i\bar{U}^{(1)}_i)dV+T_0D_{ij}s\int_V(\bar{\varepsilon}^{(2)}\bar{e}^{(1)}_{ij}-\bar{\varepsilon}^{(1)}\bar{e}^{(2)}_{ij})dV\\+T_0s\beta^f_{ij}\bar{L}^f_1\int_V(\bar{\theta}^{f(2)}\bar{e}^{(1)}_{ij}-\bar{\theta}^{f(1)}\bar{e}^{(2)}_{ij})dV+K^s_{ij}\bar{L}^s_1\int_A(\bar{\Theta}^{s(1)}\bar{\theta}^{s(2)}_{,i}n_j-\bar{\Theta}^{s(2)}\bar{\theta}^{s(1)}_{,i}n_j)dA\\+s\beta^sT_0\bar{L}^s_1\int_V(\bar{\varepsilon}^{(1)}\bar{\theta}^{s(2)}-\bar{\varepsilon}^{(2)}\bar{\theta}^{s(1)})dV=0.\end{aligned} \quad (5.2.76)$$

Also, by eliminating $\beta^fT_0\int_V(\bar{\varepsilon}^{(2)}\bar{\theta}^{s(1)}-\bar{\varepsilon}^{(1)}\bar{\theta}^{s(2)})dV$ from Eqs. (5.2.69) and (5.2.75), it gives

$$\begin{aligned}T_0s\int_A(\bar{\Gamma}^{f(1)}_i\bar{U}^{(2)}_i-\bar{\Gamma}^{f(2)}_i\bar{U}^{(1)}_i)dA+\rho^fsT_0\int_V(\bar{b}^{f(1)}_i\bar{U}^{(2)}_i-\bar{b}^{f(2)}_i\bar{U}^{(1)}_i)dV\\+\rho_{12}T_0s^3\int_V(\bar{U}^{(2)}_i\bar{u}^{(1)}_i-U^{(1)}_i\bar{u}^{(2)}_i)dV+D_{ij}sT_0\int_V(\bar{\varepsilon}^{(2)}e^{(1)}_{ij}-\bar{\varepsilon}^{(1)}e^{(2)}_{ij})dV\\+\beta^sT_0\bar{L}^s_1\int_V(\bar{\theta}^{s(2)}\bar{\varepsilon}^{(1)}-\bar{\theta}^{s(1)}\bar{\varepsilon}^{(2)})dV+K^f_{ij}\bar{L}^f_1\int_A(\bar{\Theta}^{f(1)}\bar{\theta}^{f(2)}_{,i}n_i-\bar{\Theta}^{f(2)}\bar{\theta}^{f(1)}_{,i}n_idA\\+\beta^f_{ij}T_0\bar{L}^f_1\int_V(\bar{e}^{(1)}_{ij}\bar{\theta}^{s(2)}-\bar{e}^{(2)}_{ij}\bar{\theta}^{s(1)})dV=0.\end{aligned} \quad (5.2.77)$$

Further, we combine the Eqs. (5.2.76) and (5.2.77) to obtain the final reciprocity relation in terms of thermal as well as mechanical loads as

$$
\begin{aligned}
T_0 s \int_A (\bar{\Gamma}_i^{s(1)} \bar{u}_i^{(2)} - \bar{\Gamma}_i^{s(2)} \bar{u}_i^{(1)}) dA + T_0 s \int_A (\bar{\Gamma}_i^{f(1)} \bar{U}_i^{(2)} - \bar{\Gamma}_i^{f(2)} \bar{U}_i^{(1)}) dA \\
+\rho^s s T_0 \int_V (\bar{b}_i^{s(1)} \bar{u}_i^{(2)} - \bar{b}_i^{s(2)} \bar{u}_i^{(1)}) dV + \rho^f s T_0 \int_V (\bar{b}_i^{f(1)} \bar{U}_i^{(2)} - \bar{b}_i^{f(2)} \bar{U}_i^{(1)}) dV \\
+K_{ij}^s \bar{L}_1^s \int_A (\bar{\Theta}^{s(1)} \bar{\theta}_{,i}^{s(2)} n_j - \bar{\Theta}^{s(2)} \bar{\theta}_{,i}^{s(1)} n_j) dA \\
+K_{ij}^f \bar{L}_1^f \int_A (\bar{\Theta}^{f(1)} \bar{\theta}_{,i}^{f(2)} n_j - \bar{\Theta}^{f(2)} \bar{\theta}_{,i}^{f(1)} n_j) dA = 0.
\end{aligned}
\tag{5.2.78}
$$

Hence, by applying the inverse Laplace transform to the Eq. (5.2.78), we obtain the required convolutional reciprocity relation as given by Eq. (5.2.62).

5.2.6 Conclusion

This subchapter established some theoretical results based on the TRDPE theory. In the uniqueness theorem, it is shown that more than one solution to any mixed initial boundary value problem of porothermoelasticity is not possible. Further, a variational principle is established for the anisotropic porothermoelastic medium with the effects of two thermal relaxation terms. The variational principles are important in the sense that they provide a theoretical basis for numerical techniques, and therefore an approximation of the solution can be given. They also provide an alternative way to determine the state of the dynamics of the thermomechanical systems. The variational principle for a porothermoelastic system has not been found in the literature. Therefore, a general variational principle for anisotropic medium under the temperature-rate dependent porothermoelastic system is established. Also, a reciprocity result is derived for the temperature-rate dependent porothermoelasticity theory of an anisotropic medium. The reciprocity relations are helpful in the theoretical development of BEMs. Hence, in this subchapter, some theoretical results are established that can be helpful in investigating the porothermoelastic problems with FEM and BEM.

5.3 A Half Space Problem of Porothermoelasticity Based on the Temperature-Rate Dependent Theory [3]

5.3.1 Introduction

This subchapter is devoted to investigating the thermomechanical interactions inside a porous medium under the TRDPTE theory. A TRDPTE theory is developed in the previous subchapters, and some theoretical results are established under this theory. Now, it is worth investigating the effects of the present TRDPTE theory for a physical problem of porous medium with thermal and mechanical boundary loadings.

The effect of thermal loading due to laser pulse on thermoelastic porous medium under a thermoelasticity has been discussed by Othman and Marin (2017). Sherief and Hussein (2012) have studied a half space problem to study liquid filled solid medium under applied thermal shock. Sharma (2008a) has discussed the results of wave propagation under porothermoelasticity. Ezzat and Ezzat (2016) have solved a porothermoelastic problem under the fractional order theory. Carcione et al. (2019) have developed a numerical simulation to study the wave propagation in the porothermoelastic solid. Sur (2020) have analyzed the wave propagation in porous asphalts on accounts of a memory response. Alzahrani and Abbas (2020a) have implemented the finite element method to solve a one dimensional problem of porothermoelasticity under Green and Naghdi theory. Saeed et al. (2020) have stated the governing the equations for temperature-rate dependent theory (GL theory) of porothermoelasticity for isotropic medium and solved a one dimensional problem using FEM. Alzahrani and Abbas (2020b) and Guo et al. (2019) have analyzed different thermomechanical problems in this context.

Here, a one dimensional half space problem of a porothermoelastic medium is investigated under the TRDPTE theory, where a thermal shock is applied at the boundary of half space. The behaviours of the field variables are analyzed for the two-phase

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medium. Further, the effect of the present TRDPTE theory is studied by comparing the behaviour of the field variables under four different theories, namely Biot theory (Classical theory) of thermoelasticity, GL theory (TRDTE theory), Biot's theory of porothermoelasticity (BTPTE) and the present TRDPTE theory.

5.3.2 Formulation of the Problem

Consider one dimensional half space ($x \geq 0$) composed of saturated porothermoelastic medium that is saturated with non-viscous liquid. The boundary of the medium are assumed to be traction free. Therefore, all the field variables are assumed to be the function of x and t only. Now, by following Eqs. (5.2.1-5.2.4), and (5.2.11-5.2.12), in the absence of external heat source and body force, the governing equations of the present temperature-rate dependent porothermoelasticity theory (TRDPTE) in the context of this problem are considered as

$$\frac{\partial \tau_{xx}}{\partial x} = \rho_{11}\ddot{u} + \rho_{12}\ddot{U}, \tag{5.3.1}$$

$$\frac{\partial \tau}{\partial x} = \rho_{12}\ddot{u} + \rho_{22}\ddot{U}, \tag{5.3.2}$$

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} + D\frac{\partial U}{\partial x} - B^s\left(1 + t_1^s\frac{\partial}{\partial t}\right)\vartheta^s - B^f\left(1 + t_1^f\frac{\partial}{\partial t}\right)\vartheta^f, \tag{5.3.3}$$

$$\tau = D\frac{\partial u}{\partial x} + C\frac{\partial U}{\partial x} - \beta^s\left(1 + t_1^s\frac{\partial}{\partial t}\right)\vartheta^s - \beta^f\left(1 + t_1^f\frac{\partial}{\partial t}\right)\vartheta^f, \tag{5.3.4}$$

$$K^s\frac{\partial^2 \vartheta^s}{\partial x^2} = \rho c_E^s(\dot{\vartheta}^s + t_2^s\ddot{\vartheta}^s) + \vartheta_0\left(B^s\frac{\partial \dot{u}}{\partial x} + \beta^s\frac{\partial \dot{U}}{\partial x}\right), \tag{5.3.5}$$

$$K^f\frac{\partial^2 \vartheta^f}{\partial x^2} = \rho c_E^f(\dot{\vartheta}^f + t_2^f\ddot{\vartheta}^f) + \vartheta_0\left(B^f\frac{\partial \dot{u}}{\partial x} + \beta^f\frac{\partial \dot{U}}{\partial x}\right). \tag{5.3.6}$$

where B^s and B^f are the thermoelastic constants for isotropic medium..

The set of Eqs. (5.3.1-5.3.6) will reduce to the corresponding equations that represent different theories if we set different values of the parameters as follows:

- Biot's theory of porothermoelasticity (BTPTE): when $t_1^s = t_2^s = t_1^f = t_2^2 = 0$

- Biot's theory of thermoelasticity(BTE): when $t_1^s = t_2^s = t_1^f = t_2^2 = 0$, $D = C = B^f = \beta^f = \beta^s = K^f = c_E^f = 0$
- Temperature-rate dependent thermoelasticity theory (TRDTE): when $D = C = B^f = \beta^f = \beta^s = K^f = c_E^f = 0$.

Now, to make the above equations dimensionless, we consider the non-dimensional quantities corresponding the physical quantities and parameters as

$$(x', u', U') = c_1 n(x, u, U),\ (t', t_1^{s'}, t_2^{s'}, t_1^{f'}, t_2^{f'}) = c_1^2 n(t, t_1^s, t_2^s, t_1^f, t_2^f),$$

$$(\tau'_{xx}, \tau') = (\tau_{xx}, \tau)/R,\ (\vartheta^{s'}, \vartheta^{f'}) = \tfrac{B^s}{R}(\vartheta^s, \vartheta^f),$$

where $c_1 = \sqrt{R/\rho_{11}}$,$n = \rho c_E^s / K^s$ and $R = (\lambda + 2\mu)$.

Therefore, by using above dimensionless quantities in the Eqs. (5.3.1-5.3.6), the corresponding non-dimensional form of Eqs. (5.3.1-5.3.6) are obtained as

$$\frac{\partial \tau_{xx}}{\partial x} = \ddot{u} + m_1 \ddot{U}, \tag{5.3.7}$$

$$\frac{\partial \tau}{\partial x} = m_1 \ddot{u} + m_2 \ddot{U}, \tag{5.3.8}$$

$$\tau_{xx} = \frac{\partial u}{\partial x} + a_1 \frac{\partial U}{\partial x} - \left(1 + t_1^s \frac{\partial}{\partial t}\right) \vartheta^s - b_1 \left(1 + t_1^f \frac{\partial}{\partial t}\right) \vartheta^f, \tag{5.3.9}$$

$$\tau = a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial U}{\partial x} - b_2 \left(1 + t_1^s \frac{\partial}{\partial t}\right) \vartheta^s - b_3 \left(1 + t_1^f \frac{\partial}{\partial t}\right) \vartheta^f, \tag{5.3.10}$$

$$\frac{\partial^2 \vartheta^s}{\partial x^2} = (\dot{\vartheta}^s + t_2^s \ddot{\vartheta}^s) + \gamma_1 (\frac{\partial \dot{u}}{\partial x} + b_2 \frac{\partial \dot{U}}{\partial x}) + \gamma_2 (\vartheta^s - \vartheta^f), \tag{5.3.11}$$

$$\frac{\partial^2 \vartheta^f}{\partial x^2} = F w (\dot{\vartheta}^f + t_2^f \ddot{\vartheta}^f) + w \gamma_1 (b_1 \frac{\partial \dot{u}}{\partial x} + b_3 \frac{\partial \dot{U}}{\partial x}) - \gamma_2 w (\vartheta^s - \vartheta^f), \tag{5.3.12}$$

where $m_1 = \rho_{12}/\rho_{11}$, $m_2 = \rho_{22}/\rho_{11}$,$a_1 = D/R$,$a_2 = C/R$, $b_1 = B^f/B^s$, $b_2 = \beta^s/B^s$, $b_3 = \beta^f/B^s$, $F = c_E^f/c_E^f$, $w = K^f/K^s$, $\gamma_1 = \vartheta_0 (B^s)^2/(\rho c_E^s R)$, $\gamma_2 = K K^s \rho_{11}/(\rho c_E^s)^2/R$.

In the Eqs. (5.3.7-5.3.12), we have neglected the prime notation (') for simplicity.

For the present problem, the homogeneous initial conditions along with the boundary conditions are taken as

$$\tau_{xx}(0,t)=0,\ \tau(0,t)=0,\ \text{for } t>0 \text{ and} \, t\in\mathbb{R}^{+}, \tag{5.3.13}$$

$$q^{s}+a^{s}\vartheta^{s}(0,t)=H(t)r(t),\ q^{f}+a^{f}\vartheta^{f}(0,t)=H(t)r(t),\ \text{for } t>0 \text{ and} \, t\in\mathbb{R}^{+}, \tag{5.3.14}$$

where $H(t)$ is the unit step function, $a^s$and a^f are the Biot numbers, and $r(t)$ is surface heating magnitude (see ref. Sherief and Hussein (2012)).

Next, by applying the Laplace transformation for the time domain in the Eqs. (5.3.7-5.3.12), results

$$D\grave{\tau}_{xx}=s^{2}(\grave{u}+m_{1}\grave{U}), \tag{5.3.15}$$

$$D\grave{\tau}=s^{2}(m_{1}u+m_{2}U), \tag{5.3.16}$$

$$\grave{\tau}_{xx}=D\grave{u}+a_{1}D\grave{U}-z_{0}\grave{\vartheta}^{s}-b_{1}z_{1}\grave{\vartheta}^{f}, \tag{5.3.17}$$

$$\grave{\tau}=a_{1}D\grave{u}+a_{2}D\grave{U}-b_{2}z_{0}\grave{\vartheta}^{s}-b_{3}z_{1}\grave{\vartheta}^{f}, \tag{5.3.18}$$

$$D^{2}\grave{\vartheta}^{s}=pz_{2}\grave{\vartheta}^{s}+\gamma_{1}sD\grave{u}+\gamma_{1}sb_{2}D\grave{U})+\gamma_{2}(\grave{\vartheta}^{s}-\grave{\vartheta}^{f}), \tag{5.3.19}$$

$$D^{2}\grave{\vartheta}^{f}=Fwpz_{2}\grave{\vartheta}^{s}+w\gamma_{1}sb_{1}D\grave{u}+w\gamma_{1}sb_{3}D\grave{U})+w\gamma_{2}(\grave{\vartheta}^{s}-\grave{\vartheta}^{f}), \tag{5.3.20}$$

where Laplace transform of for any function $f(t)$ is defined by $\grave{f}(s)=\int_{0}^{\infty}e^{-st}f(t)dt$, s being the Laplace transform parameter. Here we use the notation $D=\frac{\partial}{\partial x}$ and
$z_{0}=(1+t_{1}^{s}s)$,$z_{1}=(1+t_{1}^{f}s)$, $z_{2}=(1+t_{2}^{s}s)$,$z_{3}=(1+t_{1}^{f}s)$.

Now, the Eqs. (5.3.17) and (5.3.18) can be rewritten as

$$aD\grave{u}=-a_{2}\grave{\tau}_{xx}+a_{1}\grave{\tau}-(a_{2}-b_{2}a_{1})z_{0}\grave{\vartheta}^{s}-(a_{2}b_{1}-a_{1}b_{3})z_{1}\grave{\vartheta}^{f}, \tag{5.3.21}$$

$$aD\grave{U}=a_{1}\grave{\tau}_{xx}-\grave{\tau}+(a_{1}-b_{2})z_{0}\grave{\vartheta}^{s}+(a_{1}b_{1}-b_{3})z_{1}\grave{\vartheta}^{f}. \tag{5.3.22}$$

Further, substituting Eqs. (5.3.21-5.3.22) in Eqs. (5.3.15-5.3.16,5.3.19-5.3.20) to eliminate $\grave{u}$ and $\grave{U}$, and thus obtained a system of differential equations in the matrix form as

$$D^{2}\grave{X}(x,s)=A(s)\grave{X}(x,s), \tag{5.3.23}$$

where $\grave{X} = \begin{bmatrix} \grave{\tau}_{xx}(x,s) \\ \grave{\tau}(x,s) \\ \grave{\vartheta}^s(x,s) \\ \grave{\vartheta}^f(x,s) \end{bmatrix}$, and A is the coefficient matrix of order 4×4 such that

$A_{11} = p^2(m_1a_1 - a_2)$, $A_{12} = p^2(a_1 - m_1)$, $A_{13} = p^2z_0(m_1a_1 - m_1b_2 - a_2 + a_1b_2)$, $A_{14} = p^2z_1(m_1a_1b_1 - m_1b_3 - a_2b_1 + a_1b_3)$, $A_{21} = p^2(m_2a_1 - m_1a_2)$, $A_{22} = a^2(m_1a_1 - m_2)$, $A_{23} = p^2z_0(m_2a_1 - m_2b_2 - m_1a_2 + m_1a_1b_2)$, $A_{24} = p^2z_1(m_1a_1b_1 - m_2b_3 - m_1a_2b_1 + m_1a_1b_3)$, $A_{31} = p\gamma_1(a_1b_2 - a_2)$, $A_{32} = p\gamma_1(a_1 - b_2)$, $A_{33} = p\gamma_1z_0(b_2a_1 - b_2^2 - a_2 + b_2a_1) + \gamma_2 + pz_2$, $A_{34} = p\gamma_1z_1(b_2a_1b_1 - b_2b_3 - a_2b_1 + a_1b_3) - \gamma_2$, $A_{41} = wp\gamma_1(a_1b_3 - a_2)$, $A_{42} = wp\gamma_1(a_1b_1 - b_3)$, $A_{43} = wp\gamma_1z_0(a_1b_3 - b_2b_3 - a_1b_1 + a_1b_1b_2)$, $A_{44} = wp\gamma_1z_1(2a_1b_1b_3 - b_3^2 - a_2b_1^2) + Fwz_3 - w\gamma_2$.

Hence, the solution of the Eq. (5.3.23), can written as

$$\grave{X}(x,s) = \grave{N}(x,s)e^{-\sqrt{A(s)}x} + \grave{M}(x,s)e^{\sqrt{A(s)}x}. \tag{5.3.24}$$

In the present case, all the field variables can be assumed to be bounded and tends to 0 as $x \to \infty$. Therefore, the simplified form of Eq. (5.3.24) can be written as

$$\grave{X}(x,s) = \grave{X}(0,s)e^{-\sqrt{A(s)}x}, \tag{5.3.25}$$

where $\grave{X}(0,s) = [\grave{\tau}_{xx}(0,s), \grave{\tau}(0,s), \grave{\vartheta}^s(0,s), \grave{\vartheta}^f(0,s)] = [0,0,1/s,1/s]^T$ from boundary conditions (5.3.13-5.3.14).

Now, we apply the Cayley-Hamilton theorem to determine the values of $e^{-\sqrt{A(s)}x}$.

The characteristic equation of A is given by

$$\det(A - \omega I) = 0, \tag{5.3.26}$$

here ω is the characteristic parameter, and we denote ω_1, ω_2, ω_3 and ω_4 to be the roots of Eq. (5.3.26). Hence the expansion $e^{-\sqrt{A(s)}x} = \Sigma_{i=0}^{\infty}(-\sqrt{A(s)}x)^i/i!$ can be expressed as

$$e^{-\sqrt{A(s)}x} = \zeta_0 I + \zeta_1 A + \zeta_2 A^2 + \zeta_3 A^3 \tag{5.3.27}$$

where ζ_0, ζ_1, ζ_2 and ζ_3 are the parameters dependent of s only.

Now, in view of Eq. (5.3.26), Eq. (5.3.27) yields

$$e^{-\sqrt{\omega_1}x} = \zeta_0 + \zeta_1\omega_1 + \zeta_2\omega_1^2 + \zeta_3\omega_1^3, \tag{5.3.28}$$

$$e^{-\sqrt{\omega_2}x} = \zeta_0 + \zeta_1\omega_2 + \zeta_2\omega_2^2 + \zeta_3\omega_2^3, \tag{5.3.29}$$

$$e^{-\sqrt{\omega_3}x} = \zeta_0 + \zeta_1\omega_3 + \zeta_2\omega_3^2 + \zeta_3\omega_3^3, \tag{5.3.30}$$

$$e^{-\sqrt{\omega_4}x} = \zeta_0 + \zeta_1\omega_4 + \zeta_2\omega_4^2 + \zeta_3\omega_4^3. \tag{5.3.31}$$

After solving the linear algebraic Eqs. (5.3.28-5.3.31), the values of ζ_0, ζ_1, ζ_2 and ζ_3 are obtained and therefore, by using these values along with Eqs. (5.3.25,5.3.27), an analytical expression for the distribution of temperature and stress fields of both the solid and liquid phases can be determined. Finally, using the solutions of temperature and stress field, the solution for displacement field in the Laplace transform domain can also be derived from the Eqs. (5.3.15-5.3.16). Hence, the solution of all physical quantities in the Laplace transformed domain is derived.

5.3.3 Numerical Discussion

The analytical solution for the field variables derived in Section 1.3.2 can be transformed in the real time domain by applying the inverse Laplace transformation. Therefore, we apply numerical inversion of the Laplace transform method proposed by Stehfest (1970) to get the solution in the real time domain.

For the numerical calculations, we have considered a sandstone saturated with kerosene, and by following Sherief and Hussein (2012) we take the values of the two phase material as

$\lambda = 453.69 \times 10^6$Pa, $\mu = 280.94 \times 10^6$Pa, $D = 75.79 \times 10^6$Pa, $C = 33.25 \times 10^6$Pa,

$\alpha^s = 2.257 \times 10^{-6}/^o$C, $\alpha^{sf} = -0.33 \times 10^{-6}/^o$C, $\alpha^{fs} = -0.88 \times 10^{-6}/^o$C,

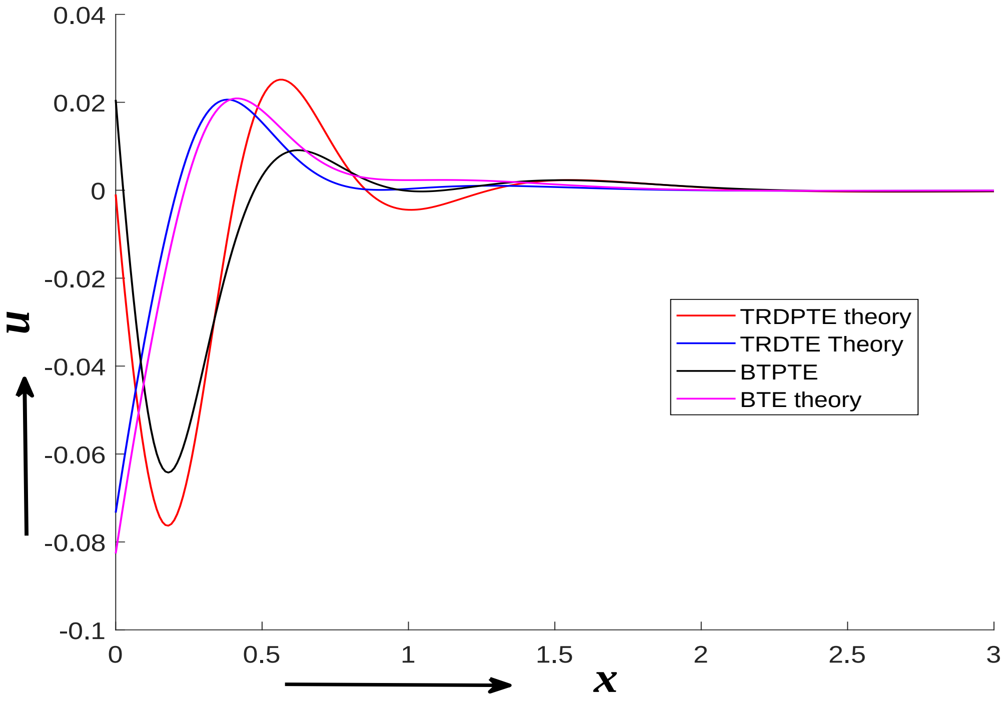
$\alpha^f = 0.932 \times 10^{-6}/^o$C, $K^s = 1.83$ W/m^{o}C, $K^f = 0.148$ W/m^{o}C, $c_E^s = 1.1715$ K J/kg^{o}C,

$c_E^f = 2.092$ K J/kg^{o}C, $K = 1.84 \times 10^{-3}$W/m^{2o}C, $\rho^s = 2360$ kg/m^3, $\rho^f = 723$ kg/m^3, $\rho_{12} = -0.001\rho$, $\beta = 0.26$, $a^s = a^f = 1$, $r(t) = 1$.

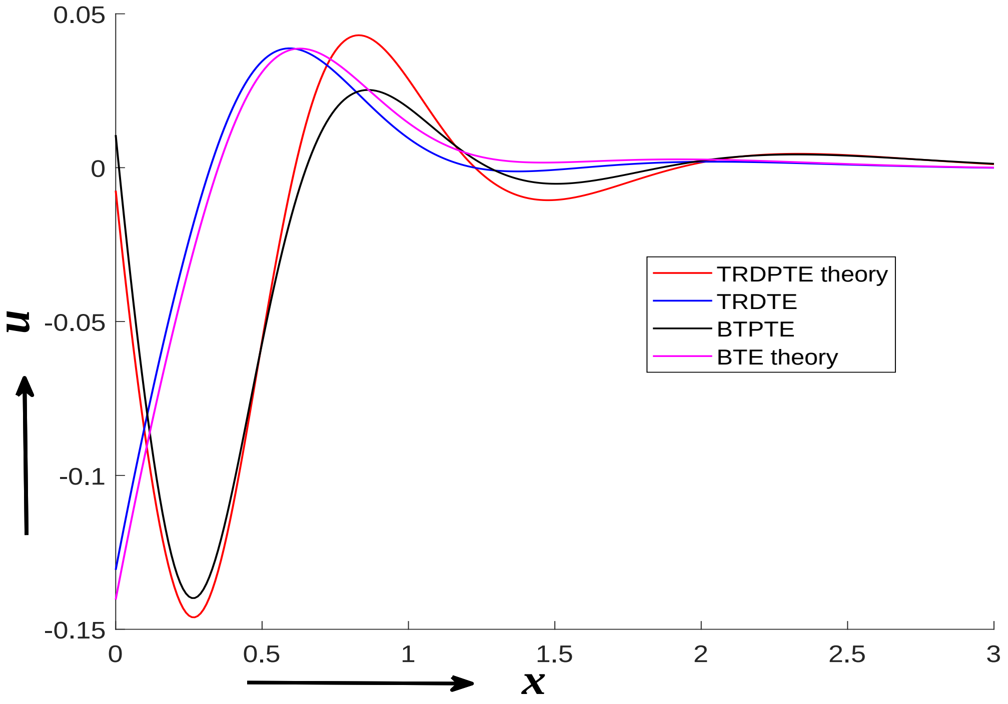
The numerical computations are performed by using MATLAB software. The dimensionless values of the relaxation parameters are taken as $t_1^s = 0.05$, $t_2^s = 0.03$, $t_1^f = 0.04$, and $t_2^f = 0.02$. The distributions of different fields in the contexts of four different theories are shown in various figures.

Figures 5.3.1 (a,b)-5.3.6(a,b) show the distributions of displacement of solid phase, displacement of liquid phase, temperature of solid phase, temperature of liquid phase, stress due to solid phase, and stress (pressure) due to fluid phase, respectively for TRDPTE and BTPTE theories with porosity and without porosity. Sherief (1993) has solved a half space problem based on TRDTE and Biot's thermoelasticity theories. Therefore the comparison of the results of TRDPTE with the BTE and TRDTE theories as observed from these figures demonstrate the agreement of obtained results.

All the field variables show an almost similar trend of variation under all four theories (i.e., TRDPTE, BTPTE, BTE, TRDTE). However, there is a significant difference in the values of fields, and there is a significant disagreement of different theories. The effect of porosity is noted to be much prominent as it is observed that the distribution of fields under the theories with porosity (TRDPTE and BTPTE) have a closer match of behaviour as compared to the plots in the context of the theories without porosity (TRDTE and BTE). The effects of porosity are significantly pronounced in the distribution of the stress fields. The stress field due to the solid phase is noticed to be compressive under the theories without porosity (TRDTE and BTE). However, TRDPTE and BTPTE, i.e., the theories of porothermoelasticity, indicate that in the presence of porosity, the stress due to solid phase is initially tensile for a range, and then it becomes compressive (see Figs 5.3.5 (a,b)). The stress due to the liquid phase is noted to be compressive, as observed in Figs 5.3.6 (a,b). Furthermore, it is revealed that the effect of temperature-rate terms is more prominent in the presence of porosity.

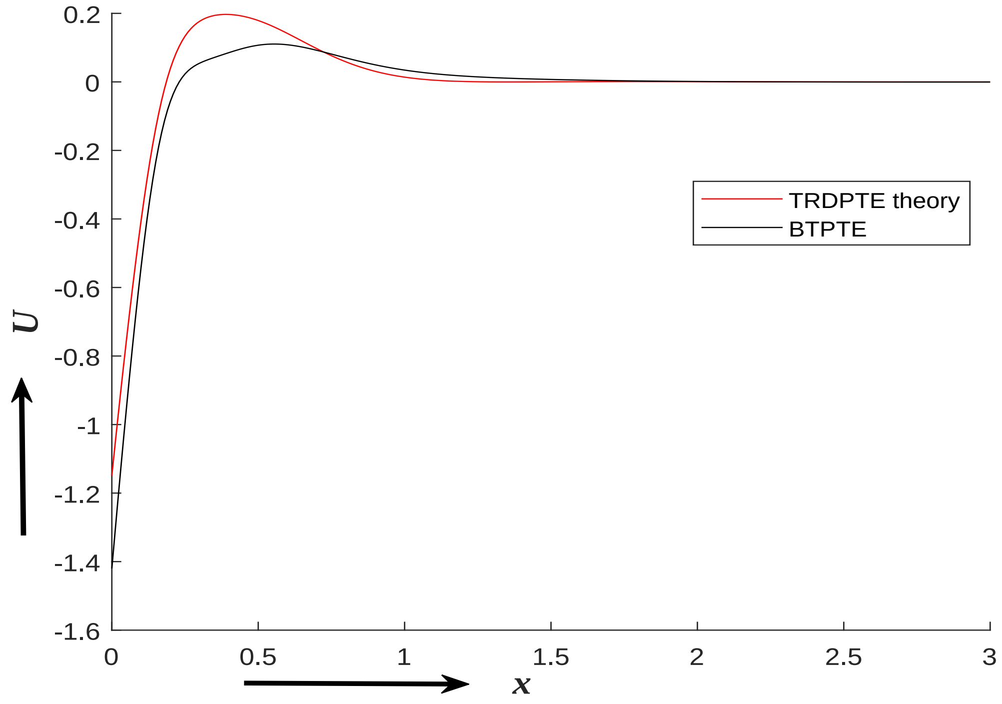


(a) $t = 0.36$

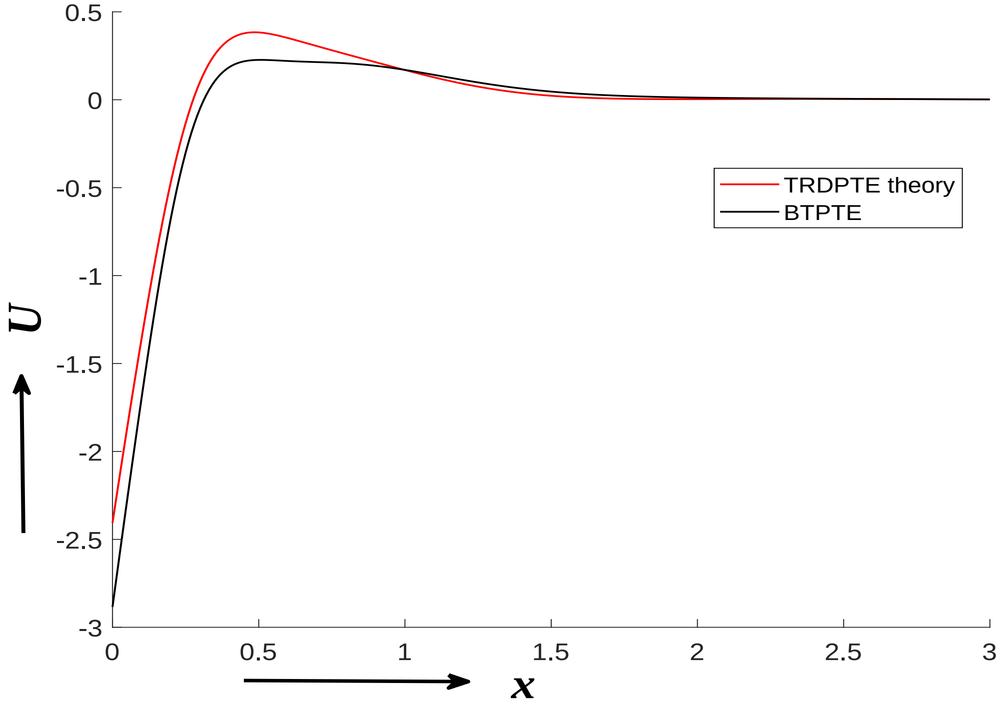


(b) $t = 0.57$

Figure 5.3.1: Variation of displacement (u) due to solid phase at different time

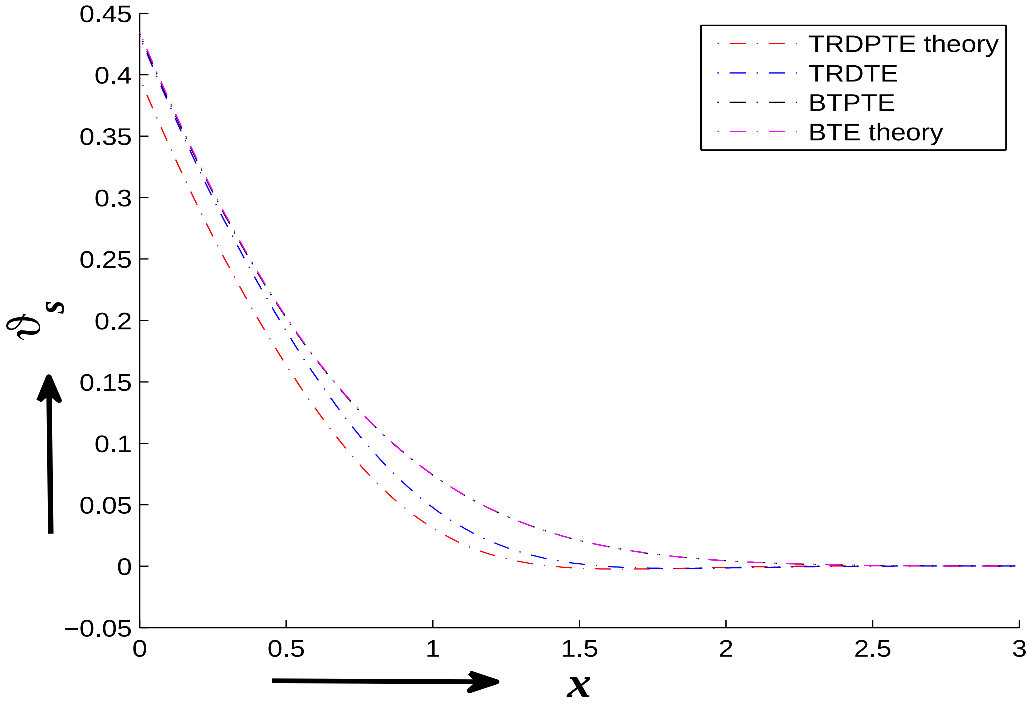


(a) $t = 0.36$

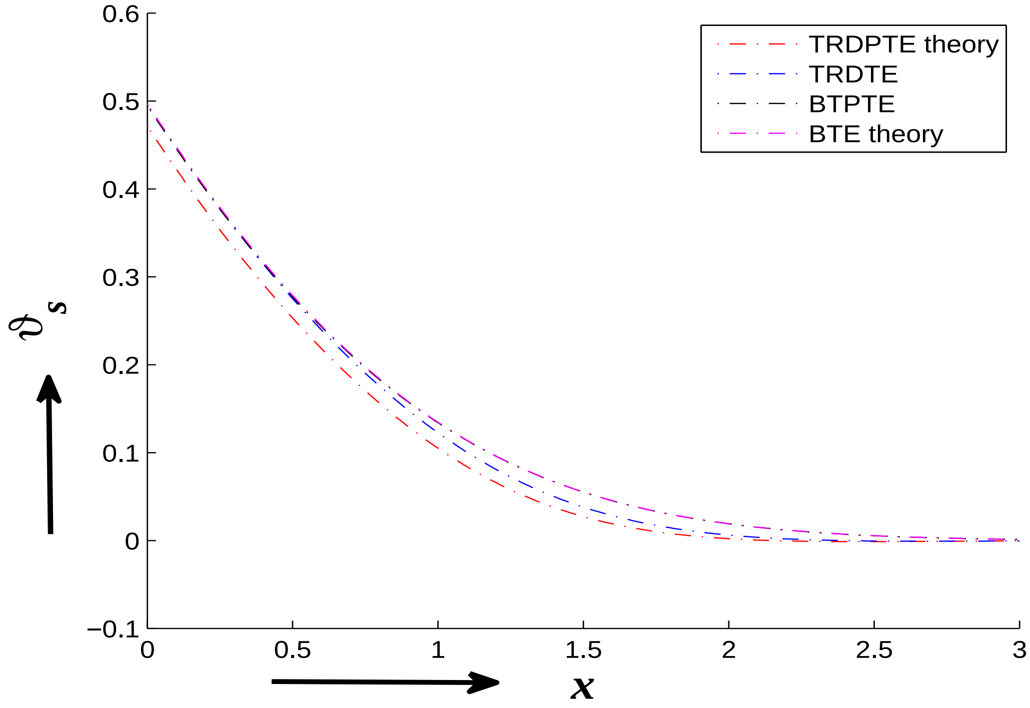


(b) $t = 0.57$

Figure 5.3.2: Variation of displacement (U) due to liquid phase at different time

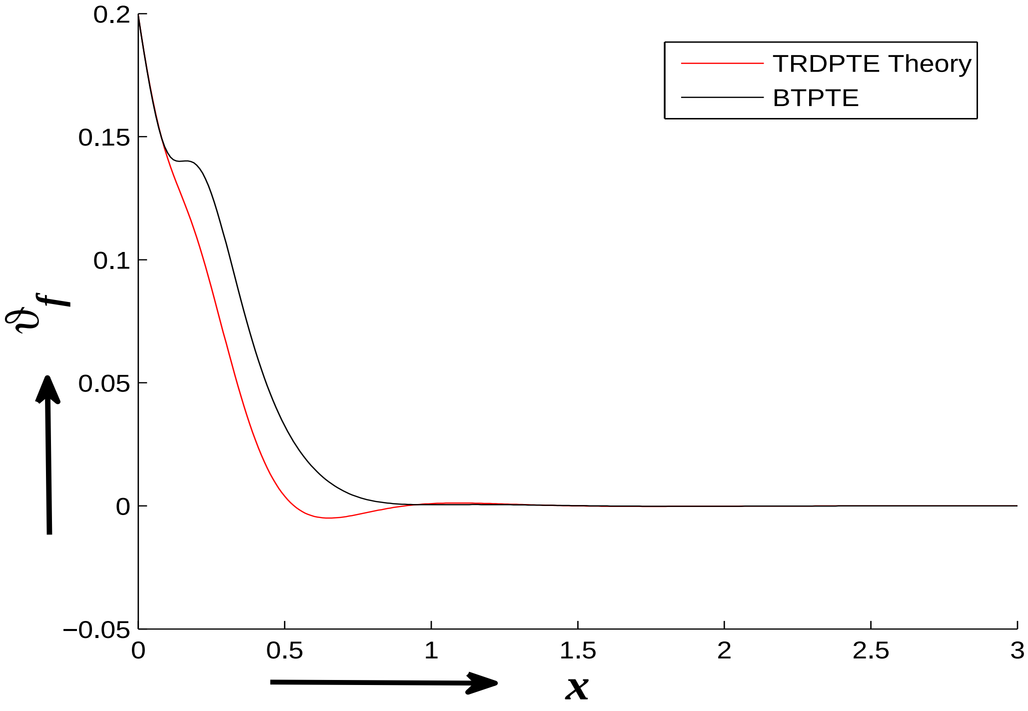


(a) $t = 0.36$

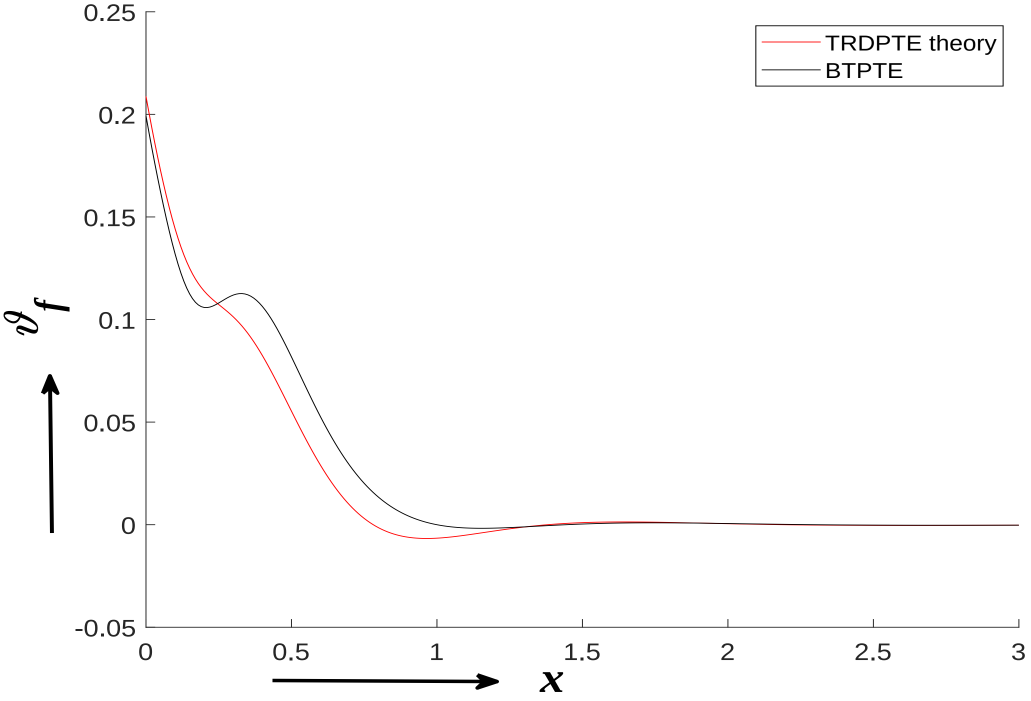


(b) $t = 0.57$

Figure 5.3.3: Variation of temperature (ϑ_s) due to solid phase at different time

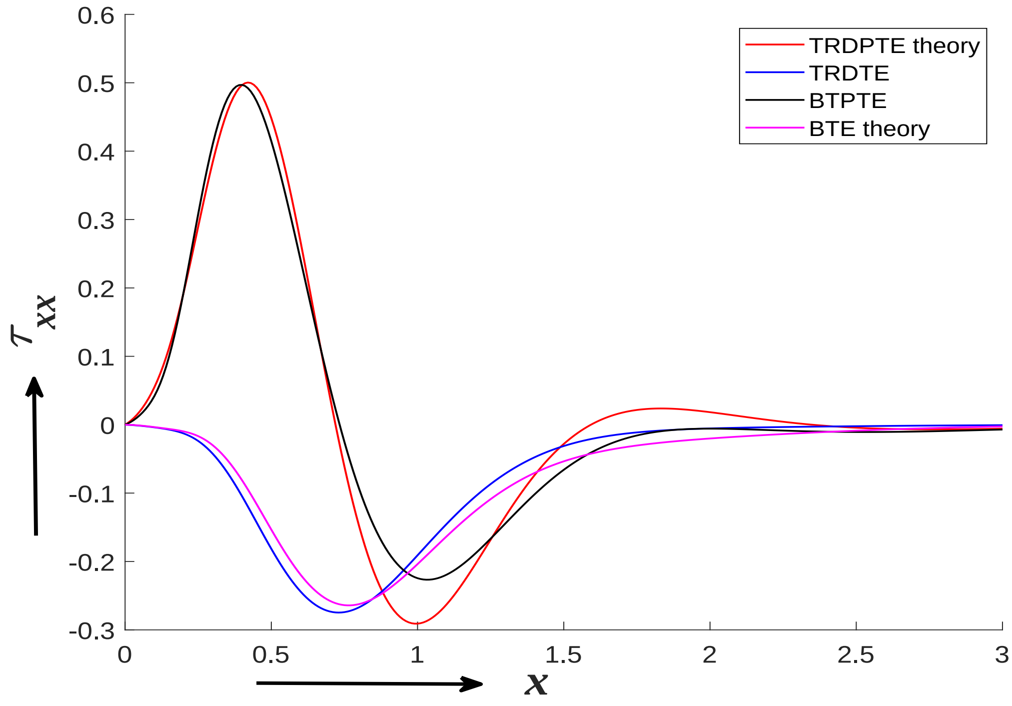


(a) $t = 0.36$

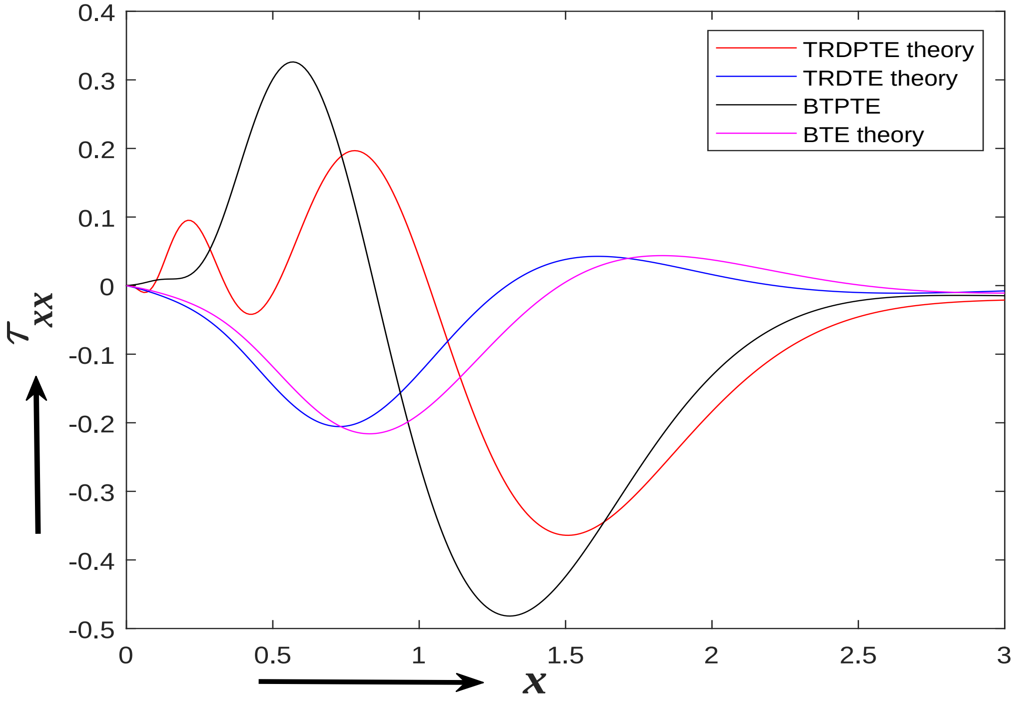


(b) $t = 0.57$

Figure 5.3.4: Variation of temperature (ϑ_f) due to liquid phase at different time

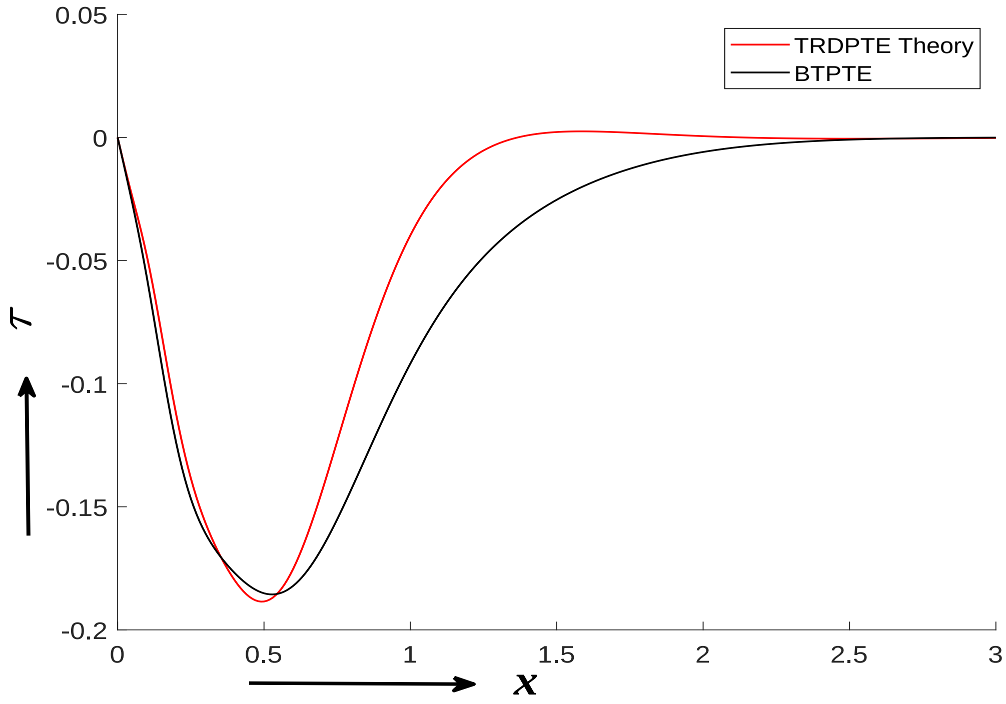


(a) $t = 0.36$

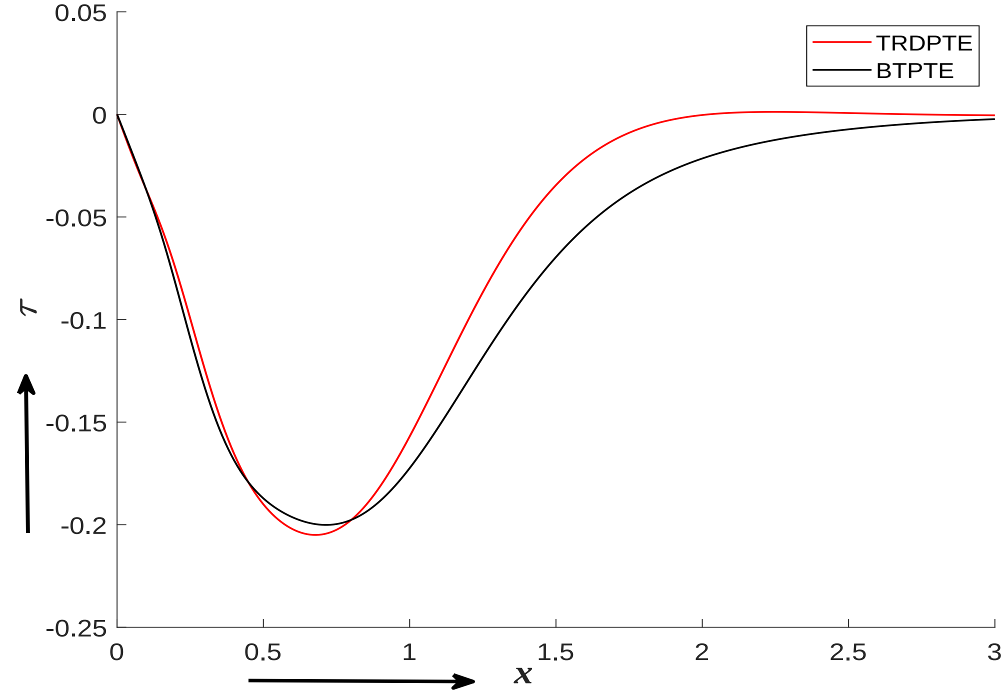


(b) $t = 0.57$

Figure 5.3.5: Variation of stress (τ_{xx}) due to solid phase at different time



(a) $t = 0.36$



(b) $t = 0.57$

Figure 5.3.6: Variation of stress (τ) due to liquid phase at different time

This is more pronounced for the displacement and stress fields. The numerical values of field variables increases under the influence of porosity, and the peak values are attained at a larger distance in the presence of porosity.

Figures 5.3.1 (a)-5.3.6 (a) and 5.3.1 (b)-5.3.6 (b) represent the distribution of all physical variable at two different time. From these figures, the effect of thermal shock can be observed for all physical fields, and it is clear that the effective domain of influence increases with time. The effective domain is larger for theories of BTPTE and BTE predicting infinite speed behaviour as compared to the theories of TRDPTE and TRDTE with finite speed of heat signals.

From Figs. 5.3.3 (a,b)-5.3.4 (a,b), we note that the temperature in solid phase has a larger effect of thermal shock, which verifies that heat propagation is much efficient in the sandstone as compared to kerosene liquid.

5.3.4 Conclusion

The implementation of the present porothermoelasticity theory is demonstrated in this subchapter by solving a half space problem using the state-space approach. By studying this problem, the effects of thermoelasticity and porosity coupling have been highlighted by the comparison of thermoelasticity theories with and without porosity. The two phase medium has been investigated under the thermal shock at the boundary of the half space, and it is noted that the physical variables exhibit a larger magnitude for solid phase quantities as compared to the fluid phase. Furthermore, it is revealed that the effects of temperature rate terms are more prominent in the presence of porosity. This is more pronounced for the displacement and stress fields as compared to the temperature field for the present problem. The absolute values of field variables increase under the influence of porosity, and the peak values are attained at a larger distance in this case. The effective domain of influence is larger for the theory of classical porothermoelasticity compared to the temperature-rate dependent porothermoelasticity, which is due to the infinite speed behaviour of classical theory compared to the theory of TRDPTE that admits finite speed of heat signals. It is believed that these results will be helpful in understanding the implementation of the present theory for studying dynamical problems and their prediction compared to other existing theories and understand the thermomechanical coupling effects in the presence of porosity.