Chapter 2

Mathematical Preliminaries

Throughout the thesis, the notation used adheres to standard conventions. The symbol $\mathbb{R}$ represents the set of real numbers, while $\mathbb{R}_+$ denotes the set of positive real numbers, i.e., $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z > 0\}$, $\mathbb{R}_{\geq 0}$ set of non-negative real numbers. Similarly, $\mathbb{R}_{>k}$ and $\mathbb{R}_{\geq k}$ represent the sets of real numbers greater than k and greater than or equal to k, respectively, where k is a real constant. The collection of non-negative integers is denoted by $\mathbb{Z}_{\geq 0}$. The notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean field. For any $p \geq 1$, the l_p-norm of a vector $y \in \mathbb{R}^n$ is defined as $\|y\|_p = (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}}$. When p is not explicitly mentioned, the norm is assumed to be the $2-$norm, denoted as $|y|$ which represents $\|y\|_2$. For matrices, we represent the transpose of matrix A as $A^\top$. Moreover, for a given vector $a = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^n$, $\text{diag}(a)$ denotes the diagonal matrix with elements $a_i \in \mathbb{R}$, where $i = 1, 2, ..., n$. In the context of a given set S, $\min\{S\}$ represents the smallest element within the set, whereas $\sup\{S\}$ denotes the supremum, which is the least upper bound, over the elements of S. If S_1, S_2, and S_3 are any sets, and we have two functions: $g_1 : S_1 \to S_2$ and $g_2 : S_2 \to S_3$, then their composition, denoted as $g_2 \circ g_1 : S_1 \to S_3$, is defined as $(g_2 \circ g_1)(\cdot) = g_2(g_1(\cdot))$. An n-dimensional open ball B_δ with radius δ is defined as the set of points in n-dimensional Euclidean space that are at a distance less than δ from a fixed point z_0. Mathematically, it can be represented as $B_\delta = \{z \in \mathbb{R}^n : \|z - z_0\| < \delta\}$, where z_0 is the fixed point. The absolute value function is symbolized as $|\cdot|$. The transpose of a vector is represented as $(\cdot)^\top$, and $\lceil z \rceil$ stands for the ceiling function (smallest integer greater than or equal to z). $\text{sign}(z) = \frac{z}{|z|}$, for $z \neq 0$ and $\text{sign}(0) = 0$. l_∞ denotes the space of bounded functions. $\min(\cdot, \cdot)$ represents the minimum function and yields the minimum value at any given instant of time.

To grasp the insights conveyed in this research, it is imperative for the reader to have a foundational understanding of the language and terminology commonly used in the examination of stability within discrete-time dynamical systems. The definitions and concepts elucidated in this chapter hold pivotal significance throughout the subsequent sections of this thesis. The majority of the foundational information in this segment has been adjusted and rephrased from the renowned textbook by H. K. Khalil, titled "Nonlinear Systems" [81] and Wasim Haddad, titled "Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach" [82], with appropriate citations included to acknowledge the source of this material.

2.1 Comparison Functions

We recall some definitions related to comparison functions that will be helpful in providing important definitions of stability.

Definition 2.1 ***(Class $\mathcal{K}$ function)*** *[81]*
A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class $\mathcal{K}$ function, if it is continuous, strictly increasing and $\alpha(0) = 0$. In addition, if $\lim_{s \to \infty} \alpha(s) = \infty$, then it is $\mathcal{K}_\infty$ function.

Definition 2.2 ***(Class $\mathcal{L}$ function)*** *[81]*
A function $\iota : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{L}$ function, if it is continuous, strictly decreasing and $\lim_{s \to \infty} \iota(s) = 0$.

Definition 2.3 ***(Class $\mathcal{KL}$ function)*** *[81]*
A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if it is a class $\mathcal{K}$ function in its first argument and a class $\mathcal{L}$ function in its second argument.

Lemma 2.4 *[81] Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be a class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then,*

- *α_1^{-1} is defined on $[0,\, \alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- *α_3^{-1} is defined on $[0,\, \infty)$ and belongs to class $\mathcal{K}_\infty$.*
- *$\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$.*
- *$\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$.*

Definition 2.5 ***(Generalized-$\mathcal{K}$ function)*** *[83]*

A map $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ *is a generalized-$\mathcal{K}$ function, if it is continuous, strictly increasing,* $\alpha(0) = 0$ *and satisfies:*

$$\begin{cases} \alpha(s_1) > \alpha(s_2), \textit{if } \alpha(s_1) > 0, s_1 > s_2 \\ \alpha(s_1) = \alpha(s_2), \textit{if } \alpha(s_1) = 0, s_1 > s_2. \end{cases}$$

Definition 2.6 ***(Generalized-$\mathcal{KL}$ function)*** *[83]*

A mapping $\Phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ *is a generalized-$\mathcal{KL}$ type ($\mathcal{GKL}$-function) if, for every fixed* $t \geq 0$*, the mapping* $s \mapsto \Phi(s, t)$ *is a generalized $\mathcal{K}$-function, and for each fixed* $s \geq 0$*, the mapping* $t \mapsto \Phi(s, t)$ *is continuous and decreases to zero as* $t \to T$ *for some* $T \leq \infty$*.*

2.2 Basic Stability Notions for Discrete-Time Systems

In this section, we revisit fundamental stability notions and definitions pertaining to the discrete-time dynamical systems.

Let us consider the following dynamical system

$$z(k + 1) = f(z(k)), \; z(0) = z_0 \tag{2.1}$$

where $z \in D \subseteq \mathbb{R}^n$ denotes the system state vector, $k \in \mathbb{Z}_{\geq 0}$, $f : D \to D$, and $f(0) = 0$. It is assumed that $f(\cdot)$ is continuous on D.

Definition 2.7 *[82] The zero solution* $z(k) \equiv 0$ *to* (2.1) *is Lyapunov stable if for all* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon) > 0$ *such that if* $\|z(0)\| < \delta$*, then* $\|z(0)\| < \epsilon$*,* $k \in \mathbb{Z}_{\geq 0}$*.*

Definition 2.8 *[82] The zero solution* $z(k) \equiv 0$ *to* (2.1) *is asymptotically stable if it is Lyapunov stable and there exists* $\delta > 0$ *such that if* $\|z(0)\| < \delta$*, then* $\lim_{k \to \infty} z(k) = 0$*.*

Definition 2.9 *[82] The zero solution* $z(k) \equiv 0$ *to* (2.1) *is globally asymptotically stable if it is Lyapunov stable and for all* $z(0) \in \mathbb{R}^n$*,* $\lim_{k \to \infty} z(k) = 0$*.*

Definition 2.10 *[82] The zero solution* $z(k) \equiv 0$ *to* (2.1) *is geometrically stable if there exists positive constants* α*,* $\beta > 1$*, and* δ *such that if* $\|z(0)\| < \delta$*, then* $\|z(k)\| \leq \alpha \|z(0)\| \beta^{-k}$*,* $k \in \mathbb{Z}_{\geq 0}$*.*

Definition 2.11 *[82] The zero solution* $z(k) \equiv 0$ *to* (2.1) *is globally geometrically stable if there exists positive constants* $\alpha,\ \beta > 1$ *such that for all* $z(0) \in \mathbb{R}^n$, $\|z(k)\| \leq \alpha\|z(0)\|\beta^{-k}$, $k \in \mathbb{Z}_{\geq 0}$.

Finally, the zero solution $z(k) \equiv 0$ to (2.1) is unstable if it is not Lyapunov stable.

2.3 Finite-Time Stability of Discrete-Time Systems

Let us consider the following dynamical system

$$z(k+1) = f(z(k)),\ z(0) = z_0 \tag{2.2}$$

where $z \in D \subseteq \mathbb{R}^n$ denotes the system state vector, $k \in \mathbb{Z}_{\geq 0}$, $f : D \to D$, and $f(0) = 0$.

Definition 2.12 *(Finite-Time Stability) [37]*
Consider the nonlinear dynamical system (2.2). *The zero solution* $z(k) \equiv 0$ *to* (2.2) *is finite-time stable if there exist an open neighborhood* $N \subseteq D$ *of the origin and a function* $\mathcal{K} : N\backslash\{0\} \to \mathbb{Z}_+$, *called the settling time function, such that the following statements hold:*

1. *Finite-time convergence: For every* $z \in N\backslash\{0\}$ *and* $k \geq \mathcal{K}(z)$, $z(k)$ *is contained in* $N \cap \{0\}$.
2. *Lyapunov stability: For every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $B_\delta(0) \subset N$ *and for every* $z \in B_\delta(0)\backslash\{0\}$, $z(k) \in B_\epsilon(0)$ *for all* $k \in \{0, \ldots, \mathcal{K}(z_0) - 1\}$.

The zero solution $z(k) \equiv 0$ *to* (2.2) *is globally finite-time stable if it is finite-time stable with* $N = D = \mathbb{R}^n$.

The following lemma is proposed in [37] to characterize the finite-time stability.

Lemma 2.13 *[37] Consider the system* (2.2). *Assume the existence of a continuous map* $V : D \to \mathbb{R}_{\geq 0}$, *which satisfies:*

$$V(z) > 0, \forall\ z \in D\backslash\{0\}, \tag{2.3}$$

$$V(z = 0) = 0 \ \textit{and} \tag{2.4}$$

$$\Delta V(z) \leq -\min\{V(z), l\}, \forall\ z \in D\backslash\{0\} \tag{2.5}$$

where $l \in \mathbb{R}_+$. *Then the zero solution* $z(k) \equiv 0$ *to* (2.2) *exhibits finite-time stability. Moreover, there exist an open neighborhood* N *of the origin and a settling-time function* $\mathcal{K} : N \to \mathbb{Z}_+$ *such that* $\mathcal{K}(z_0) \leq \lceil \frac{V(z_0)}{l} \rceil$, $z_0 \in N$, *where* $\mathcal{K}(\cdot)$ *is lower semicontinuous on* N. *If, in addition,* $N = D = \mathbb{R}^n$, $V(\cdot)$ *is radially unbounded, and* (2.5) *holds on* $\mathbb{R}^n$, *then zero solution* $z(k) \equiv 0$ *to* (2.2) *is globally finite-time stable.*

Similarly, another characterization of finite-time stability of system (2.2) is provided below.

Lemma 2.14 *[37] Consider the system* (2.2). *Assume the existence of a continuous map* $V : D \to \mathbb{R}_{\geq 0}$, *which satisfies:*

$$V(z) > 0, \forall\, z \in D \backslash \{0\}, \tag{2.6}$$

$$V(z = 0) = 0 \text{ and} \tag{2.7}$$

$$\Delta V(z) \leq -l \min \left\{ \frac{V(z)}{l}, (V(z))^{\sigma} \right\}, \forall\, z \in D \backslash \{0\} \tag{2.8}$$

where $l \in \mathbb{R}_+$ *and* $\sigma \in (0, 1)$. *Then the zero solution* $z(k) \equiv 0$ *to* (2.2) *exhibits finite-time stability. Moreover, there exist an open neighborhood* N *of the origin and a settling-time function* $\mathcal{K} : N \to \mathbb{Z}_+$ *such that* $\mathcal{K}(z_0) \leq \lceil \log_{[1-lV(z_0)^{\sigma-1}]} \frac{l^{\frac{1}{1-\sigma}}}{V(z_0)} \rceil + 1$, $z_0 \in D$, *for* $V(z_0) > l^{\frac{1}{1-\sigma}}$ *or,* $\mathcal{K}(z_0) = 1$, $z_0 \in D \backslash \{0\}$, *for* $V(z_0) \leq l^{\frac{1}{1-\sigma}}$, *where* $\mathcal{K}(\cdot)$ *is lower semicontinuous on* N. *If, in addition,* $N = D = \mathbb{R}^n$, $V(\cdot)$ *is radially unbounded, and* (2.8) *holds on* $\mathbb{R}^n$, *then zero solution* $z(k) \equiv 0$ *to* (2.2) *is globally finite-time stable.*

2.4 Finite-Time Input-to-State Stability of Discrete-Time Systems

Now, we discuss the notions of the finite-time input-to-state stability of discrete-time systems (see [84] for more detail), useful in developing the Lyapunov stability conditions for algorithms presented in the subsequent chapters, when discrete-time systems are subjected to matched type bounded perturbations (MTBP). Consider the following dynamical system

$$z(k+1) = \psi(z(k), \nu(k)) \tag{2.9}$$

where $z \in D \subseteq \mathbb{R}^n$ denotes the state vector of the system, $k \in \mathbb{Z}_{\geq 0}$, $\nu(k) \in \mathbb{R}^m$ represents the exogenous input to the system. The mapping $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous on its domain satisfying $\psi(0, 0) = 0$. We define the solution to (2.9) as $z(k) := z(k, z_0, \nu(k))$ for any initial condition z_0.

Definition 2.15 *(Finite-Time Input-to-State Stability) [84]*
System (2.9) *is said to be finite-time input-to-state stable if there exist* $\beta(\cdot, \cdot) \in \mathcal{GKL}$ *and* $\gamma(\cdot) \in \mathcal{K}_\infty$ *such that, for every* z_0 *and* $\nu(k)$*, the following holds:* $\|z(k)\| \leq \beta(\|z_0\|, k - k_0) + \gamma(\|\nu\|)$*, with* $\beta(\zeta, k) = 0$ *for* $k \geq \mathcal{K}(\zeta)$*, where* $\mathcal{K}(\zeta)$ *is continuous on its argument with* $\mathcal{K}(0) = 0$*.*

Lemma 2.16 *[84] Consider the system* (2.9)*. Assume that there exist a continuous function* $V : \mathbb{Z}_{\geq 0} \times D \to \mathbb{R}_{\geq 0}$*, which satisfies:*

$$\alpha_1(\|z(k)\|) \leq V(k, z) \leq \alpha_2(\|z(k)\|), \forall\ k \in \mathbb{Z}_{\geq 0}, \forall\ z \in D \backslash \{0\}, \tag{2.10}$$

$$V(k, 0) = 0, \forall\ k\ \in \mathbb{Z}_{\geq 0},\ \textit{and} \tag{2.11}$$

$$\Delta V(k, z) \leq -\min\{(V(k, z), (V(k, z))^a\} + \alpha_3(|\nu(k)|), \forall\ k \in \mathbb{Z}_{\geq 0}, \forall\ z \in D \backslash \{0\}, \tag{2.12}$$

where $a \in (0, 1)$*,* α_1, α_2 *and* $\alpha_3 \in$ *class* $\mathcal{K}$ *functions. Then the system* (2.9) *is finite-time input-to-state stable. Moreover, if* $D = \mathbb{R}^2$*,* α_1 *and* $\alpha_2 \in$ *class* $\mathcal{K}_\infty$ *functions and* $\Delta V(k, z) \leq -\min\{V(k, z),\ (V(k, z))^a\} + \alpha_3(|\nu(k)|), \forall\ k \in \mathbb{Z}_{\geq 0}, \forall z \in \mathbb{R}^2 \backslash \{0\}$*, then the system* (2.9) *is global finite-time input-to-state stable.*